Barvinok's enumeration algorithm and applications to Statistics

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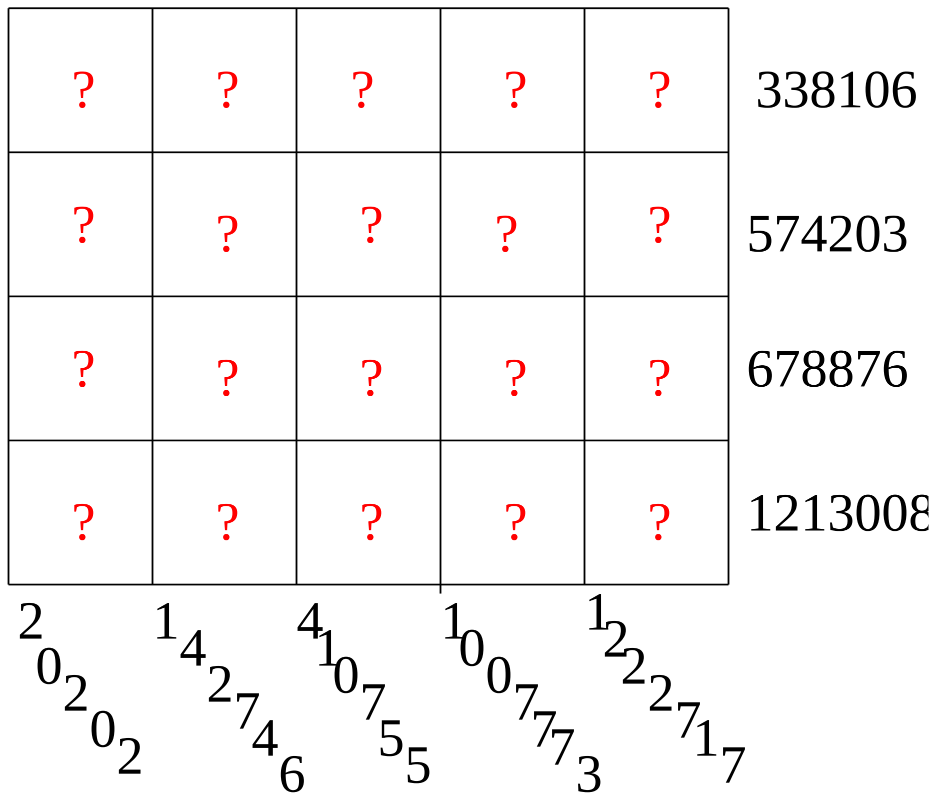
Getting started...

HOW MANY WAYS are there?



How to solve

Let $P = \{x \in \mathbb{R}^d | Ax = a,\ Bx \leq b\}$, where A, B are integral matrices and a, b are integral vectors.

Tool: The multivariate generating function

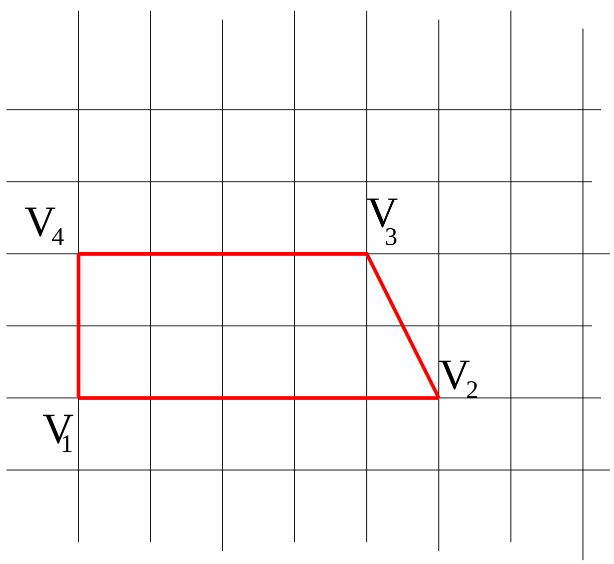
$$f(P, z) = \sum_{\alpha \in P \cap \mathbb{Z}^d} z^{\alpha},$$

where $z^{\alpha} = z_1^{\alpha_1} z_2^{\alpha_2} \ldots z_d^{\alpha_d}$.

This is an infinite formal power series if P is not bounded, but if P is a polytope it is a polynomial.

Example for $f(P, z)$

Let $V_1 = (0, 0)$, $V_2 = (5, 0)$, $V_3 = (4, 2)$, and $V_4 = (0, 2)$.



Each vertex is represented by the following monomials:

For $V_1 = (0,0)$, $z^{V_1} = z_1^0 z_2^0 = 1$.

For $V_2 = (5,0)$, $z^{V_2} = z_1^5 z_2^0 = z_1^5$.

For $V_3 = (4,2)$, $z^{V_3} = z_1^4 z_2^2$.

For $V_4 = (0,2)$, $z^{V_4} = z_1^0 z_2^2 = z_2^2$.

In this manner, we have $f(P,z)$ as the following:

$$f(P,z) = {z_1}^5 + {z_1}^4 z_2 + {z_1}^4 + {z_1}^4 {z_2}^2 + z_2 {z_1}^3 + {z_1}^3 + {z_1}^3 {z_2}^2 + z_2 {z_1}^2 + {z_1}^2 + {z_1}^2 {z_2}^2 + z_1 z_2 + z_1 + z_1 {z_2}^2 + {z_2}^2 + z_2 + 1.$$

However...

The multivariate generating function $f(P, z)$ has exponentially many monomials even in fixed the dimension.

Question: How can we encode $f(P, z)$ in polynomial size if we fix the dimension?

Answer: We can encode $f(P, z)$ as a short sum of rational functions.

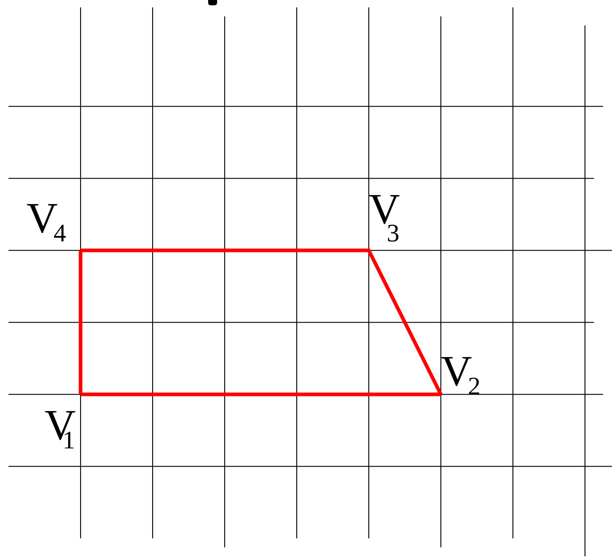
Theorem: [Barvinok (1993)]

Assume that we fix the dimension d and suppose we have a rational convex polyhedron $P = \{ u \in \mathbb{R}^d \ : \ Au \leq b \}$, where $A \in \mathbb{Z}^{m \times d}$ and $b \in \mathbb{Z}^m$. Then there exists a polynomial time algorithm to compute $f(P, z)$ in the following form of:

$$f(P, z) \quad = \quad \sum_{i \in I} \pm \frac{z^{u_i}}{(1 - z^{v_{1i}})(1 - z^{v_{2i}}) \ldots (1 - z^{v_{di}})}$$

where u_i, v_{1i}, $\ldots$ $v_{di} \in \mathbb{Z}^d$ for all $i \in I$.

From the previous example



$$f(P,z) = {z_1}^5 + {z_1}^4 z_2 + {z_1}^4 + {z_1}^4 {z_2}^2 + z_2 {z_1}^3 + {z_1}^3 + {z_1}^3 {z_2}^2 + z_2 {z_1}^2 + {z_1}^2 + {z_1}^2 {z_2}^2 + z_1 z_2 + z_1 + z_1 {z_2}^2 + {z_2}^2 + z_2 + 1$$

$$= \frac{1}{(1-z_1)(1-z_2)} + \frac{{z_1}^5}{(1-{z_1}^{-1})(1-z_2)} + \frac{z_1^2}{(1-z_1)(1-z_2^{-1})} + \frac{z_1^5}{(1-z_1^{-1}z_2)(1-z_2^{-1})} + \frac{{z_1}^4 {z_2}^2}{(1-{z_2}^{-1})(1-z_1)} - \frac{z_1^4 z_2^2}{(1-z_1^{-1}z_2^2)(1-z_1^{-1})}.$$

LattE

- **INPUT**: Integral matrices A, B and integral vectors a, b for the polytope P.

- **OUTPUT**: $f(P, z)$ written as a short sum of rational functions.

- **APPLICATIONS**:

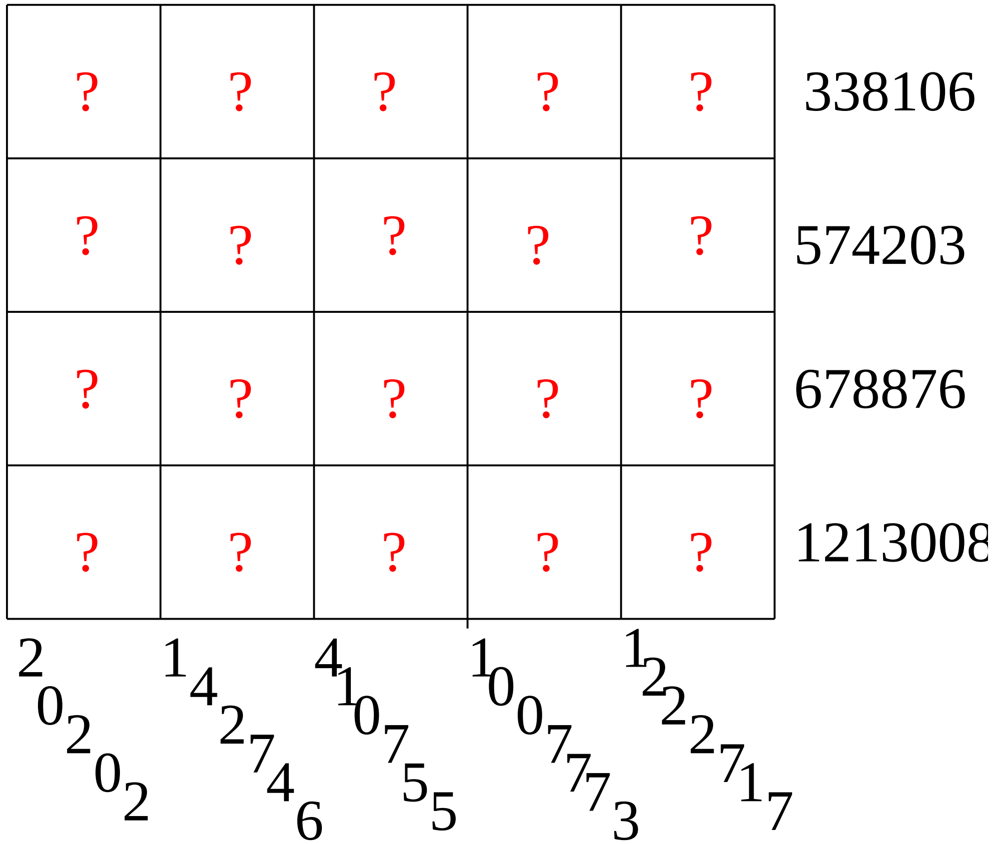
  (A) Counting Problem,

  (B) Integer Programming,

  (C) Integer Feasibility Problem,

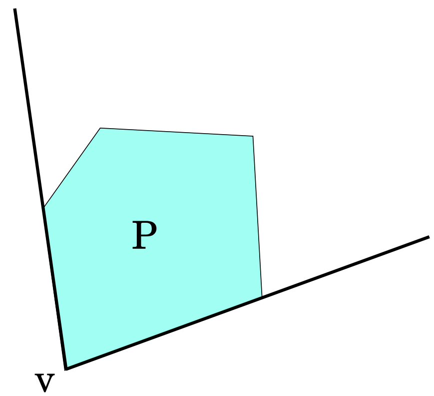
  (D) Computing a universal test set of a given integral matrix A.

Answer to the puzzle



3160528209301169094598220490521497877480049630580229972623 97.

Theory behind the implementation



Theorem[Brion, Lawrence]

Let P be a convex polyhedron and let $V(P)$ be the vertex set of P. Let K_v be the tangent cone at $v \in V(P)$. Then

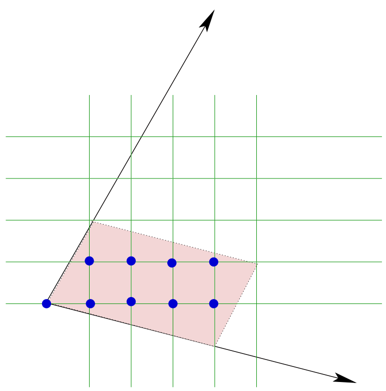
$$f(P, z) = \sum_{v \in V(P)} f(K_v, z).$$

If we have a simple cone...

For a simple cone $K \subset \mathbb{R}^d$,

$$f(K, z) = \frac{\sum_{u \in \Pi \cap \mathbb{Z}^d} z^u}{(1 - z^{c_1})(1 - z^{c_2}) \ldots (1 - z^{c_d})}$$

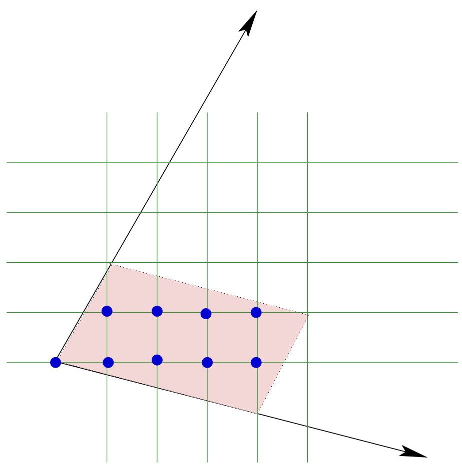
where Π is the half open parallelepiped spanned by the rays of the cone K, $c_1, \ldots, c_d \in \mathbb{Z}^d$.



Example

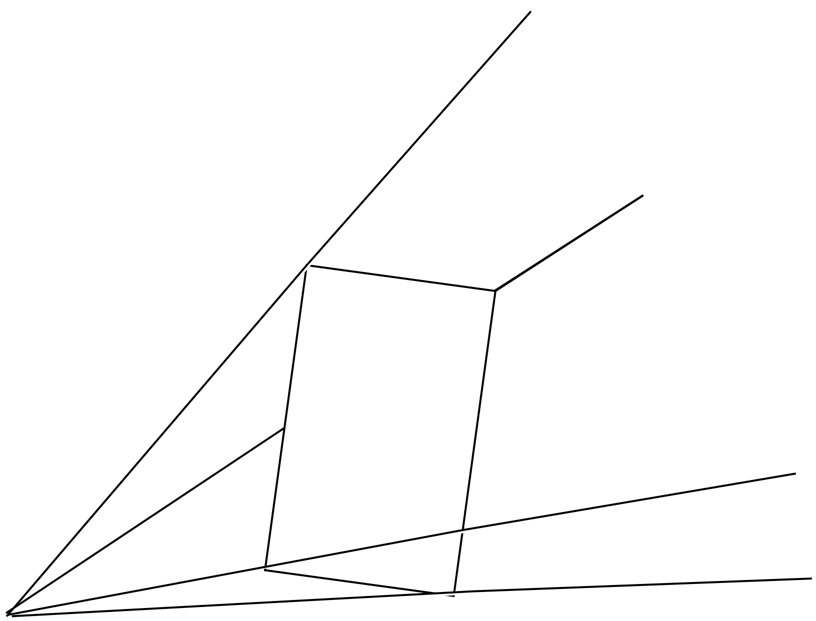
In this case, we have $d = 2$ and $c_1 = (1, 2)$, $c_2 = (4, -1)$. We have:

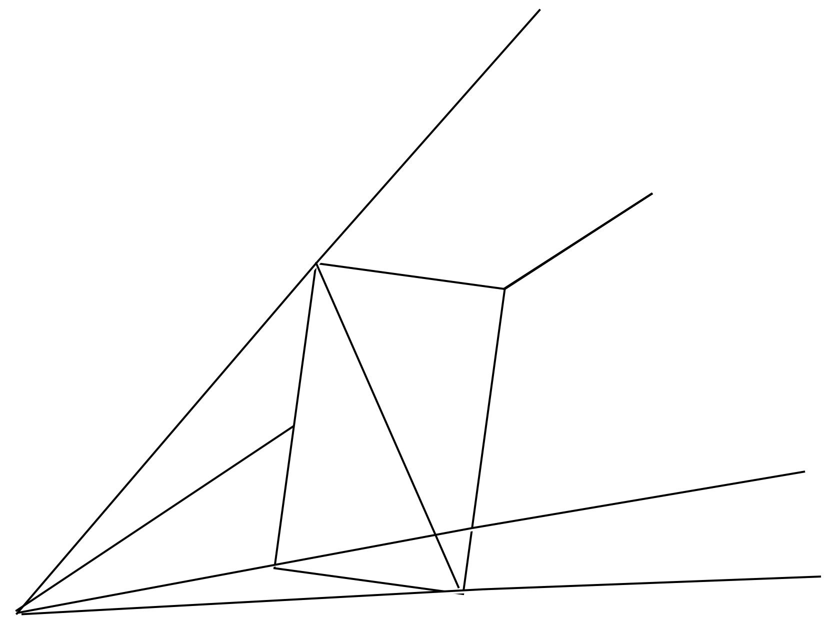
$$f(K, z) = \frac{z_1^4 z_2 + z_1^3 z_2 + z_1^2 z_2 + z_1 z_2 + z_1^4 + z_1^3 + z_1^2 + z_1 + 1}{(1 - z_1 z_2^2)(1 - z_1^4 z_2^{-1})}.$$



If a cone K is not simple....

We triangulate the cone into simple cones.





Fact: If the parallelepiped Π for a simple cone K has only one integral point u, then

$$f(K, z) = \frac{z^u}{\prod_{i=1}^{d}(1 - z^{c_i})}.$$

Goal: Want to decompose K into simple cones whose parallelepipeds Π have ONLY one integral point.

Definition: A *unimodular cone* K is a simple cone such that the half open parallelepiped generated by the rays of K contains only one lattice point

Barvinok's cone decomposition

Theorem [Barvinok] Fix the dimension d. Then there exists a polynomial time algorithm which decomposes a rational polyhedral cone $K \subset \mathbb{R}^d$ into unimodular cones K_i with numbers $\epsilon_i \in \{-1, 1\}$ such that

$$f(K, z) = \sum_{i \in I} \epsilon_i f(K_i, z), \quad |I| < \infty.$$

We have so far...

1. Find the vertices of the given polytope and their defining tangent cones.

2. Triangulate and apply Barvinok's cone decomposition to each of the cones.

3. Obtain the signed rational function of each cone and sum them up.

We need to include lower dimensional cones and also apply the inclusion-exclusion principle, to get the multivariate generating function $f(P, z)$.

Brion's polarization trick

Lemma: Let $K \subset \mathbb{R}^d$ be a cone. If K contains a straight line then $f(K, z) \equiv 0$.

By this lemma we can avoid using the inclusion-and-exclusion principle and save time and memory. The trick is the following:

1. Polarize the tangent cone.

2. Decompose the polar into unimodular cones.

3. Polarize back each of the unimodular cones.

If we do this process, the multivariate function $f(K, z)$ for each lower dimensional cone K becomes zero.

Example

$d = 2$ and $K = \text{cone}\{(1,\, 0),\ (1,\, k)\}$ for some large $k \in \mathbb{Z}$.

The dual cone $K^* = \text{cone}\{(-k,\, 1),\ (0,\, -1)\}$.

Applying Barvinok's cone decomposition, we have

$$f(K^*, z) = f(K_1^*, z) + f(K_2^*, z) + f(K_3^*, z),$$

where $K_1^* = \text{cone}\{(-k,\, 1),\ (-1,\, 0)\}$, $K_2^* = \text{cone}\{(0,\, -1),\ (-1,\, 0)\}$, and $K_3^* = \text{cone}\{(-1,\, 0)\}$.

Applying Brion's trick, we have

$$f(K, z) = f(K_1, z) + f(K_2, z),$$

where $K_1 = \text{cone}\{(0,\, -1), (1,\, k)\}$ and $K_2 = \text{cone}\{(0,\, 1), (1,\, 0)\}$.

Outline of the implementation

1. Find the vertices of the given polytope and their defining tangent cones.

2. Compute the polar cone to each of the cones.

3. Triangulate and apply Barvinok's cone decomposition to each of the polar cones.

4. Polarize back each of the full dimensional unimodular cones.

5. Obtain the signed rational function of each cone and sum them up.

Applications to statistics

Computing Markov basis

Suppose $P = \{x \in \mathbb{R}^d | Ax = b,\ x \geq 0\} \neq \emptyset$ and let M be a finite set such that $M \subset \{x \in \mathbb{Z}^d | Ax = 0\}$.

We define the graph G_b such that:

- Nodes of G_b are all the lattice points inside of P.
- We draw an undirected edge between a node u and a node v iff $u - v \in M$.

Definition :

M is called a Markov basis if G_b is a connected graph for all b.

Example

				Total
	? ? ?	? ? ?	? ? ?	6
	? ? ?	? ? ?	? ? ?	6
Total	4	4	4	

Table 1: 2×3 tables with 1-marginals.

There are 19 tables with these marginals.

$\pm$

1	-1	0
-1	1	0

$\pm$

0	1	-1
0	-1	1

$\pm$

1	0	-1
-1	0	1

There are 3 elements in a Markov basis modulo signs.

4	0	2
0	4	2

+

-1	0	1
1	0	-1

=

3	0	3
1	4	1

A table with the marginals plus an element of a Markov basis is also a table with the given marginals.

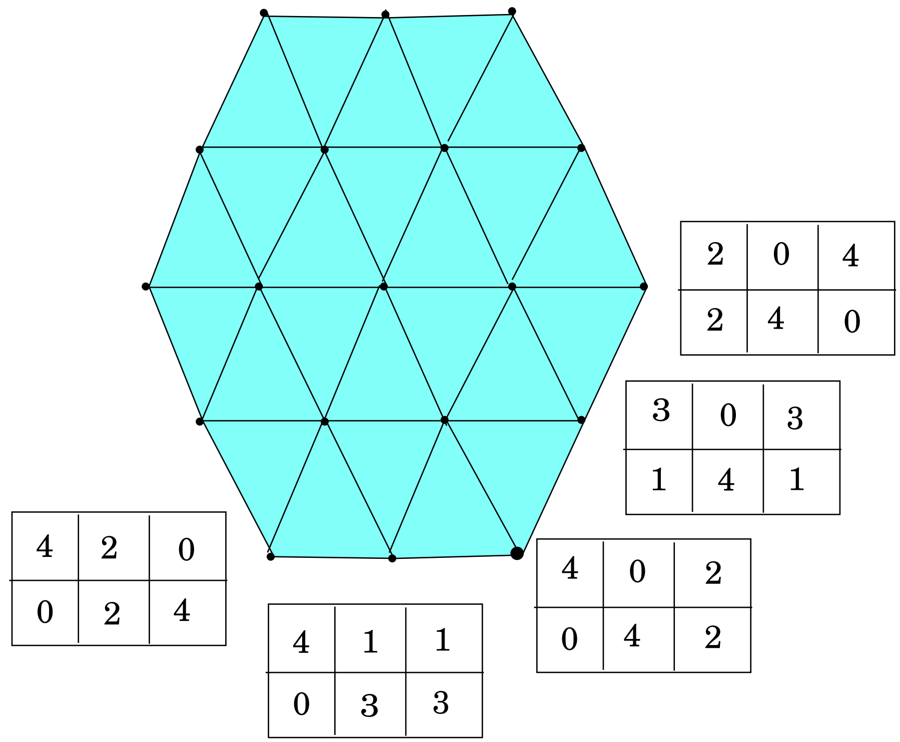


Figure 1: A Markov basis for 2×3 tables. An element of the Markov basis is a undirected edge between integral points in the polytope.

Problem

Input: An integral $m \times d$ matrix A for the polytope $P = \{x | Ax = b,\ x \geq 0\}$.

Note: The cardinarity of a Markov basis for tables can be really large. (It is exponential even if we fix the number of cells).

Problem: Want to compute a Markov basis of A in polynomial time if we fix the number of cells.

Main Theorem

Theorem [De Loera, Haws, Hemmecke, Huggins, Sturmfels, Y.]

Let $A \in \mathbb{Z}^{d \times n}$, where d and n are fixed.

Then there is a polynomial time algorithm to compute the multivariate generating function for a Markov basis associated to A as a short sum of rational functions.

Theorem [De Loera, Haws, Hemmecke, Huggins, Sturmfels, Y.]

Let $A \in \mathbb{Z}^{d \times n}$, $b \in \mathbb{Z}^d$, $W \in \mathbb{Z}^{n \times n}$, where d and n are fixed.

Suppose the term order $\prec_W$ is given. Then there is a polynomial time algorithm to compute the multivariate generating function G for the reduced Gröbner basis of the toric ideal associated to A with the term order $\prec_W$ as a short sum of rational functions.

Example (universal Gröbner basis)

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix}.$$

Encode binomials $x^u - x^v$ in n variables as $x^u y^v$ in $2n$ variables $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots, x_n^{\alpha_n}$ and $y^\beta = y_1^{\beta_1} y_2^{\beta_2}, \ldots, y_n^{\beta}$.

The output is a sum of 538 simple rational functions such as $\left(1 - \frac{x_3 y_4}{x_1 y_2}\right)\left(1 - \frac{x_1 x_4 y_2}{x_3}\right)(1 - x_1 y_1)\left(1 - x_1 x_3 {y_2}^2\right)(1 - x_3 y_3)(1 - x_2 y_2)$.

The number of binomials is 207188785065 and CPU time is 12 seconds.

Another theorem via LattE...

Theorem: [De Loera, Haws, Hemmecke, Huggins, Sturmfels, Y.]

The generating function for the number of $3 \times 3 \times 3 \times 3$ magic cubes is given the rational function $p(t)/q(t)$ where

$$p(t) = t^{54} + 150\,t^{51} + 5837\,t^{48} + 63127\,t^{45} + 331124\,t^{42} + 1056374\,t^{39} + 2326380\,t^{36} + 3842273\,t^{33} + 5055138\,t^{30} + 5512456\,t^{27} + 5055138\,t^{24} + 3842273\,t^{21} + 2326380\,t^{18} + 1056374\,t^{15} + 331124\,t^{12} + 63127\,t^{9} + 5837\,t^{6} + 150\,t^{3} + 1$$

$$q(t) = \left(t^3+1\right)^4 \left(t^{12}+t^9+t^6+t^3+1\right) \left(1-t^3\right)^9 \left(t^6+t^3+1\right).$$

$$\frac{p(t)}{q(t)} = 1 + 153t^3 + 6297t^6 + 82161t^9 + 582377t^{12} + 2823169t^{15} + \ldots$$

Theorem: [De Loera, Haws, Hemmecke, Huggins, Sturmfels, Y.]

The generating function in t for the number $f(n)$ of 5×5 magic squares of magic sum n is given by the rational function $p(t)/q(t)$ with

$$\begin{aligned}
p(t) = {}& t^{76}+28\,t^{75}+639\,t^{74}+11050\,t^{73}+136266\,t^{72}+1255833\,t^{71}+9120009\,t^{70}+54389347\,t^{69}+ \\
& 274778754\,t^{68} + 1204206107\,t^{67} + 4663304831\,t^{66} + 16193751710\,t^{65} + 51030919095\,t^{64} + \\
& 147368813970\,t^{63} + 393197605792\,t^{62} + 975980866856\,t^{61} + \\
& 2266977091533\,t^{60} + 4952467350549\,t^{59} + 10220353765317\,t^{58} + 20000425620982\,t^{57} + \\
& 37238997469701\,t^{56} + \\
& 66164771134709\,t^{55} + 112476891429452\,t^{54} + 183365550921732\,t^{53} + 287269293973236\,t^{52} + \\
& 433289919534912\,t^{51} + 630230390692834\,t^{50} + 885291593024017\,t^{49} + 1202550133880678\,t^{48} + \\
& 1581424159799051\,t^{47} + 2015395674628040\,t^{46} + 2491275358809867\,t^{45} + \\
& 2989255690350053\,t^{44} + 3483898479782320\,t^{43} + 3946056312532923\,t^{42} + \\
& 4345559454316341\,t^{41} + 4654344257066635\,t^{40} + 4849590327731195\,t^{39} + \\
& 4916398325176454\,t^{38} + 4849590327731195\,t^{37} + 4654344257066635\,t^{36} + \\
& 4345559454316341\,t^{35} + 3946056312532923\,t^{34} + 3483898479782320\,t^{33} + \\
& 2989255690350053\,t^{32} + 2491275358809867\,t^{31} + 2015395674628040\,t^{30} + \\
& 1581424159799051\,t^{29} + 1202550133880678\,t^{28} + 885291593024017\,t^{27} + \\
& 630230390692834\,t^{26} + 433289919534912\,t^{25} + 287269293973236\,t^{24} + 183365550921732\,t^{23} + \\
& 112476891429452\,t^{22} + 66164771134709\,t^{21} + 37238997469701\,t^{20} + 20000425620982\,t^{19} +
\end{aligned}$$

$10220353765317\,t^{18} + 4952467350549\,t^{17} + 2266977091533\,t^{16} + 975980866856\,t^{15} + 393197605792\,t^{14} + 147368813970\,t^{13} + 51030919095\,t^{12} + 16193751710\,t^{11} + 4663304831\,t^{10} + 1204206107\,t^{9} + 274778754\,t^{8} + 54389347\,t^{7} + 9120009\,t^{6} + 1255833\,t^{5} + 136266\,t^{4} + 11050\,t^{3} + 639\,t^{2} + 28\,t + 1$

$$q(t) = \left(t^2 - 1\right)^{10} \left(t^2 + t + 1\right)^7 \left(t^7 - 1\right)^2 \left(t^6 + t^3 + 1\right) \left(t^5 + t^3 + t^2 + t + 1\right)^4 \cdot (1 - t)^3 \left(t^2 + 1\right)^4$$

$$\frac{p(t)}{q(t)} = 1 + 20t + 449t^2 + 6792t^3 + 67063t^4 + 484419t^5 + \cdots.$$

Question??

If you are interested in our software Lattice Point Enumeration, please go to:

`http://www.math.ucdavis.edu/~latte`.

If you have any questions about `LattE`, please send me email at

ruriko@math.duke.edu.