

# B1. Fourier Analysis of Discrete Time Signals

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## Objectives

- Introduce discrete time periodic signals
- Define the Discrete Fourier Series (DFS) expansion of periodic signals
- Define the Discrete Fourier Transform (DFT) of signals with finite length
- Determine the Discrete Fourier Transform of a complex exponential

## 1. Introduction

In the previous chapter we defined the concept of a signal both in continuous time (analog) and discrete time (digital). Although the time domain is the most natural, since everything (including our own lives) evolves in time, it is not the only possible representation.

In this chapter we introduce the concept of expanding a generic signal in terms of elementary signals, such as complex exponentials and sinusoids. This leads to the frequency domain representation of a signal in terms of its Fourier Transform and the concept of frequency spectrum so that we characterize a signal in terms of its frequency components.

First we begin with the introduction of periodic signals, which keep repeating in time. For these signals it is fairly easy to determine an expansion in terms of sinusoids and complex exponentials, since these are just particular cases of periodic signals. This is extended to signals of a finite duration which becomes the Discrete Fourier Transform (DFT), one of the most widely used algorithms in Signal Processing.

The concepts introduced in this chapter are at the basis of spectral estimation of signals addressed in the next two chapters.

## 2. Periodic Signals

**[VIDEO: Periodic Signals \(19:45\)](#)**

[http://faculty.nps.edu/rcristi/eo3404/b-discrete-fourier-transform/videos/chapter1-seg1\\_media/chapter1-seg1-0.wmv](http://faculty.nps.edu/rcristi/eo3404/b-discrete-fourier-transform/videos/chapter1-seg1_media/chapter1-seg1-0.wmv)

[http://faculty.nps.edu/rcristi/eo3404/b-discrete-fourier-transform/videos/b1\\_02\\_periodicSignals.mp4](http://faculty.nps.edu/rcristi/eo3404/b-discrete-fourier-transform/videos/b1_02_periodicSignals.mp4)

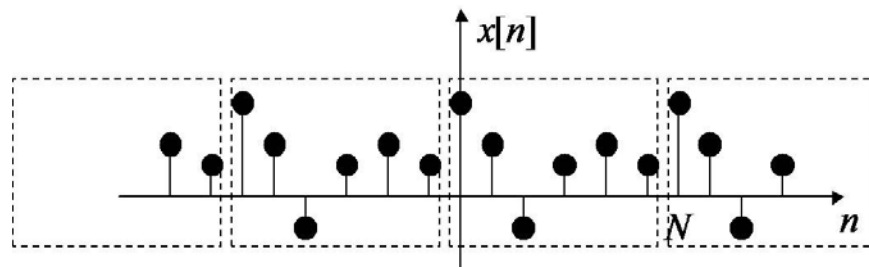
In this section we define a class of discrete time signals called Periodic Signals. There are two reasons why these signals are important:

- A number of signals in nature exhibit periodic repetition. Think of vibrations, ocean waves, seismic waves or electromagnetic waves;
- It is pretty easy to believe that periodic signals are made of (periodic) sinusoids of different frequencies, which is the goal of the Fourier analysis of the rest of the chapter.

**Definition:** a discrete time signal  $x[n]$  is periodic if and only if there exists a positive integer  $N > 0$  such that

$$x[n] = x[n + N] \text{ for all } n \quad (1)$$

In other words, a periodic signal keeps repeating itself for all values of the index  $n$  from  $-\infty$  to  $+\infty$ . The smallest positive integer  $N$  satisfying (1) is called the *period* of the signal.



**Figure 1: periodic signal**

Example: consider the signal

$$x[n] = 2 \cos(0.2\pi n + 0.9\pi)$$

It is periodic with period  $N = 10$  since

$$\begin{aligned} x[n+10] &= 2 \cos(0.2\pi(n+10) + 0.9\pi) \\ &= 2 \cos(0.2\pi n + 0.9\pi + 2\pi) = x[n] \end{aligned}$$

for all  $n$ .

In general a sinusoidal signal of the form

$$x[n] = A \cos\left(2\pi \frac{k}{N} n + \alpha\right) \quad (2)$$

with  $k, N$  integers is periodic with period  $N$ , since

$$\begin{aligned} x[n+N] &= A \cos\left(2\pi \frac{k}{N}(n+N) + \alpha\right) \\ &= A \cos\left(2\pi \frac{k}{N} n + \alpha + k \times 2\pi\right) = x[n] \end{aligned} \quad (3)$$

for all  $n$ .

Example: consider the sinusoid

$$x[n] = 5 \cos(0.3\pi n - 0.1\pi)$$

This can be written as

$$x[n] = 5 \cos\left(2\pi \frac{3}{20}n - 0.1\pi\right)$$

Comparing this with (2) we can see that the signal is periodic with period  $N = 20$ . In fact we can easily write

$$\begin{aligned} x[n+20] &= 5 \cos(0.3\pi(n+20) - 0.1\pi) \\ &= 5 \cos(0.3\pi n - 0.1\pi + 3 \times 2\pi) = x[n] \end{aligned}$$

for all  $n$ .

In a very similar way a complex exponential of the form

$$x[n] = Ae^{j\left(k\frac{2\pi}{N}\right)n} \quad (4)$$

is periodic with period  $N$  since

$$\begin{aligned} x[n+N] &= Ae^{j\left(k\frac{2\pi}{N}\right)(n+N)} \\ &= Ae^{j\left(k\frac{2\pi}{N}\right)n} \times e^{jk2\pi} = x[n] \end{aligned}$$

Example: consider the complex exponential

$$x[n] = (1 + 2j)e^{j0.1\pi n}$$

This can be written as

$$x[n] = (1 + 2j)e^{j\left(2\pi\frac{1}{20}\right)n}$$

and it is periodic with period  $N = 20$ .

### 3. Expansion of Periodic Signals: the Discrete Fourier Series (DFS).

**VIDEO: Reference Frames (21:25)**

[http://faculty.nps.edu/rcristi/eo3404/b-discrete-fourier-transform/videos/chapter1-seg1\\_media/chapter1-seg1-1.wmv](http://faculty.nps.edu/rcristi/eo3404/b-discrete-fourier-transform/videos/chapter1-seg1_media/chapter1-seg1-1.wmv)

[http://faculty.nps.edu/rcristi/eo3404/b-discrete-fourier-transform/videos/b1\\_03\\_referenceFrames.mp4](http://faculty.nps.edu/rcristi/eo3404/b-discrete-fourier-transform/videos/b1_03_referenceFrames.mp4)

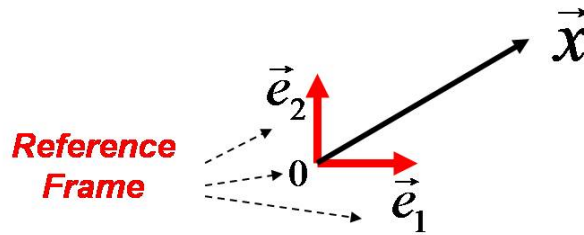
Now the problem we want to address is to determine a “reference frame” for discrete time periodic signals. In other words the issue is to determine a set of periodic signals which serve as a basis of any other periodic signal.

The concept of a “reference frame” is very important in a number of fields, including geometry, navigation and signal processing. It extends simple, intuitive geometric ideas to more complex abstract problems.

Going back to geometry, recall that, given a vector  $\vec{x}$  we can represent it in terms of a reference frame defined by:

- An origin “0”;
- Reference vectors  $\vec{e}_1, \vec{e}_2, \dots$

Figure 2 below shows an example of a vector in the plane.



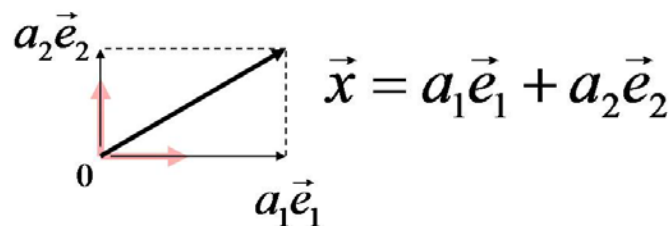
**Figure 2: a vector and a reference frame in the plane**

Given this definition we can see that a vector (say) in the plane is always represented by the components along the directions of the basis vectors. In this case, as shown in figure 3, the vector  $\vec{x}$  is represented as shown in figure 3 as

$$\vec{x} = a_1\vec{e}_1 + a_2\vec{e}_2$$

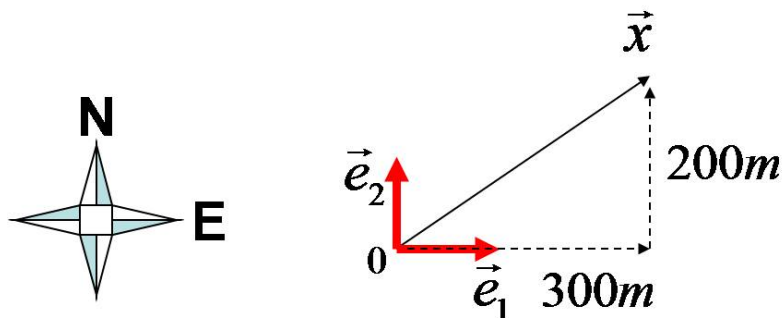
where

- $a_1$  is the projection of the vector  $\vec{x}$  along  $\vec{e}_1$
- $a_2$  is the projection of the vector  $\vec{x}$  along  $\vec{e}_2$



**Figure 3: vector representation in terms of its projections on the reference frame**

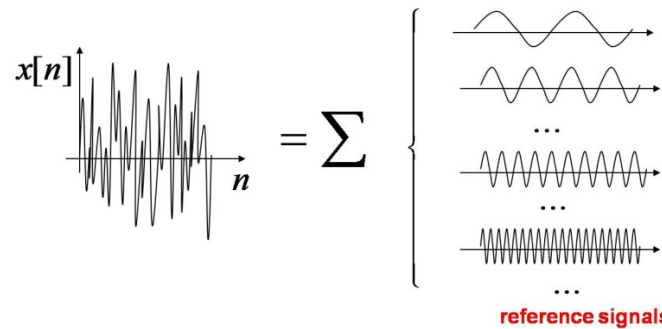
For example you want to locate yourself with respect to a reference point and you say that you are 300m East and 200m North of your reference. In this case the reference vectors  $\vec{e}_1, \vec{e}_2$  are in the East and North directions respectively, of unit length (1 meter) and the components  $a_1, a_2$  of your position are 300m and 200m respectively. This is shown in figure 4.



**Figure 4: example of a geographical position as a vector in the Earth reference frame.**

For signals we use a very similar concept. We want to express any arbitrary signal  $x[n]$  as the sum of reference signals having appropriate physical properties, significant for our problems. In

our case and in a lot of applications, the reference signals are sinusoids or complex exponentials, as shown in figure 5.



**Figure 5: a signal as a sum of reference signals**

Given any periodic signal with period  $N$  a very attractive set of reference candidates is the set of complex exponentials with the same period  $N$ . These are defined as

$$e_k[n] = e^{j2\pi\frac{k}{N}n}, \quad k = 0, \dots, N-1 \quad (5)$$

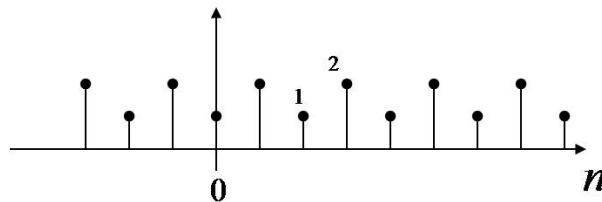
It turns out that we can show the following

**Fact: any discrete time periodic signal with period  $N$  can be written as**

$$x[n] = \sum_{k=0}^{N-1} a_k e_k[n]$$

**for some constants  $a_0, a_1, \dots, a_{N-1}$ .**

**Example:** take the periodic signal shown in figure 6 below. It is easy to see that it is periodic with period  $N = 2$ , since it keeps repeating the value 1,2,1,2,... periodically.



**Figure 6: periodic signal with period  $N = 2$**

Since the period is  $N = 2$  the reference exponentials  $e_k[n]$  in (5) are given by

$$e_0[n] = e^{j2\pi\frac{0}{2}n} = 1^n = 1$$

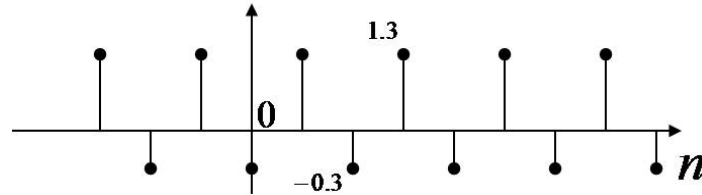
$$e_1[n] = e^{j2\pi\frac{1}{2}n} = (-1)^n$$

We can easily verify (try to believe!) that the given periodic signal in figure 6 can be written as

$$\begin{aligned} x[n] &= 1.5e_0[n] + 0.5e_1[n] \\ &= 1.5 \times 1^n - 0.5 \times (-1)^n \end{aligned}$$

In fact this expression yields  $x[n] = 1.5 - 0.5 = 1$  for  $n$  even and  $x[n] = 1.5 + 0.5 = 2$  for  $n$  odd.

Example: consider another example of a periodic signal with period  $N = 2$  shown in figure 7.



**Figure 7: periodic signal with period  $N = 2$**

Again we use the same reference signals  $e_k[n]$ ,  $k = 0, 1$  defined as

$$e_0[n] = e^{j2\pi\frac{0}{2}n} = 1^n = 1$$

$$e_1[n] = e^{j2\pi\frac{1}{2}n} = (-1)^n$$

It can be easily verified that this signal can be written as

$$\begin{aligned} x[n] &= 0.5e_0[n] + 0.8e_1[n] \\ &= 0.5 \times 1^n + 0.8 \times (-1)^n \end{aligned}$$

## 4. Orthogonality of the Reference Signals

**VIDEO: Orthogonal Reference Signals (20:52)**

[http://faculty.nps.edu/rcristi/eo3404/b-discrete-fourier-transform/videos/chapter1-seg2\\_media/chapter1-seg2-0.wmv](http://faculty.nps.edu/rcristi/eo3404/b-discrete-fourier-transform/videos/chapter1-seg2_media/chapter1-seg2-0.wmv)

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From the simple geometric examples in the previous section (figures 2,3,4) we see that the vectors in the reference frames are orthogonal to each other. It turns out that this makes it easier in determining the expansion of the vectors in terms of reference frames.

A very similar approach extends to reference signals. In particular we can verify that the

complex exponentials  $e_k[n] = e^{jk\frac{2\pi}{N}n}$  are all “orthogonal” to each other in the sense that

$$\sum_{n=0}^{N-1} e_k^*[n]e_m[n] = \begin{cases} 0 & \text{if } m \neq k \\ N & \text{if } m = k \end{cases} \quad (6)$$

This is easy to show just by applying the definition of complex exponentials as follows:

$$\sum_{n=0}^{N-1} e_k^*[n]e_m[n] = \sum_{n=0}^{N-1} e^{j2\pi\frac{m-k}{N}n} = \sum_{n=0}^{N-1} \left( e^{j2\pi\frac{m-k}{N}} \right)^n \quad (7)$$

Now we can apply the geometric sum, defined as

$$\sum_{n=0}^{N-1} \alpha^n = \begin{cases} \frac{1-\alpha^N}{1-\alpha} & \text{if } \alpha \neq 1 \\ N & \text{if } \alpha = 1 \end{cases} \quad (8)$$

with  $\alpha = e^{j2\pi\frac{m-k}{N}}$ . Then applying (7) to (8) we obtain the orthogonality condition in (6).

The fact that the complex exponentials  $e_k[n]$  are orthogonal as in (6) makes the computation of the expansion coefficients  $a_k$  a very simple matter.

In fact, let  $x[n]$  be a periodic signal with period  $N$  and let's write it again as

$$x[n] = \sum_{k=0}^{N-1} a_k e_k[n]$$

Then, the  $N$  coefficients  $a_0, a_1, \dots, a_{N-1}$  of the expansion can be easily computed from the orthogonality of the complex exponentials as

$$\sum_{n=0}^{N-1} x[n] e_k^*[n] = \sum_{n=0}^{N-1} \underbrace{\left( \sum_{m=0}^{N-1} a_m e_m[n] \right)}_{x[n]} e_k^*[n] = \sum_{m=0}^{N-1} a_m \left( \sum_{n=0}^{N-1} e_k^*[n] e_m[n] \right) = N a_k$$

### [VIDEO: Discrete Fourier Series \(16:08\)](#)

[http://faculty.nps.edu/rcristi/eo3404/b-discrete-fourier-transform/videos/chapter1-seg2\\_media/chapter1-seg2-1.wmv](http://faculty.nps.edu/rcristi/eo3404/b-discrete-fourier-transform/videos/chapter1-seg2_media/chapter1-seg2-1.wmv)

[http://faculty.nps.edu/rcristi/eo3404/b-discrete-fourier-transform/videos/b1\\_05\\_discreteFourierSeries.mp4](http://faculty.nps.edu/rcristi/eo3404/b-discrete-fourier-transform/videos/b1_05_discreteFourierSeries.mp4)

The rightmost expression comes from the orthogonality property, for which the summation is nonzero only when  $k = m$ . It is customary to define  $X[k] = N a_k$  for  $k = 0, \dots, N-1$  and call it the Discrete Fourier Series (DFS) of the periodic sequence  $x[n]$ . This can be computed as

$$X[k] = DFS \{x[n]\} = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}, \quad k = 0, \dots, N-1 \quad (9)$$

Conversely, the original the signal  $x[n]$  is called the Inverse Discrete Fourier Series (IDFS) and it can be written as

$$x[n] = IDFS \{X[k]\} = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}kn} \quad (10)$$

**Example Revisited:** consider the example we have seen above (figure 6), with period  $N = 2$ . Then the Discrete Fourier Series (DFS) of the given signal is computed as

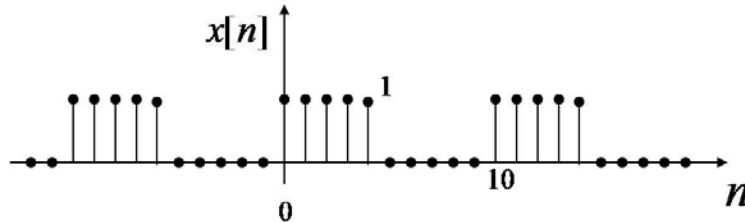
$$X[k] = \sum_{n=0}^1 x[n] e^{-j\frac{2\pi}{2}nk} = x[0] + x[1](-1)^k = 1 + 2 \times (-1)^k, \quad k = 0, 1$$

Therefore this yields  $X[0] = 3$  and  $X[1] = -1$ . The given signal can then be expressed as

$$x[n] = IDFS\{X[k]\} = \frac{1}{2} \sum_{k=0}^1 X[k] e^{j\frac{2\pi}{2}kn} = 1.5 - 0.5 \times (-1)^n$$

same as in the example above.

Example: Consider the periodic signal  $x[n]$  shown in figure 8 below.

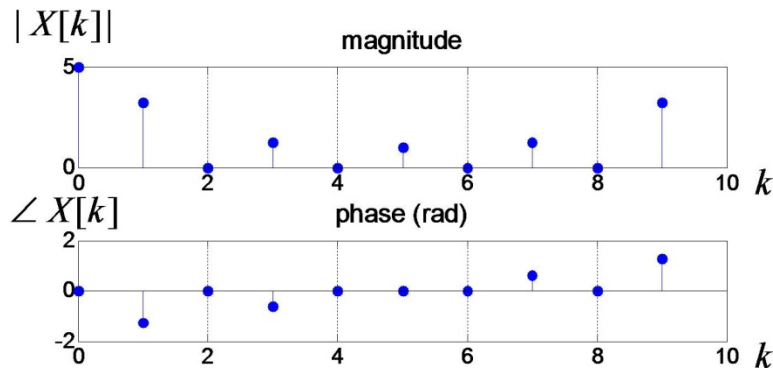


**Figure 8:** a periodic signal with period  $N = 10$ .

Then we compute its Discrete Fourier Series (DFS) as

$$X[k] = \sum_{n=0}^9 x[n] e^{-j\frac{2\pi}{10}kn} = \sum_{n=0}^4 e^{-j\frac{2\pi}{10}kn} = \begin{cases} \frac{1 - e^{-j\frac{2\pi}{10}5k}}{1 - e^{-j\frac{2\pi}{10}k}} & \text{if } k = 1, 2, \dots, 9 \\ 5 & \text{if } k = 0 \end{cases}$$

This sequence can be plotted in terms of magnitude and phase as



**Figure 9:** DFS of the signal in figure 8

Example: Let the signal  $x[n]$  be defined as

$$x[n] = 2|\cos(0.5\pi n)|$$

It is periodic and let's see what the period is. Just compute a few values to obtain

$x[0] = 2, x[1] = 0, x[2] = 2, x[3] = 0, \dots$  This shows that the period is  $N = 2$ , which yields its DFS as

$$X[k] = \sum_{n=0}^1 x[n] e^{-j\frac{2\pi}{2}kn} = x[0] + x[1] \times (-1)^k, \quad k = 0, 1$$

Substituting for  $x[0], x[1]$  we obtain  $X[0] = X[1] = 2$ .



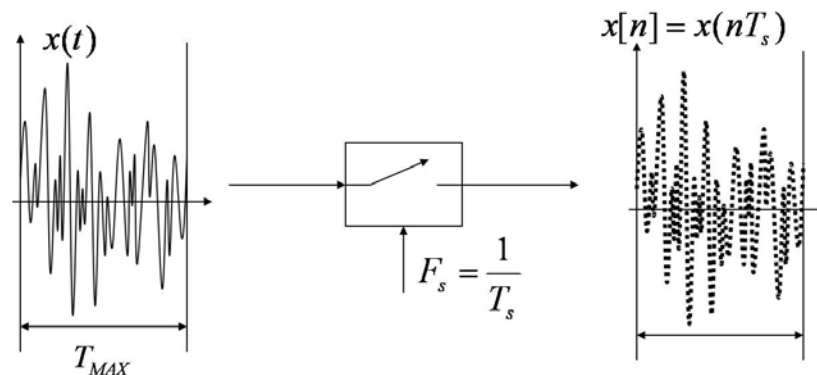
## 5. Signals of a Finite Length and the Discrete Fourier Transform (DFT)

**[VIDEO: the Discrete Fourier Transform \(12:20\)](#)**

[http://faculty.nps.edu/rcristi/eo3404/b-discrete-fourier-transform/videos/chapter1-seg3\\_media/chapter1-seg3-0.wmv](http://faculty.nps.edu/rcristi/eo3404/b-discrete-fourier-transform/videos/chapter1-seg3_media/chapter1-seg3-0.wmv)

[http://faculty.nps.edu/rcristi/eo3404/b-discrete-fourier-transform/videos/b1\\_06\\_signalsFiniteLength.mp4](http://faculty.nps.edu/rcristi/eo3404/b-discrete-fourier-transform/videos/b1_06_signalsFiniteLength.mp4)

In all data processing experiments we analyze data of a finite length. A typical situation is shown in figure 10 below, where a segment of data of length  $T_{MAX}$  (in seconds, for example) is digitized at a rate  $F_s$  samples/second, thus yielding a vector of  $N = T_{MAX} \times F_s$  samples. For example we have  $T_{MAX} = 30msec$  (ie 0.03sec) of data, sampled at  $F_s = 20kHz$ , ie 20,000 samples/sec. Then we obtain a numerical file with length  $N = 0.03 \times 20,000 = 600$  samples.



**Figure 10: a sampled signal of finite length**

As a consequence we obtain a data vector  $x[n], n = 0, \dots, N - 1$  with  $N$  data points.

In many applications we want the frequency content of this data which allows us to extract relevant information. Think about a music signal: the frequencies are related to the notes in the music score. In general any sort of audio signal, be it speech or from a hydrophone, has a signature in frequency which tells a lot about the nature of the signal itself. We can use the frequency signature in speech recognition, for example. In other words it is very important to determine the spectrum of a signal numerically directly from the samples.

In order to do this, we can make use of what we have developed for periodic signals. In particular we can look at a signal  $x[n], n = 0, \dots, N - 1$  of length  $N$  samples as one period of a periodic signal with period  $N$ . In this way we can use exactly the same formulas of the Discrete Fourier Series (DFS) and define the Discrete Fourier Transform (DFT) in exactly the same way:

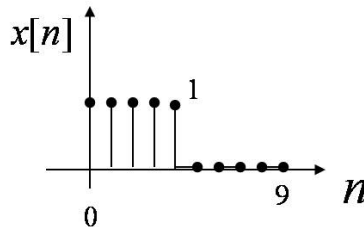
$$X[k] = DFT \{x[n]\} = \sum_{n=0}^{N-1} x[n] e^{-j \frac{2\pi}{N} kn}, \quad k = 0, \dots, N - 1$$

$$x[n] = IDFT \{X[k]\} = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j \frac{2\pi}{N} kn}, \quad n = 0, \dots, N - 1$$

These expressions are the same as the DFS with the only difference that the DFS is for periodic signals, thus  $x[n]$  is periodic and defined for all samples  $n$ , while the DFT is for finite length signals, thus  $x[n]$  is defined only for  $n = 0, \dots, N - 1$ .

The meaning of this expansion is that any finite sequence can be expanded in terms of complex exponentials with frequencies  $k2\pi / N$ .

Example. Consider the signal  $x[n], n = 0, \dots, 9$  of length  $N = 10$  shown in figure 11.

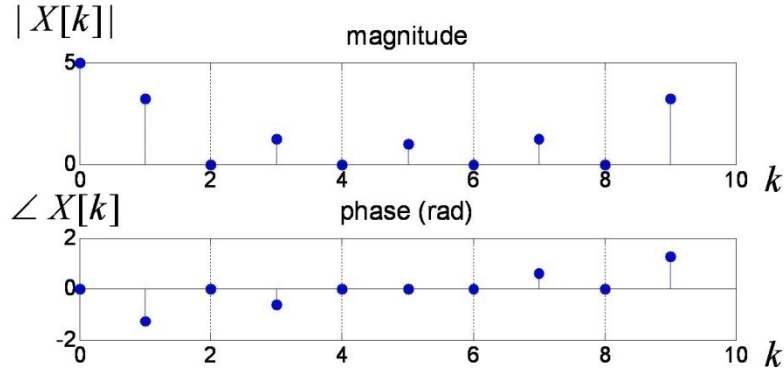


**Figure 11: example of a signal of length 10**

Its DFT can be computed analytically using the geometric series as

$$X[k] = \sum_{n=0}^9 x[n] e^{-j\frac{2\pi}{10}kn} = \sum_{n=0}^4 e^{-j\frac{2\pi}{10}kn} = \begin{cases} \frac{1 - e^{-j\frac{2\pi}{10}5k}}{1 - e^{-j\frac{2\pi}{10}k}} & \text{if } k = 1, 2, \dots, 9 \\ 5 & \text{if } k = 0 \end{cases}$$

This can be plotted in terms of magnitude and phase as shown in figure 12 below.



**Figure 12: DFT of the signal in figure 11.**

## 6. DFT of Complex Exponentials

**VIDEO: DFT of Complex Exponentials (20:35)**

[http://faculty.nps.edu/rcristi/eo3404/b-discrete-fourier-transform/videos/chapter1-seg3\\_media/chapter1-seg3-1.wmv](http://faculty.nps.edu/rcristi/eo3404/b-discrete-fourier-transform/videos/chapter1-seg3_media/chapter1-seg3-1.wmv)  
[http://faculty.nps.edu/rcristi/eo3404/b-discrete-fourier-transform/videos/b1\\_07\\_dftComplexExponential.mp4](http://faculty.nps.edu/rcristi/eo3404/b-discrete-fourier-transform/videos/b1_07_dftComplexExponential.mp4)

Since we use the DFT to detect spectral components, it is particularly useful to see the DFT of a complex exponential signal of finite length  $N$  defined as

$$x[n] = Ae^{j\omega_0 n}, \quad n = 0, \dots, N-1$$

where  $\omega_0$  represents the digital frequency in radians. Applying the definition of its DFT we obtain the following expression

$$\begin{aligned} X[k] &= DFT \{x[n]\} = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn} \\ &= \sum_{n=0}^{N-1} Ae^{j\omega_0 n} e^{-j\frac{2\pi}{N}kn} = \sum_{n=0}^{N-1} Ae^{-j\left(k\frac{2\pi}{N} - \omega_0\right)n}, \quad k = 0, \dots, N-1 \end{aligned}$$

Notice that this expression has a particular structure and it can be written in the following way:

$$X[k] = A \times W_N \left( k \frac{2\pi}{N} - \omega_0 \right)$$

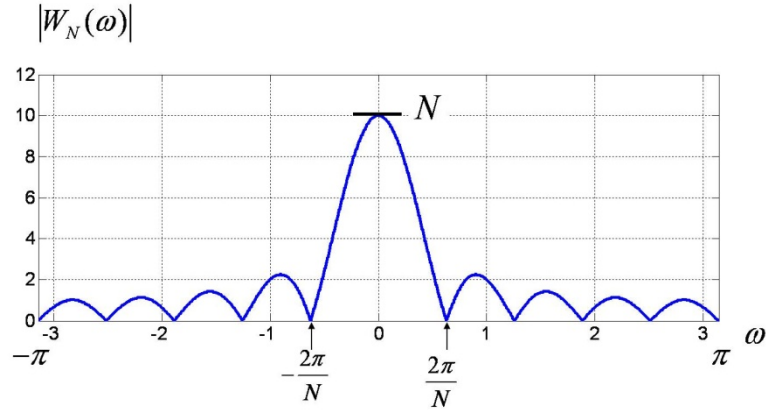
where the function  $W_N(\omega)$  depends only on the data length  $N$  and it is defined as

$$W_N(\omega) = \sum_{n=0}^{N-1} e^{-j\omega n} = \frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}}$$

Again the variable  $\omega$  indicates digital frequency and it is in radians. As we will show below, the same expression can be written in a different way as

$$W_N(\omega) = e^{-j\omega(N-1)/2} \frac{\sin\left(\frac{\omega N}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}$$

The plot of its magnitude for  $N = 10$  is shown in figure 13.



**Figure 13: magnitude of  $W_{10}(\omega)$**

Notice that the largest values are concentrated around  $\omega = 0$ . Also we can distinguish the “main lobe” within the interval  $-\frac{2\pi}{N} < \omega < \frac{2\pi}{N}$  and all the other “sidelobes”. Most of the energy is on the main lobe.

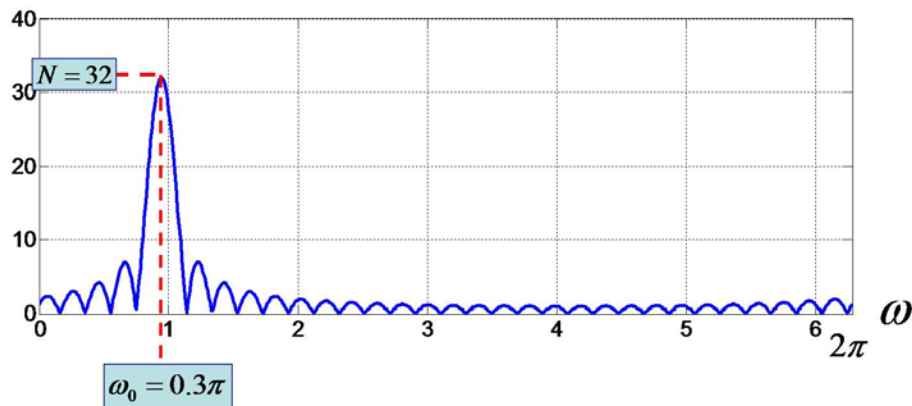
Example: consider the sequence

$$x[n] = e^{j0.3\pi n}, n = 0, \dots, 31$$

In this case  $\omega_0 = 0.3\pi$ ,  $N = 32$ . Then its DFT becomes

$$X[k] = W_{32}(\omega - 0.3\pi) \Big|_{\omega = k \frac{2\pi}{32}}, k = 0, \dots, 31$$

In order to see how this looks like, let’s plot first  $|W_{32}(\omega - 0.3\pi)|$  as in figure 14

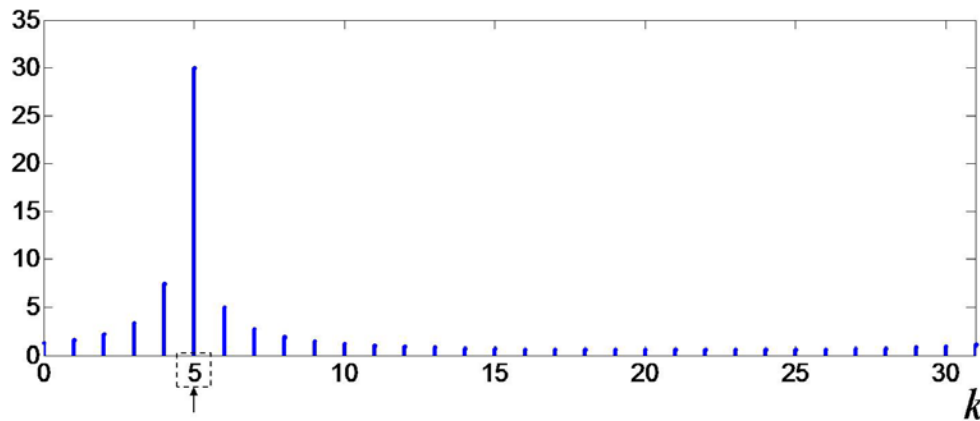


**Figure 14: plot of  $|W_{32}(\omega - 0.3\pi)|$**

Then, to see its DFT, we take samples as

$$X[k] = DFT \{x[n]\} = W_N(\omega - 0.3\pi) \Big|_{\omega = k \frac{2\pi}{32}}, k = 0, \dots, 31$$

This is shown in figure 15.



**Figure 15: DFT of the example. Notice that the maximum corresponds to the index  $k = 5$  which yields the digital frequency  $\omega = 5 \times 2\pi / 32 = 0.312\pi \cong 0.3\pi$**

## 7. The Fast Fourier Transform in Matlab

One of the most important reason (if not the most important) reasons why the DFT has become a very popular algorithm is the extremely high efficiency of its implementation. In particular the Fast Fourier Transform (FFT) can compute the DFT of a large data set in a very short time. Just to give an idea, if we have  $N$  data points, and  $N$  is a power of 2, we can compute its DFT in an amount of time proportional to  $N \log_2 N$ , using the FFT, versus  $N^2$  by “brute force” computation. For example let (say)  $N = 2^{10} = 1024$ . Then the time taken by the FFT is proportional to  $1024 \log_2 1024 \cong 10^3$ , while the brute force approach would take an amount of time proportional to  $1024^2 \cong 10^6$ . As a consequence, all computer applications, including Matlab, use the FFT rather than the DFT. The results of both operations are identical, but the FFT computes it in a fraction of the time taken by the brute force DFT.

**[VIDEO: FFT in Matlab \(20:43\)](#)**

[http://faculty.nps.edu/rcristi/eo3404/b-discrete-fourier-transform/videos/chapter1-seg4\\_media/chapter1-seg4.wmv](http://faculty.nps.edu/rcristi/eo3404/b-discrete-fourier-transform/videos/chapter1-seg4_media/chapter1-seg4.wmv)

[http://faculty.nps.edu/rcristi/eo3404/b-discrete-fourier-transform/videos/b1\\_08\\_fftSinMatlab.mp4](http://faculty.nps.edu/rcristi/eo3404/b-discrete-fourier-transform/videos/b1_08_fftSinMatlab.mp4)  
(CHECK IT)

Again we want to determine the DFT of the sequence  $x[n] = e^{j0.3\pi n}$  with  $n = 0, \dots, 31$ . Then:

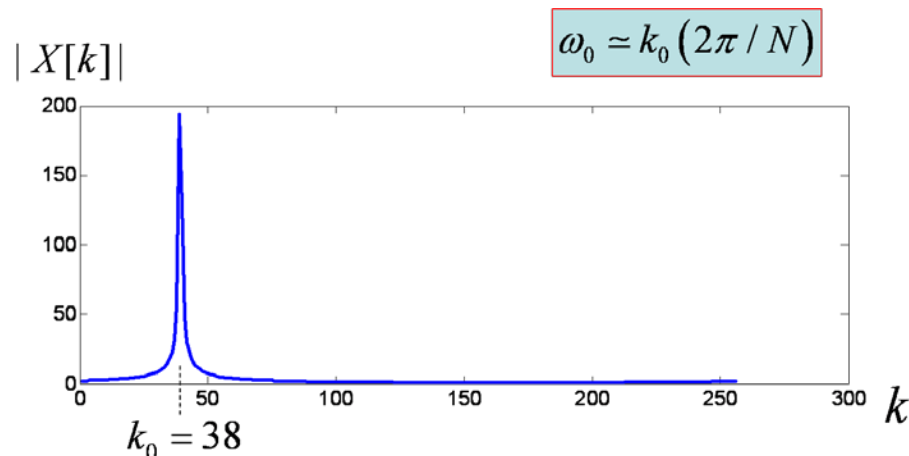
```
% Data Length:
N=32;

% 1. generate data
n=0:N-1;
x=exp(j0.3*pi*n);
```

```
% 2. take the FFT
X=fft(x);
```

```
% 3. plot it (choose one)
stem(abs(X)) % if you want to see the individual values
plot(abs(X)) % to see a continuous plot
```

Example: let us take the same sequence, but with more data points. Say, let  $N=256$ . Then the same Matlab program yields the graph shown in the figure below.



**Figure 16. DFT of the same sequence, with  $N=256$  samples.**

The peak can be verified to be at  $k = 38$  so that the estimated frequency is

$$\omega = 38 \frac{2\pi}{256} = 0.2969\pi \cong 0.3\pi$$

as expected.

In the following example, we look at the opposite problem. Given a signal, we want to determine the frequency content looking at its DFT. Each peak yields an estimate of the dominant frequencies.

Consider a signal  $x[n], n = 0, \dots, 255$  of length  $N = 256$ . Its DFT  $X[k]$  has its magnitude as shown in the figure below. So the problem is what can we say about this signal. Since it has two distinct and well defined peaks, at  $k = 27$  and  $k = 81$  the signal has two frequencies at

$$\omega = 27 \frac{2\pi}{256} = 0.2109\pi$$

$$\omega = 81 \frac{2\pi}{256} = 0.6328\pi$$

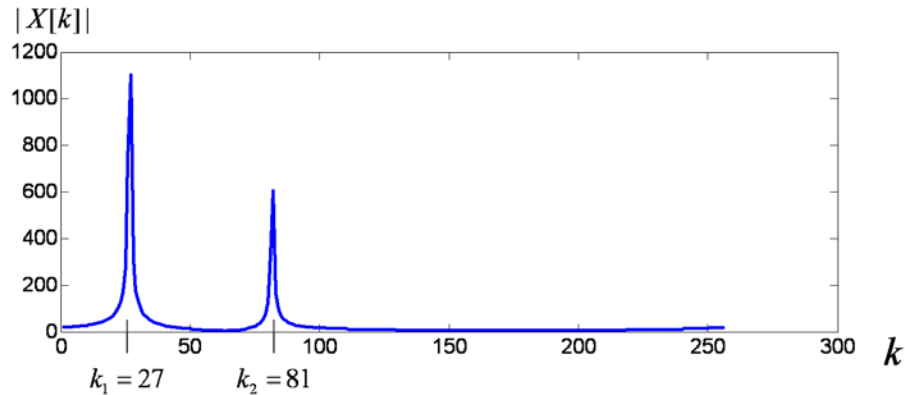


Figure 17. Given DFT for the Example.

## 8. DFT of Sinusoidal Signals.

**VIDEO: DFT of a Sinusoid (18:54)**

[http://faculty.nps.edu/rcristi/eo3404/b-discrete-fourier-transform/videos/chapter1-seg5\\_media/chapter1-seg5-0.wmv](http://faculty.nps.edu/rcristi/eo3404/b-discrete-fourier-transform/videos/chapter1-seg5_media/chapter1-seg5-0.wmv)

[http://faculty.nps.edu/rcristi/eo3404/b-discrete-fourier-transform/videos/b1\\_09\\_dftSinusoidalSignals.mp4](http://faculty.nps.edu/rcristi/eo3404/b-discrete-fourier-transform/videos/b1_09_dftSinusoidalSignals.mp4)

Recall that a sinusoid is the sum of two complex exponentials, as

$$x[n] = A \cos(\omega_0 n + \alpha) = \left(\frac{A}{2} e^{j\alpha}\right) e^{j\omega_0 n} + \left(\frac{A}{2} e^{-j\alpha}\right) e^{-j\omega_0 n}$$

In the spectral domain it is represented as two frequencies as shown in the figure below.

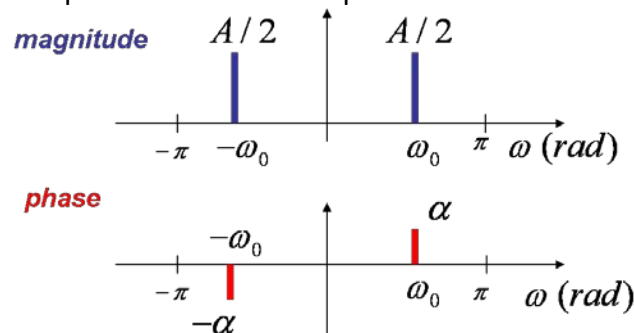


Figure 18. Spectral Representation of a Sinusoid

Now we want to see what is its DFT when we take a finite data length  $x[n], n = 0, \dots, N - 1$ .

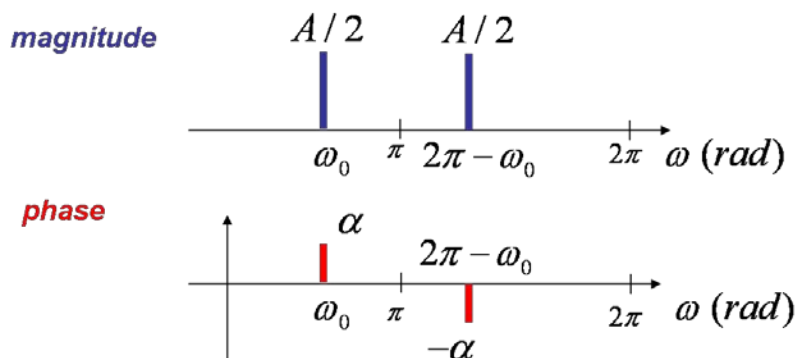
Define again  $X[k] = DFT\{x[n]\}$  and since it is the sum of two complex exponentials, its DFT is the sum of the respective DFT's, as

$$X[k] = DFT \left\{ \left( \frac{A}{2} e^{j\alpha} \right) e^{j\omega_0 n} \right\} + DFT \left\{ \left( \frac{A}{2} e^{-j\alpha} \right) e^{-j\omega_0 n} \right\}$$

The rightmost term, with negative frequency  $-\omega_0$  requires some consideration since the frequencies  $k2\pi/N, k=0, \dots, N-1$  computed by the DFT cannot be negative. We can easily see that  $e^{-j\omega_0 n} = e^{j(2\pi-\omega_0)n}$ , with  $2\pi - \omega_0 \geq 0$  so that the above DFT becomes

$$X[k] = DFT \left\{ \left( \frac{A}{2} e^{j\alpha} \right) e^{j\omega_0 n} \right\} + DFT \left\{ \left( \frac{A}{2} e^{-j\alpha} \right) e^{j(2\pi-\omega_0)n} \right\}$$

The two frequencies  $\omega_0$  and  $2\pi - \omega_0$  are shown in the figure below.



**Figure 19. Frequency domain representation of a sinusoidal signal, using positive frequencies only.**

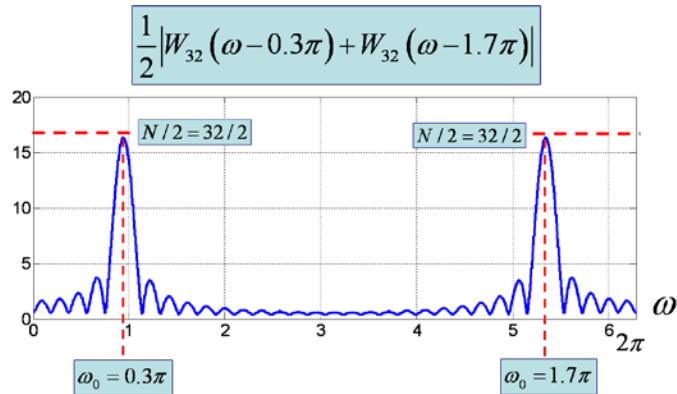
As a consequence of this, the DFT of a sinusoidal signal at frequency  $0 < \omega_0 < \pi$  is going to show two peaks: one at  $\omega_0$  and one at  $2\pi - \omega_0$ . Both components have the same magnitude  $A/2$  and phase  $\pm \alpha$  of opposite sign.

**Example.** Suppose we have a signal  $x[n] = 2 \cos(0.3\pi n + \alpha)$  with  $n = 0, \dots, 31$ . In terms of complex exponentials, it has two frequencies, at  $\omega = 0.3\pi$  and  $\omega = -0.3\pi$ . The negative frequency is equivalent to  $\omega = 2\pi - 0.3\pi = 1.7\pi$  so that its DFT becomes

$$X[k] = W_{32}(\omega - 0.3\pi) + W_{32}(\omega - 1.7\pi) \Big|_{\omega = k2\pi/32}$$

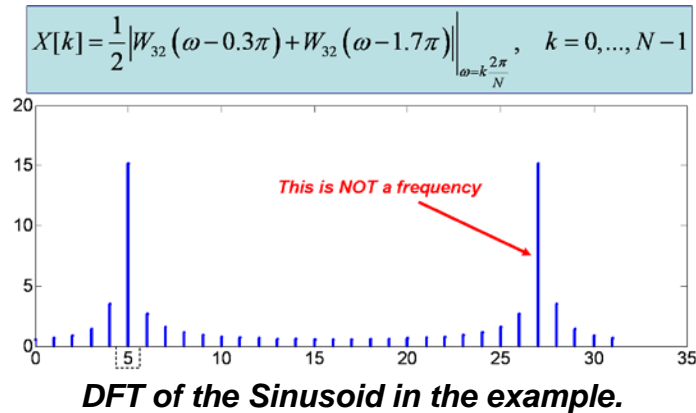
The magnitude of  $|W_{32}(\omega - 0.3\pi) + W_{32}(\omega - 1.7\pi)|$  is shown in the figure below.





**Figure 20. Magnitude Plot of  $\left|W_{32}(\omega - 0.3\pi) + W_{32}(\omega - 1.7\pi)\right|$ .**

Notice the two peaks, as expected, and the maximum value  $N/2 = 16$  at the peaks. The DFT now is computed as samples of the values at  $\omega = k2\pi/32$  with  $k = 0, \dots, 31$ . This yields the 32 values shown below.



**DFT of the Sinusoid in the example.**

We can easily see the two peaks at  $k = 5$  and  $k = 32 - 5 = 27$ . The corresponding frequencies are  $\omega = 5 \times 2\pi/32 \cong 0.3\pi$  and  $\omega = 27 \times 2\pi/32 \cong 1.7\pi$  as expected.

By the definition of the DFT and the example we see an important property called Symmetry.

### **VIDEO: Symmetry of the DFT (09:22)**

[http://faculty.nps.edu/rcristi/eo3404/b-discrete-fourier-transform/videos/chapter1-seg5\\_media/chapter1-seg5-1.wmv](http://faculty.nps.edu/rcristi/eo3404/b-discrete-fourier-transform/videos/chapter1-seg5_media/chapter1-seg5-1.wmv)  
[http://faculty.nps.edu/rcristi/eo3404/b-discrete-fourier-transform/videos/b1\\_10\\_symmetryDFT.mp4](http://faculty.nps.edu/rcristi/eo3404/b-discrete-fourier-transform/videos/b1_10_symmetryDFT.mp4)

**Symmetry of the DFT. Given a real signal  $x[n], n = 0, \dots, N - 1$ , its DFT  $X[k] = \text{DFT}\{x[n]\}, k = 0, \dots, N - 1$  is such that**

$$X[N - k] = X^*[k]$$

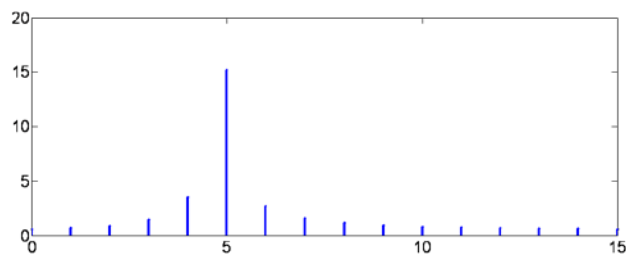
**Proof.** From the definition of the DFT replacing the index  $k$  with  $N - k$  we obtain

$$X[N - k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}(N-k)n} = \sum_{n=0}^{N-1} x[n] e^{j\frac{2\pi}{N}kn}$$

Since  $x[n]$  is real by assumption, the rightmost expression is the complex conjugate of  $X[k]$  which shows the property.

The consequence of this property is that, when the signal is real, the whole information of the frequency spectrum is in the first half as  $X[k], k = 0, \dots, (N/2) - 1$ , while the information in the second half of the DFT is redundant.

Example. Referring to the previous example, since the signal is real the whole information of its frequency spectrum is shown in the plot for  $k = 0, \dots, 15$  as shown below.



**Figure 21. DFT of a Sinusoid: first half of the plot.**

## 9. FFT of a Sinusoid in Matlab

This section (on video only) shows an implementation of the FFT of a sinusoid in Matlab.

### **VIDEO: FFT of a Sinusoid in Matlab (13:01)**

<http://faculty.nps.edu/rcristi/eo3404/b-discrete-fourier-transform/videos/chapter1-seg6.html>

[http://faculty.nps.edu/rcristi/eo3404/b-discrete-fourier-transform/videos/b1\\_11\\_fftSinusoidMatlab.mp4](http://faculty.nps.edu/rcristi/eo3404/b-discrete-fourier-transform/videos/b1_11_fftSinusoidMatlab.mp4)