# D1. Fourier Transform

## Objectives

- Understand the concept of Fourier Transform to analyze continuous time signals
- Define the Bandwidth of a signal
- Define a Complex Signal and its use in Modulation

## 1. Introduction

In the previous units we presented the theory of discrete time signals and a number of applications. The general idea is that, by sampling a signal at a rate sufficiently large, there is no loss of information. The discrete time nature of a discrete time signal is very attractive in signal processing since all operations become numerical thus implemented by digital computers. Also the use of high level computer software such as Matlab allows the processing of digital signals in a fairly simple way.

In this chapter we look back at continuous time signals which are at the basis of how physical quantities evolve in time. In particular we define the Fourier Transform as the extension of the Discrete Time Fourier Transform (DTFT) to continuous time signals. The importance is that some fundamental operations in the time domain, such as convolution and modulation, have a very simple correspondent in the frequency domain.

Continuous time signals are defined in the next section, as the limit of discrete time signals with infinite sampling frequency. Along these lines it is shown that the DTFT converges to a continuous time formulation which becomes the Fourier Transform (FT). The use of the Fast Fourier Transform (FFT) previously seen to compute the FT of a arbitrary bandlimited signal is addressed in the following section. Finally the application to Amplitude Modulation of complex signals is the subject of the last section.

## 2. Continuous Time Signals

### VIDEO: Continuous Time Signals (8:45)
http://faculty.nps.edu/rcristi/eo3404/d-beamforming/videos/d1_02_continuousTimeSignals.mp4

In a continuous time signal $x(t)$ the independent variable $t$ is continuous. All signals in nature are continuous time. A continuous time signal is shown in the figure below.

$x(t)$
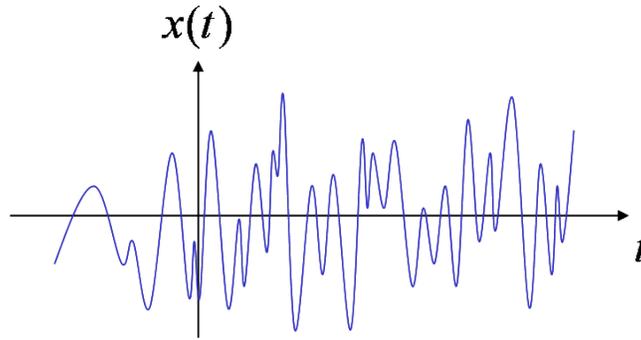
**A continuous time signal**

In the previous sections of the course we have seen that we can characterize discrete time signals $x[n]$ using the Discrete Time Fourier Transform (DTFT). In particular, if the discrete time signal comes from sampling a continuous time signal every $T_S$ seconds, as $x[n] = x(nT_S)$, the DTFT/IDTFT are recalled here for convenience:

$$X_{DTFT}(\omega) = DTFT\{x(nT_S)\} = \sum_{n=-\infty}^{+\infty} x(nT_S)e^{-j\omega n}$$

$$x(nT_S) = IDTFT\{X(\omega)\} = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_{DTFT}(\omega)e^{j\omega n}d\omega$$

The extension to continuous time signals is very simple. Just think of a continuous time signal $x(t)$ as the limit of its samples $x(nT_S)$ as the sampling interval $T_S \to 0$ (ie the sampling frequency $F_S = 1/T_S \to \infty$). This is shown graphically in the figure below.

$x(nT_S)$

$T_S \to 0$

$x(t)$
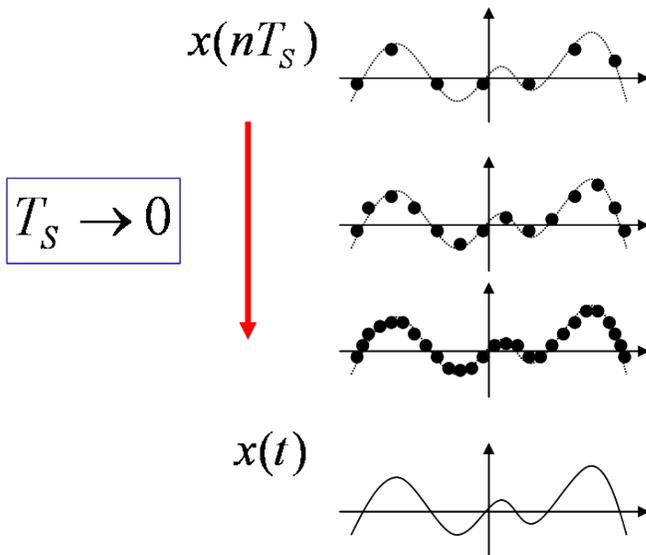


**A continuous time $x(t)$ can be viewed as the limit of a sampled signal $x(nT_S)$ as $T_S \to 0$.**

In order to define an appropriate frequency domain representation of a continuous time signal, let's rewrite the expressions of the DTFT and IDTFT as follows

$$T_S X_{DTFT}\left(2\pi F\, T_S\right) = \sum_{n=-\infty}^{+\infty} x(nT_S)e^{-j2\pi F\, nT_S} T_S$$

$$x(nT_S) = \frac{1}{2\pi}2\pi \int_{-F_S/2}^{F_S/2} T_S X_{DTFT}(2\pi F\, T_S)e^{j2\pi F\, nT_S} dF \tag{1}$$

where we replaced the digital frequency $\omega$ with its own definition as

$$\omega = 2\pi\frac{F}{F_S} = 2\pi F\, T_S$$

Now, in (1) above, take the limit for $T_S \to 0$. Then $nT_S \to t$, ie the discrete time becomes continuous time, $F_S \to +\infty$, ie the sampling frequency becomes infinity and $\sum ...T_S \to \int ...dt$, ie the summation becomes an integral. These facts lead to the following expressions and definitions

$$X(F) = FT\{x(t)\} = \int_{-\infty}^{+\infty} x(t)e^{-j2\pi Ft} dt$$

$$x(t) = IFT\{X(F)\} = \int_{-\infty}^{+\infty} X(F)e^{j2\pi Ft} dF$$

Which define the Fourier Transform (FT) and its inverse (IFT) of a continuous time signal. Notice that, now, the frequencies can span the whole range from $-\infty < F < +\infty$. Recall in the discrete time case that there was a limit in the frequencies to be interval $\pm F_S/2$.

See an Example.

Example. Consider the signal $x(t)$ to be the rectangular pulse of duration $T_0$ defined below.
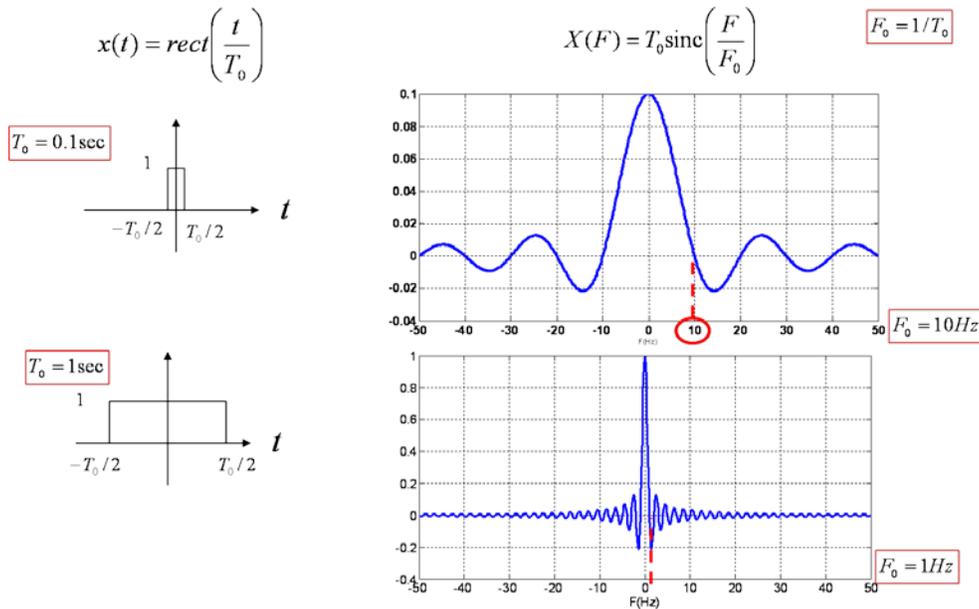
$$x(t) = rect\left(\frac{t}{T_0}\right)$$



1
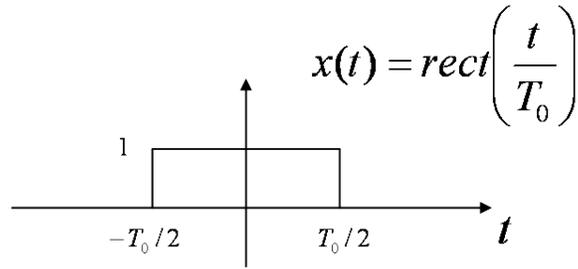
$-T_0/2$     $T_0/2$     $t$

**Rectangular pulse of duration $T_0$ in the example**

Then its Fourier Transform is determined by simple computation as

$$X(F) = \int_{-T_0/2}^{T_0/2} e^{-j2\pi Ft} dt = \frac{e^{-j2\pi FT_0/2} - e^{j2\pi FT_0/2}}{-j2\pi F}$$

$$= \frac{\sin(\pi FT_0)}{\pi F} = T_0 \operatorname{sinc}\left(\frac{F}{F_0}\right)$$

where again $\operatorname{sinc}(x) = \sin(\pi x)/\pi x$. Although usually the FT is complex (same as the DFT and the DTFT we saw before) in this particular case it is real and it can be shown by a simple plot. Two examples for two different numerical cases are shown below.

$$x(t) = rect\left(\frac{t}{T_0}\right)$$

$$X(F) = T_0 \operatorname{sinc}\left(\frac{F}{F_0}\right)$$

$\boxed{F_0 = 1/T_0}$



$\boxed{T_0 = 0.1\text{sec}}$

$\boxed{F_0 = 10Hz}$

$\boxed{T_0 = 1\text{sec}}$

$\boxed{F_0 = 1Hz}$

**Two rectangular pulses and their Fourier Transforms**

What we see from this example is that a short pulse $(T_0 = 0.1\text{sec})$ has a wide spectrum, with the strongest frequency components in the interval $-10Hz < F < +10Hz$. A longer pulse $(T_0 = 1\text{sec})$ has a narrower frequency range, with the dominant components being in the interval $-1Hz < F < +1Hz$. This shows that by "stretching" a signal in time (ie making it of longer duration) it makes it narrower in frequency and viceversa.

In the next section we see some of the relevant properties of the FT, which we need to understand for the rest of unit.

# 3. Properties of the Fourier Transform

The general theory of the Fourier Transform has been the subject of an incredible number of textbooks. In these notes we concentrate only on what we need. In particular we have the following:

**Property 1: Symmetry. The Fourier Tranform of a real signal $x(t)$ is such that**

$$X(-F) = X^*(F)$$

**In other words the magnitude has even symmetry, and the phase has odd symmetry as shown in the figure below.**
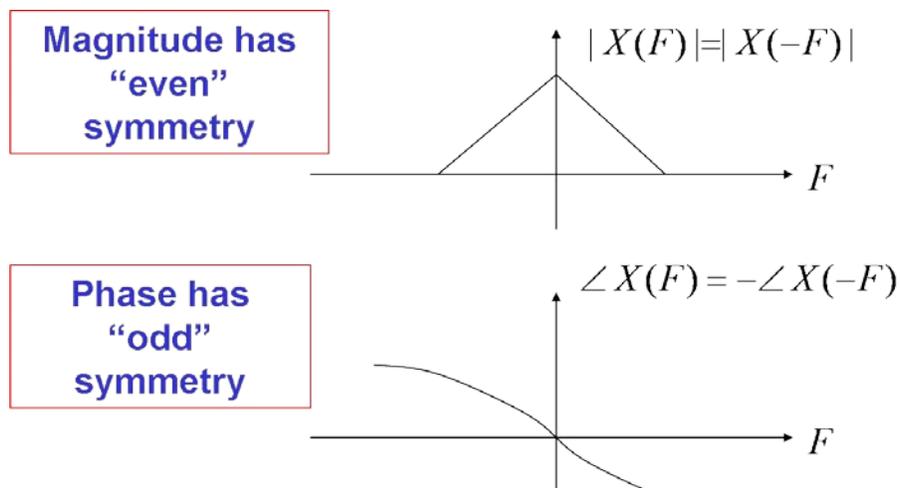


**Symmetry of a FT of a real signal $x(t)$.**

Proof. The reason is very simple, just try to believe:

$$X(-F) = \int_{-\infty}^{+\infty} x(t)e^{j2\pi Ft}\,dt = \left(\int_{-\infty}^{+\infty} x(t)e^{-j2\pi Ft}\,dt\right)^* = X^*(F)$$

and the property is readily proved.

**Property 2: Time Shift. Given any signal $x(t)$ with Fourier Transform $X(F)$ and any time shift $t_0$ we have**

$$FT\{x(t-t_0)\} = e^{-j2\pi F t_0} X(F)$$

$$FT\{x(t-t_0)\} = \int_{-\infty}^{+\infty} x(t-t_0)e^{-j2\pi F t}dt$$

$$(\text{let } t'=t-t_0) = e^{-j2\pi F t_0}\left(\int_{-\infty}^{+\infty} x(t')e^{-j2\pi F t'}dt'\right)$$

The rightmost term in brackets is $X(F)$.

## VIDEO: Bandwidth of a Signal (3:41)

http://faculty.nps.edu/rcristi/eo3404/d-beamforming/videos/d1_05_bandwidth.mp4

**Definition: Bandwidth of a Baseband Signal.** **A Baseband Signal is a signal with frequency components around $F = 0$. Its bandwidth $B$ is the frequency such that $X(F) = 0$ if $|F| > B$. In other words all frequency components are within the interval $-B < F < B$.**

An example of a Baseband signal and its bandwidth is shown in the figure below.
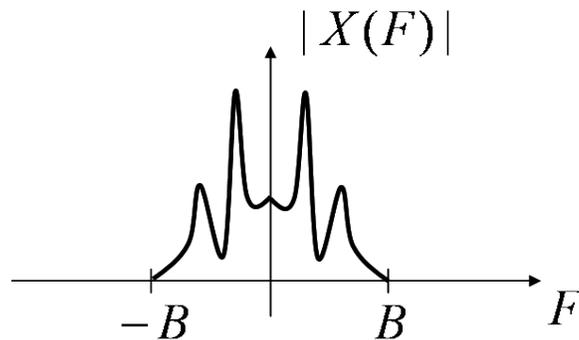


*The FT (magnitude only) of a Baseband Signal with Bandwidth B.*

The Bandwith of a signal has a very nice relationship with the time domain behavior of the signal itself. In particular we can say the following:

**Property 3: Let $x(t)$ be a Baseband Signal with Bandwidth $B$. Then, for any $|\Delta t| << 1/B$ we have**

$$x(t) \cong x(t + \Delta t)$$

**In other words the signal does not change within a time interval much smaller (say an order of magnitude) than $1/B$.**

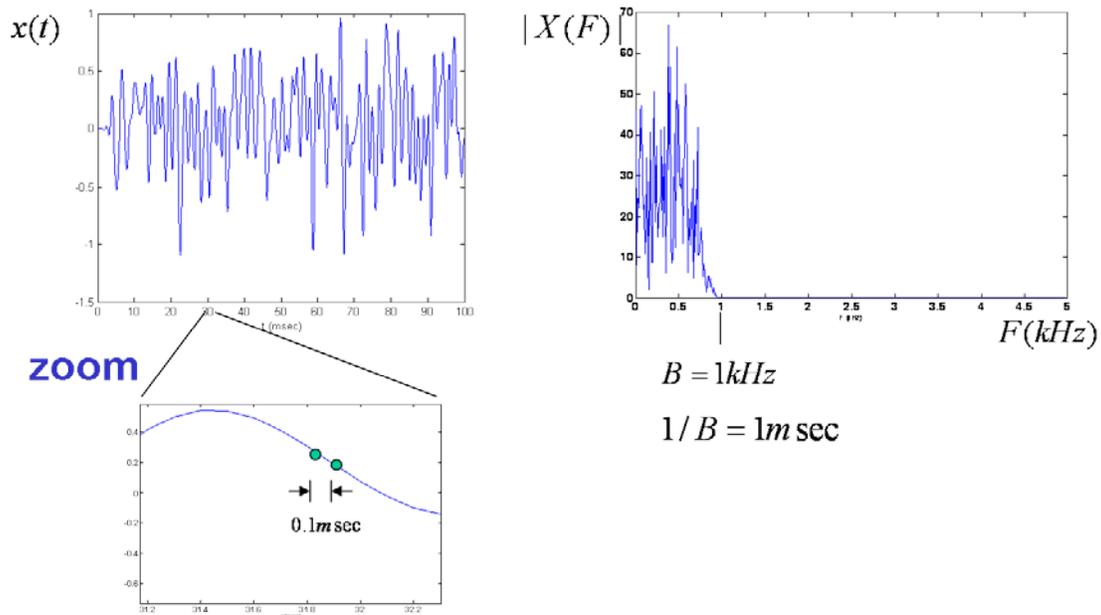Proof. From the definition of the Fourier Transform we can write:

$$x(t + \Delta t) = \int_{-\infty}^{+\infty} X(F) e^{j2\pi Ft} e^{j2\pi F\Delta t} dF$$

$$\cong \int_{-B}^{+B} X(F) e^{j2\pi Ft} e^{j2\pi F\Delta t} dF \cong x(t)$$

Since $-B < F < B$ and $|\Delta t| << 1/B$ we have $F\Delta t \cong 0$ so that $e^{j2\pi F\Delta t} \cong 1$.

This can be seen in the following example.

**Example**. Consider the signal $x(t)$ shown in the figure below, with its Fourier Transform $X(F)$ (magnitude only). The bandwidth is $B = 1kHz$ .



***Example of a signal with Bandwidth B=1kHz. The change in values within an interval $|\Delta t| \leq 0.1/B = 0.1m\sec$ is very small.***

If we zoom into the signal and we look at any time interval of length smaller than $0.1m\sec$ we see that the values are very close to each other.

# 4. Computation of the Fourier Transform by the FFT

The expression of the FT is essentially analytical. Since it is expressed as an integral, it can be applied only to signals which have an analytical expression.

A numerical way of computing the FT is by approximating it with the FFT. In this case consider a continuous time signal with finite duration

$$x(t), \ 0 \le t \le T_0$$

and let $B$ be its bandwidth. Then we can approximate the Fourier Transform numerically by taking the FFT of samples of the signals taken at sufficiently high sampling rate. In this way we can write

$$X(F) = FT\{x(t)\} = \int_0^{T_0} x(t)e^{-j2\pi Ft}dt \cong \sum_{n=0}^{M-1} x(nT_S)e^{-j2\pi FnT_S}T_S$$

where we choose the sampling interval as

$$T_S << \frac{1}{B}$$

say at least one order of magnitude smaller. Call $M$ the total number of samples as

$$M = round(T_0 / T_S)$$

Then take the $N \ge M$ point FFT

$$X[k] = FFT\{x(0),...,x(M-1),0,0,...,0\}, \ k = 0,...,N-1$$

By padding the sequence with zeros. The larger the FFT size $N$, the more frequency points are computed. Now we can map the values of the FFT into frequency components as follows

$$X\left(k\frac{F_S}{N}\right) = T_S X[N+k], \quad k = -\frac{N}{2},...,-1$$

$$X\left(k\frac{F_S}{N}\right) = T_S X[k], \qquad k = 0,...,\frac{N}{2}-1$$

Where the first epression yields the negative frequencies in the interval $-F_S/2 \le F < 0$, and the second expression yields the positive frequencies in the interval $0 \le F < F_S/2$. Notice that, as expected, due to the sampling we see the spectrum over the interval

$-F_S/2 \leq F < F_S/2$ and, in order for the approximation to work, the FT of the signal has to be contained well within this interval. If, after taking the FFT, we discover frequency components close to $\pm F_S/2$, then we need to repeat the operation with a higher sampling frequency.

Let us see an example.

**Example**. Consider a sinusoid with frequency $F_0 = 10kHz$ and duration $T_0 = 5m\sec$. Choose a sampling frequency $F_S = 200kHz$. This yields $M = 1000$ samples and let us choose an $N = 2^{12} = 4096$ point FFT. This is computed in Matlab as
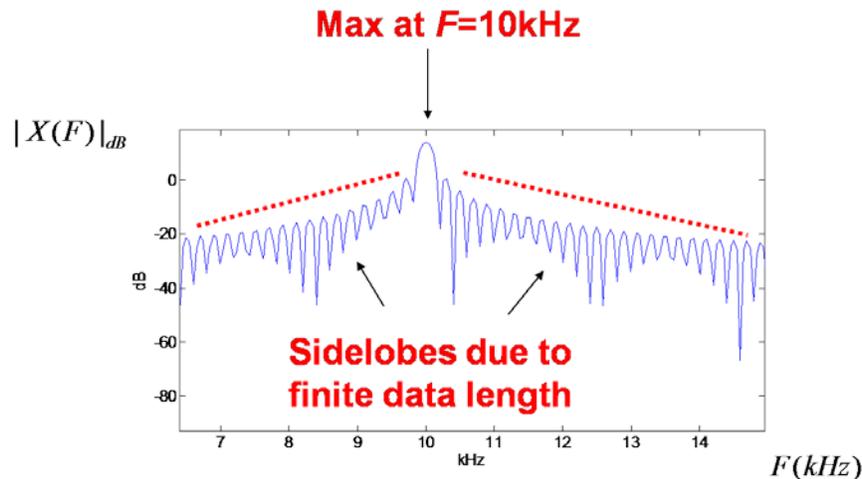
```
X=fft(x, N);
F=(-N/2:N/2 -1)*Fs/N;
plot(F,fftshift(20*log10(Ts*abs(X))))
```

and its magnitude is shown below.
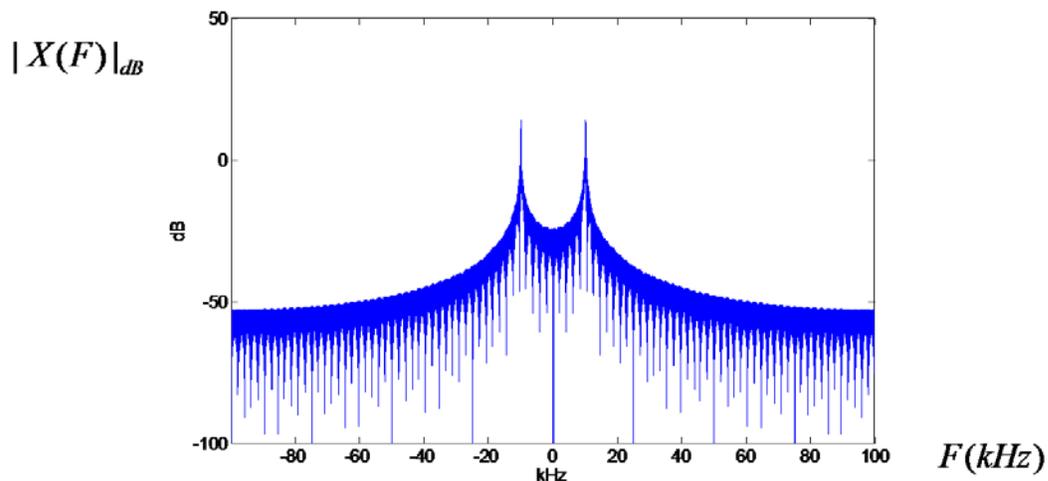


### *Example of FT by FFT*

The figure below shows the frequency spectrum around the maximum, occurring at F=10kHz. As in the case of the DFT, we see the mainlobe around the maximum and the sidelobes due to the finite data length.

**Max at *F*=10kHz**

$|X(F)|_{dB}$

*Zoom around the maximum, showing the mainlobe and sidelobes*

## 5. Complex Signals and Modulation

One areas where the FT is extremely useful is in the analysis and design of modulation systems not only for communications but also for target localization.

A very important concept is the definition of a "complex signal", defined as

$$x(t) = a(t) + jb(t)$$

where $a(t), b(t)$ are two real signals, called the Real and Imaginary parts. As usual $j = \sqrt{-1}$ is the complex constant.

In nature, there is no such a thing as a "complex signal", it is just a mathematical artifact. It is a convenient way of putting together two real signals $a(t), b(t)$ and treat them "as if" they were only one signal $s(t) = a(t) + jb(t)$.
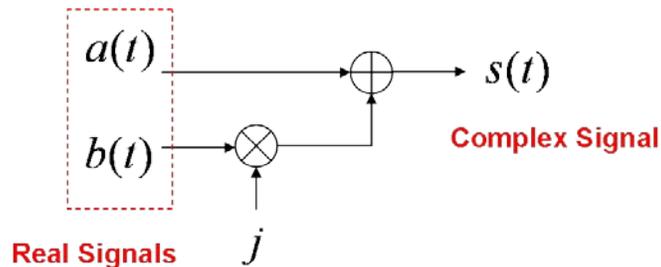
**Two real signals (** $a(t)$ **and** $b(t)$ **) can be grouped together into one complex signal** $s(t) = a(t) + jb(t)$

A very tipical example is in Digital Communications, where we transmit information in terms of bits ("ones" and "zeros") or, to keep things more balanced, as "+1" and "-1". Since we want to achieve the highest possible data rate (so we can be bombarded by more useless information!) we transmit groups of bits at once. This leads to QPSK (Quadrature Phase Shift Keying), also called 4QAM (Quadrature Amplitude Modulation) where we transmit two bits at a time. This is shown in the figure below.
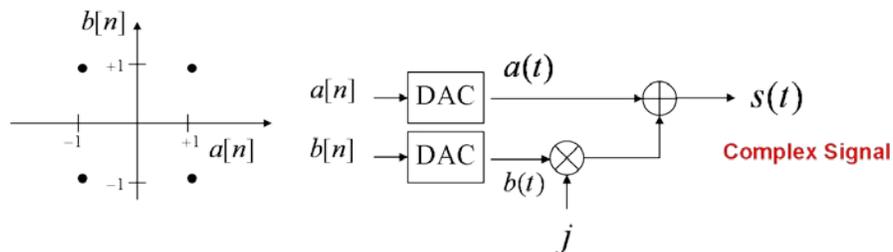


**Example of QPSK in Digital Communications: we transmit two bits (** $a[n], b[n]$ **) at a time.**
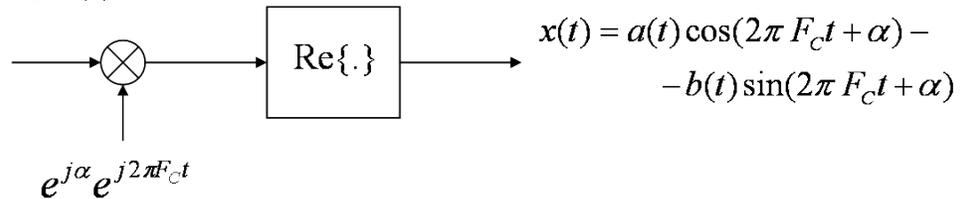
In this figure you see that two discrete time signals $a[n], b[n]$ are converted to analog signals by the Digital to Analog Converters (DAC) and then grouped into a complex signal.

The complex signal is then modulated into a real signal "shifted" in frequency as shown in the figure below.

$$s(t) = a(t) + jb(t)$$



$$x(t) = a(t)\cos(2\pi F_c t + \alpha) -$$
$$- b(t)\sin(2\pi F_c t + \alpha)$$

$$e^{j\alpha}e^{j2\pi F_c t}$$

**Modulation of a Complex Signal by a Crarrier Frequency** $F_C$

In this figure $F_C$ represent the carrier frequency and $\alpha$ its phase. Notice that the end product (ie the signal $x(t)$) is real and it contains both real signals $a(t)$ and $b(t)$.

Why do we modulate? There are two main reasons:

- In order to transmit a signal through a medium (water, air, wires, cables…) you need a time varying signal , oscillating at sufficiently high frequencies. A signal which is a constant (say) voltage or changing very slowly is not going to propagate anywhere;
- If you want to share the same medium by a number of users (typical example your radio or your TV) you need to shift the users to different frequencies (kHz for AM and MHz for FM radio) so they do not interfere which each other.
- Lastly (but not least) today frequencies are a commodity, they are auctioned by the FAC and you can transmit only at certain frequencies.

In order to see what happens in the frequency domain, we need to use some of the properties of the Fourier Transform we saw in the previous section. In particular we want to see the relation between the Fourier Transform $S(F)$ of the (complex) transmitted signal $s(t)$ and the Fourier Transform $X(F)$ of the (real) modulated signal $x(t)$.
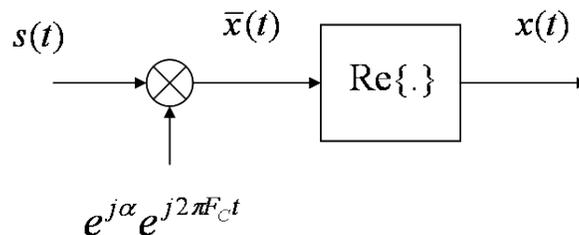
$$s(t) \quad\quad \overline{x}(t) \quad\quad x(t)$$



$$e^{j\alpha}e^{j2\pi F_c t}$$

**Modulated Signal**

Referring to the figure above, call

$$\overline{x}(t) = s(t)e^{j\alpha}e^{j2\pi F_C t}$$

and let's compute its FT as

$$\overline{X}(F) = FT\{\overline{x}(t)\} = \int_{-\infty}^{+\infty} e^{j\alpha} s(t)e^{j2\pi F_C t}e^{-j2\pi F t}dt =$$

$$= \int_{-\infty}^{+\infty} e^{j\alpha} s(t)e^{-j2\pi(F-F_C)t}dt = e^{j\alpha} S(F - F_C)$$

Now take the real part as

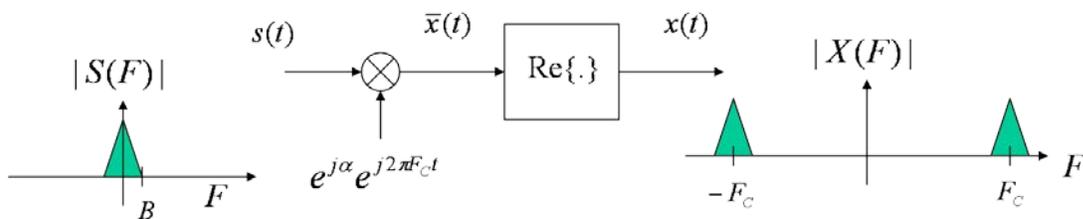$$x(t) = \text{Re}\{\overline{x}(t)\} = \frac{1}{2}\overline{x}(t) + \frac{1}{2}\overline{x}^{*}(t)$$

and compute the FT of the right most term

$$FT\{\overline{x}^{*}(t)\} = \int_{-\infty}^{+\infty} \overline{x}^{*}(t)e^{-j2\pi F t}dt = \left(\int_{-\infty}^{+\infty} \overline{x}(t)e^{j2\pi F t}dt\right)^{*}$$

$$= \overline{X}^{*}(-F) = e^{-j\alpha} S^{*}(-F - F_C)$$

Now put both terms together and come to the final result

$$X(F) = \frac{1}{2}e^{j\alpha} S(F - F_C) + \frac{1}{2}e^{-j\alpha} S^{*}(-F - F_C)$$

This relationship is shown (in magnitude) in the figure below.



***Modulation corresponds to a frequency shift of the frequency spectrum,
around the carrier frequency $F_C$***

In all applications the carrier frequency $F_C$ is much larger (several order of magnitude) than the signal bandwidth $B$, ie

$$F_C \ggg B$$

Just to get an idea how the modulated signal looks like, assume that $s(t)$ is real, we obtain the signal shown in the figure below.



$s(t)$

$x_{AM}(t) = s(t)\cos(2\pi F_c t + \alpha)$

$\cos(2\pi F_c t + \alpha)$
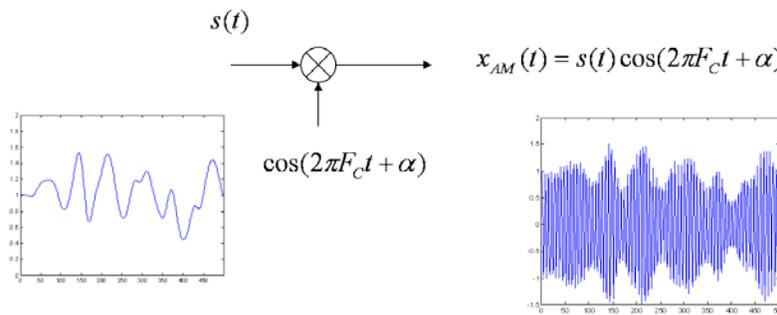
*Example of Modulated Signal: Real Case, for illustration*

It is like a sinusoid with very high frequency and time varying amplitude. In the simple case of an AM radio the receiver can be implemented with very simple hardware: maybe some of the readers had as a toy (or even better built) a simple receiver made of a diode, a capacitor and a coil.