# Computing holes in semi-groups

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## Introduction

- For a matrix $A \in \mathbb{Z}^{d\times n}$, let $C$, $L$, and $Q$ denote the cone, the lattice, and the semi-group (monoid) spanned by the columns $A_{.j}$, $j = 1,\dots,n$, of $A$.
- We assume the cone $C$ to be pointed.
- By $Q_{\text{sat}} = C \cap L$ we denote the *saturation* of $Q$ and call $Q$ *normal* if the set $H = Q_{\text{sat}} \setminus Q$ is empty.
- The elements of $H$ are called *holes* and a hole $h \in H$ is *fundamental* if there is no other hole $h' \in H$ such that $h - h' \in Q$.
- While $F$ is always finite [TY06], $H$ could be infinite.
- We call $s \in Q$ a *saturation point* of $Q$, if $s + Q_{\text{sat}} \subseteq Q$. The set of all saturation points of $Q$ is denoted by $S$.
- By $\min(S;Q)$ we denote the set of all $Q$-minimal elements of $S$, that is, the set of all $s \in S$ for which there is no other $s' \in S$ with $s - s' \in Q$. Again, it can be shown that $\min(S;Q)$ is always finite [TY06, Prop. 4.4].

**Goal.** We present an algorithm that computes an *explicit* representation of $H$.

## Example

Consider the $2 \times 4$ matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 4 \end{pmatrix}.$$

The associated semi-group $Q$ has infinitely many holes

$$H = \{(1,1)^\intercal + \alpha \cdot (1,0)^\intercal : \alpha \in \mathbb{Z}_+\},$$

out of which only $(1,1)^\intercal$ is fundamental, see Figure 2. Moreover, the semi-group $Q$ has three $Q$-minimal saturation points: $(1,2)^\intercal$, $(1,3)^\intercal$, and $(1,4)^\intercal$.

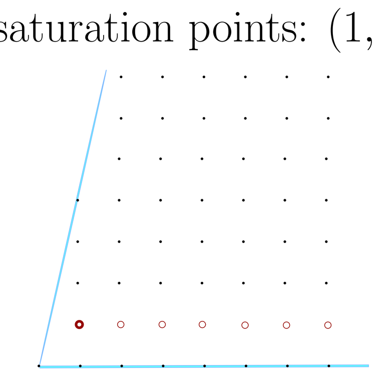

Figure 2: Non-holes, holes and fundamental hole for the example.

## Main algorithm

**Algorithm.** (Computing an *explicit* representation of $H$.)

1. Compute the set $F$ of fundamental holes.
2. For each of the finitely many $f \in F$, compute the set $\min((f+Q)\cap Q;Q)$ of $Q$-minimal elements in $(f+Q)\cap Q$. Herein, $s \in (f+Q)\cap Q$ is called $Q$-minimal if there is no other $s' \in (f+Q)\cap Q$ with $s - s' \in Q$.
3. From the $Q$-minimal elements in $(f+Q)\cap Q$, compute an explicit representation of the holes of $Q$ lying in $f+Q$.

## Computing the fundamental holes $F$

**Note.** The set $F$ of fundamental holes is finite [TY06], since it is a subset of

$$P := \left\{\sum_{j=1}^{n} \lambda_j A_{.j} : 0 \leq \lambda_1,\dots,\lambda_n < 1\right\}.$$

Let $B$ be the minimal integral generating set of $C \cap L$.

- If $B$ contains no hole of $Q$, $Q$ must be normal.
- Moreover, every hole of $Q$ appearing in $B$ must be fundamental, since $B$ is minimal.
- If $f \in F$ is not in $B$, $f$ can be written as a nonnegative integer linear combination of elements in $B$, since $f \in C \cap L$ and since $B$ is an integral generating set of $C \cap L$. This representation cannot have summands that are not fundamental holes, since otherwise $f$ is not fundamental.

**Algorithm.** (Computing the fundamental holes $F$)

1. Compute the minimal integral generating set $B$ of $C \cap L$.
2. Check each $z \in B$ whether it is a fundamental hole or not, that is, compute $B \cap F$.
3. Generate all nonnegative integer combinations of elements in $B \cap F$ that lie in $P$ and check for each such $z$ whether it is a fundamental hole or not.

## Computing the $Q$-minimal elements in $(f+Q)\cap Q$

In order to compute these $Q$-minimal elements, we have to find an explicit representation for the solutions of

$$\{\lambda \in \mathbb{Z}_+^n : \exists \mu \in \mathbb{Z}_+^n \text{ such that } f + A\lambda = A\mu\}. \tag{1}$$

Every $Q$-minimal point $z \in (f+Q)\cap Q$ must correspond to a minimal inhomogeneous solution $\lambda$ of this system.

## Computing the holes in $f+Q$

Having found the $Q$-minimal non-holes in $f+Q$, we can find an explicit representation for all holes in $f+Q$ as follows.

1. let us construct a monomial ideal $I_{A,f} \in \mathbb{Q}[x_1,\dots,x_n]$ generated by the monomials
$$I_{A,f} = \langle x^\lambda : \lambda \in \mathbb{Z}_+^n, f + A\lambda \in (f+Q)\cap Q\rangle.$$
2. Since under our assumption that $C$ is pointed, there are only finitely many $\lambda \in \mathbb{Z}_+^n$ such that $f + A\lambda = z$ for each $z \in f+Q$, by solving $f + A\lambda = z, \lambda \in \mathbb{Z}_+^n$ for all $Q$-minimal points in $(f+Q)\cap Q$, we can find a finite generating set for $I_{A,f}$.
3. While the monomial $x^\lambda$ corresponds to $z = f + A\lambda \in f+Q$, we have $z \in (f+Q)\cap Q$ if and only if $x^\lambda \in I_{A,f}$. Thus, the set of holes in $f+Q$ corresponds to the set of standard monomials of the monomial ideal $I_{A,f}$.
4. Mapping this explicit representation for the standard monomials $x^\lambda$ back to $z \in f+Q$, we get a finite representation of the holes in $f+Q$.

## Example

**Computing fundamental holes**: The lattice $L = \mathbb{Z}^2$. The minimal Hilbert basis $B$ of $C \cap L$ is:

$$B = \{(1,0)^\intercal, (1,1)^\intercal, (1,2)^\intercal, (1,3)^\intercal, (1,4)^\intercal\}.$$

$(1,1)^\intercal$ is a hole. Being in $B$, $B \cap F = \{(1,1)^\intercal\}$. Since a nonnegative integer linear combinations of elements from $B \cap F$ $2\cdot(1,1)^\intercal = (2,2)^\intercal$ is an element of $Q$, $F = \{(1,1)^\intercal\}$.

**Computing the $Q$-minimal elements in $(f+Q)\cap Q$**: `4ti2` [HHM05] only allows the computation of all minimal inhomogenous solutions of

$$\{(\lambda,\mu) \in \mathbb{Z}_+^{2n} : f + A\lambda = A\mu\}. \tag{2}$$

As every minimal solution $\lambda$ to (1) must appear in a minimal solution $(\lambda,\mu)$ of (2). Let $f = (1,1)^\intercal$ and consider $(f+Q)\cap Q$. The linear system to solve is

$$\begin{array}{l} 1 + \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = \mu_1 + \mu_2 + \mu_3 + \mu_4 \\ 1 \qquad\;\; + 2\lambda_2 + 3\lambda_3 + 4\lambda_4 = \qquad 2\mu_2 + 3\mu_3 + 4\mu_4 \end{array}$$

with $\lambda_i, \mu_j \in \mathbb{Z}_+, i,j \in \{1,2,3,4\}$.

`4ti2` gives the following 5 minimal inhomogeneous solutions $(\lambda,\mu)$ to system (2):

$$\begin{array}{rcl} (\lambda,\mu) & \rightarrow & z = f + A\lambda \\ (0,0,0,2,0,0,3,0)^\intercal & \rightarrow & (3,9)^\intercal \\ (0,1,0,0,1,0,1,0)^\intercal & \rightarrow & (2,3)^\intercal \\ (0,0,1,0,1,0,0,1)^\intercal & \rightarrow & (2,4)^\intercal \\ (0,0,1,0,0,2,0,0)^\intercal & \rightarrow & (2,4)^\intercal \\ (0,0,0,1,0,1,1,0)^\intercal & \rightarrow & (2,5)^\intercal \end{array}$$

Note that $(3,9)^\intercal$ is not $Q$-minimal, since we computed minimal inhomogeneous solutions $(\lambda,\mu)$ of system (2). The $Q$-minimal elements in $(f+Q)\cap Q$ are $\{(2,3)^\intercal, (2,4)^\intercal, (2,5)^\intercal\}$.

**Computing the holes in $f+Q$**: Let us construct the generators of $I_{A,f}$. We have to find all representations of the form $z = f + A\lambda, \lambda \in \mathbb{Z}_+^4$ for each $Q$-minimal element $z$ in $(f+Q)\cap Q$, i.e. for each $z \in \{(2,3)^\intercal, (2,4)^\intercal, (2,5)^\intercal\}$.

$$\begin{array}{rcl} z & = & f + A\lambda \\ (2,3)^\intercal & = & (1,1)^\intercal + A(0,1,0,0)^\intercal \\ (2,4)^\intercal & = & (1,1)^\intercal + A(0,0,1,0)^\intercal \\ (2,5)^\intercal & = & (1,1)^\intercal + A(0,0,0,1)^\intercal \end{array}$$

Thus, we get the monomial ideal

$$I_{A,f} = \langle x_2, x_3, x_4\rangle,$$

whose set of standard monomials is $\{x_1^\alpha : \alpha \in \mathbb{Z}_+\}$. Thus, the set of holes in $f+Q$ is

$$\{f + \alpha A_{.1} : \alpha \in \mathbb{Z}_+\} = \{(1,1)^\intercal + \alpha(1,0)^\intercal : \alpha \in \mathbb{Z}_+\}$$

## References

[HHM05] R. Hemmecke, R. Hemmecke, and P. Malkin. 4ti2 version 1.2—computation of Hilbert bases, Graver bases, toric Gröbner bases, and more. Available at www.4ti2.de, sep. 2005.

[TY06] A. Takemura and R. Yoshida. A generalization of the integer linear infeasibility problem, 2006.