A2. Sinusoids and Complex Exponentials

Objectives:

- Introduce continuous time and discrete time sinusoidal signals
- Define amplitude, frequency and phase of continuous time sinusoids
- Introduce the digital frequency of a discrete time sinuosoid
- Introduce the complex exponential representation of sinusoidal signals

1. Introduction

In the introductory chapter on acoustic signals we have seen how the acoustic waves are produced by vibrations and resonances. No matter how they are produced (by string, air or whatever medium or technique) they all have one thing in common: they are made of sinusoids. Just one sinusoid at a specific frequency gives you a wistle with a specific pitch, while if you put a few sinusoids together harmonically related (frequencies multiple of one fundamental frequency) they produce a note of an instrument or the human voice. Also a repetitive signature of a target, like a screw from a submarine, it is made of sinusoids at different frequencies.

In this chapter we define sinusoidal signals both in continuous time as well in discrete time. The goal is to pave the way to the analysis of signals in the frequency domain and their frequency spectra.

2. Sinusoids: Continuous Time

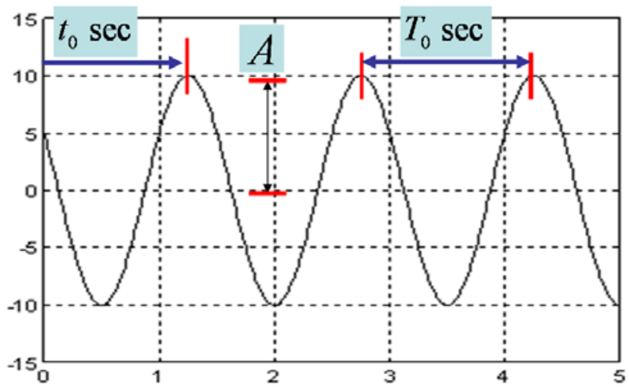
VIDEO: Continuous Time Sinusoids (08:33)

http://faculty.nps.edu/rcristi/eo3404/a-signals/videos/section2-seg1_media/section2-seg1-0.wmv
http://faculty.nps.edu/rcristi/eo3404/a-signals/videos/continuousTimeSinusoids.mp4

A continuous time sinusoid $x(t)$ is defined by the following expression

$$x(t) = A\cos(2\pi F_0 t + \alpha) = A\cos(2\pi F_0 (t - t_0))$$

Its plot and the definitions are illustrated in the figure below.

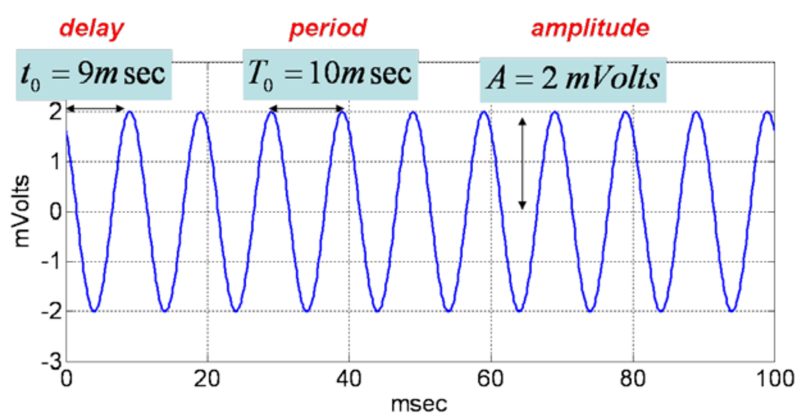


Sinusoidal signal, illustrating Amplitude** A**, period** $T_0 = 1/F_0$**, time delay** t_0**.

In this expression

- A is the amplitude, expressed in the same units as the signal (Volts, Amperes, meters …), usually positive. The minimum and the maximum value of the sinusoid are $-A$ and $+A$;
- F_0 is the frequency, expressed in Hz or in 1/sec. It is related to the period T_0, ie the time interval between two repetions, as $F_0 = 1/T_0$;
- α is the phase, expressed as an angle (say radians).

Example: take the sinusoid in the figure below. Then we can measure directly from the graph that the amplitude is $A = 2mVolts$, the period, ie the time between (say) two peaks (or two zeros, or any other two identical values) is $T_0 = 10m\sec$, and the time delay (occurrence of the first peak) is $t_0 = 9m\sec$. Then we compute the frequency $F_0 = 1/T_0 = 100Hz$, and the phase $\alpha = -2\pi F_0 t_0 = -2\pi \times 100 \times 9 \times 10^{-3} = -1.8\pi\ rad$. This yields the expression for the sinusoid as $x(t) = 2\cos(200\pi t - 1.8\pi)$ milliVolts.



Sinusoidal signal for the example.

3. Sinusoid: Discrete Time

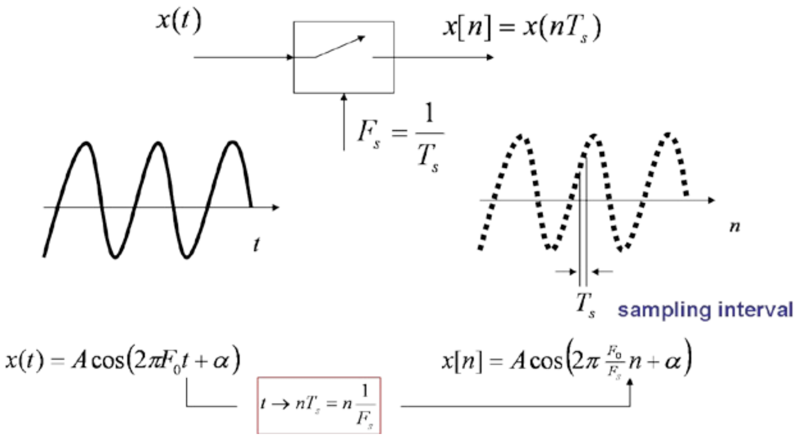
VIDEO: Discrete Time Sinusoids (05:38)

http://faculty.nps.edu/rcristi/eo3404/a-signals/videos/section2-seg1_media/section2-seg1-1.wmv
http://faculty.nps.edu/rcristi/eo3404/a-signals/videos/discreteTimeSinusoids.mp4

A sampled sinuosoid, in discrete time, is obtained by sampling a continuous time sinusoid by a sampling frequency F_S Hz or Samples per Second. Ignoring quantization and the errors from quantization effects (ie assuming infinite precision) a sampled sinusoids is a numerical sequence $x[n]$ obtained as

$$x[n] = x(nT_S) = A\cos(2\pi F_0 nT_S + \alpha)$$

where T_S is the sampling interval. This is shown in the figure below.



Sampled Sinusoid with digital frequency $\omega_0 = 2\pi F_0 / F_S$

Since $T_S = 1/F_S$ we can write the sampled sequence $x[n]$ as

$$x[n] = x(nT_S) = A\cos(\omega_0 n + \alpha)$$

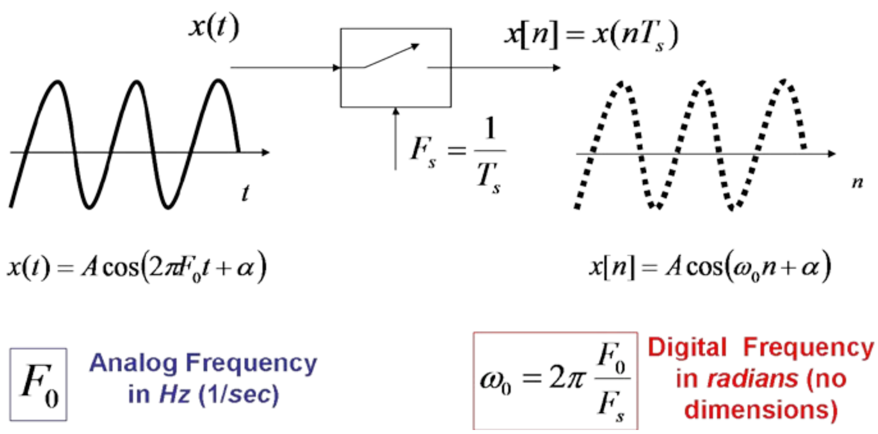
where

$$\omega_0 = 2\pi \frac{F_0}{F_S}$$

is the digital frequency expressed in radians. Notice that the digital frequency is a relative frequency (relative to the sampling frequency) and has no dimensions.

Example. A sinusoid with frequency $F_0 = 1.5kHz$ is sampled at a sampling frequency $F_S = 5.0kHz$ $F_S = 5.0kHz$, ie 5,000 samples per second. The sampled sinusoid has a digital frequency $\omega_0 = 2\pi \times 1.5/5.0 = 0.6\pi\ radians$.

The mapping between the analog frequency F_0 in Hz and the digital frequency $\omega_0 = 2\pi F_0 / F_S$ is shown in the figure below.



The analog frequency F_0 ***is mapped into the digital frequency*** ω_0.

Example. The sinusoidal signal $x(t) = 2\cos(100\pi t + 0.1\pi)$ is sampled at a sampling frequency $F_S = 500Hz$. Then the analog frequency is $F_0 = 50Hz$, the digital frequency is $\omega_0 = 2\pi \times 50/500 = 0.2\pi\ radians$ and the sampled signal can be written as $x[n] = 2\cos(0.2\pi n + 0.1\pi)$.

4. Complex Numbers and Complex Exponentials

VIDEO: Complex Numbers (21:11)

http://faculty.nps.edu/rcristi/eo3404/a-signals/videos/section2-seg2_media/section2-seg2-0.wmv
http://faculty.nps.edu/rcristi/eo3404/a-signals/videos/complexNumbers.mp4

As we mentioned several times, sinusoidal signals play a very important role in the analysis of signals. The problem with sinusoids is the fact that they do not lend themselves to easy mathematical manipulations. For example to add, or multiply two sinusoids requires the use of some trigonometric formulas which are very unintuitive and hard to remember.

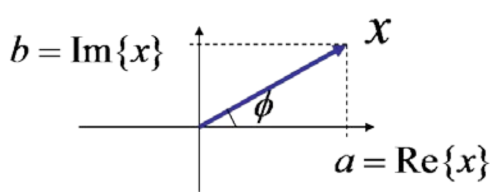
It turns out, fortunately, that with a little bit of abstraction we can represent sinusoids in a very convenient way using complex numers and, in particular, complex exponentials.

Unlike sinuosoids and trigonometric functions, exponentials are extremely easy to manipulate, and operations like differentiation and integration become just algebraic manipulations.

Since the understanding of complex exponentials is extremely important in Digital Signal Processing, let us recall briefly complex numbers and their main properties.

Complex Numbers**. A complex number x is defined as $x = a + jb$ where a is the real part, b the imaginary part and $j = \sqrt{-1}$.**

The complex number x can be represented in the complex plane as a vector, with the horizontal axis being the real part and the vertical axis the imaginary part. This can be seen in the figure below.



A complex number can be represented as a vector in the complex plane.

Looking at the vector representation in the figure above, we can define the angle ϕ with the real axis and the length $|x|$ of the vector (ie the complex number) x. Then we can easily see that

$$a = |x|\cos(\phi)$$
$$b = |x|\sin(\phi)$$

Which leads to the polar representation of the complex number in terms of magnitude $|x|$ and phase ϕ, as

$$x = |x|\left(\cos(\phi) + j\sin(\phi)\right)$$

Example. Take the complex number $x = 1 + j2$,. The real part is $a = \text{Re}\{x\} = 1$, the imaginary part is $b = \text{Im}\{x\} = 2$, the magnitude $|x|$ is the length of the vector $x = \sqrt{a^2 + b^2} = \sqrt{5}$ and the phase $\varphi = \arctan(b/a) = 1.1071\, radians$.

The most important fact at the basis of the whole complex number representation is the Complex Exponential:

VIDEO: Complex Exponentials (09:11)

http://faculty.nps.edu/rcristi/eo3404/a-signals/videos/section2-seg2_media/section2-seg2-1.wmv
http://faculty.nps.edu/rcristi/eo3404/a-signals/videos/complexExponentials.mp4

Complex Exponential. Define the complex exponentials as $e^{j\phi} = \cos(\phi) + j\sin(\phi)$

Now we can see two ways of representing a complex number:

Representation of a Complex Number. A complex number can be represented in rectangular coordinates (real and imaginary parts) as $x = a + jb$ **or polar coordinates (magnitude and phase) as** $x = |x| e^{j\phi}$ **.**

Example. The complex number in the prvious example can be written as

$$x = 1 + j2 = \sqrt{5}e^{j1.1071}.$$

The conversion from rectangular to polar coordinates is very straightforward and it is just based on simple geometric considerations. In particular, from rectangular to polar:

$$|x| = magnitude(x) = \sqrt{a^2 + b^2}$$

$$\phi = phase(x) = \arctan(b/a) \qquad \text{if } a > 0$$
$$\phi = phase(x) = \arctan(b/a) \pm \pi \quad \text{if } a < 0$$

Similarly, from polar to rectangular:

$$a = \mathrm{Re}\{x\} = |x|\cos(\phi)$$
$$b = \mathrm{Im}\{x\} = |x|\sin(\phi)$$

Example. Consider the complex number $x = 1 - 2j$. Then the ral part is $\mathrm{Re}\{x\} = 1$, the imaginary part $\mathrm{Im}\{x\} = -2$, the magnitude $|x| = \sqrt{4+1} = \sqrt{5}$ and the phase $phase\{x\} = \arctan\{-2\} = -1.1071 rad$. Then we can write it as $x = 1 - 2j = \sqrt{5}e^{-j1.1071}$.

Let us see another example, similar … but different

Example. Consider the complex number $x = -1 + 2j$. Then the ral part is $\mathrm{Re}\{x\} = -1$, the imaginary part $\mathrm{Im}\{x\} = 2$, the magnitude $|x| = \sqrt{4+1} = \sqrt{5}$ and the phase $phase\{x\} = \arctan\{-2\} + \pi = -1.1071 + \pi = 2.0344 rad$. The we can write it as $x = -1 + 2j = \sqrt{5}e^{j2.0344}$.

Now that we established a relation between complex exponentials and sinusoids, we can express trigonometric functions by complex exponentials. In fact recall

$$e^{j\phi} = \cos(\phi) + j\sin(\phi)$$

$$e^{-j\phi} = \cos(\phi) - j\sin(\phi)$$

where we used $\cos(-\phi) = \cos(\phi)$ and $\sin(-\phi) = -\sin(\phi)$. Then we can easy show the relations

$$\cos(\phi) = \frac{1}{2}\left(e^{j\phi} + e^{-j\phi}\right)$$
$$\sin(\phi) = \frac{1}{2j}\left(e^{j\phi} - e^{-j\phi}\right)$$

known as Euler Formulas.

In the treatment which follows we will see that it is much easier to deal with exponentials rather then sinusoids. Just to give you an idea, see the following example.

Example. Do you remember the formula for $\cos(\alpha)^2$? I don't either, but with a little bit of patience we can derive it using Euler Formulas as $\cos(\alpha)^2 = \frac{1}{4}\left(e^{j\alpha} + e^{-j\alpha}\right)^2 = \frac{1}{4}\left(e^{j2\alpha} + 2 + e^{-j2\alpha}\right)$ since $e^{j\alpha}e^{-j\alpha} = 1$. Use Euler Formula again to find $\cos(\alpha)^2 = \frac{1}{2} + \frac{1}{2}\cos(2\alpha)$.

Morale of the story:

It is much easier to manipulate complex exponentials than trigonometric functions.

5. Representation of Sinusoidal Signals in terms of Complex Exponentials

We can extend Euler Formulas to signals, and write a general sinusoidal signal as

$$x(t) = A\cos\left(2\pi F_0 t + \alpha\right) = \frac{A}{2}\left(e^{j(2\pi F_0 t + \alpha)} + e^{-j(2\pi F_0 t + \alpha)}\right)$$

Since the exponentials can be factored, we can lump the constant terms with amplitude and phase in a complex constant and separate it from the time varying part. This yields

$$x(t) = A\cos\left(2\pi F_0 t + \alpha\right) = \left(\frac{A}{2}e^{j\alpha}\right)e^{j2\pi F_0 t} + \left(\frac{A}{2}e^{-j\alpha}\right)e^{-j2\pi F_0 t}$$

Notice that it is the sum of two complex exponentials, one with positive frequency F_0 Hz and one with negative frequency equal to "$-F_0$" Hz.

Example. The signal $x(t) = 5\cos(1000\pi t - 1.2)$ can be written as

$$x(t) = \left(2.5e^{-j1.2}\right)e^{j1000\pi t} + \left(2.5e^{j1.2}\right)e^{-j1000\pi t}.$$

Similarly in discrete time. The discrete time sinusoid

$$x[n] = A\cos(\omega_0 n + \alpha) = \frac{A}{2}\left(e^{j(\omega_0 n + \alpha)} + e^{-j(\omega_0 n + \alpha)}\right)$$

can be written in terms of complex exponentials

$$x[n] = A\cos(\omega_0 n + \alpha) = \left(\frac{A}{2}e^{j\alpha}\right)e^{j\omega_0 n} + \left(\frac{A}{2}e^{-j\alpha}\right)e^{-j\omega_0 n}$$

Again it has two frerquencies: one positive ω_0 radians, and one negative, $-\omega_0$ radians.

Example. The signal in the previous example is sampled at a sampling frequency $F_S = 5kHz$. Then the digital frequency is $\omega_0 = 2\pi F_0 / F_S = 1000\pi / 5000 = \pi / 5$. Then the sampled sinusoid is of the form $x[n] = 5\cos(0.2\pi n - 1.2)$ which can be written in complex form as $x[n] = \left(2.5e^{-j1.2}\right)e^{j0.2\pi n} + \left(2.5e^{j1.2}\right)e^{-j0.2\pi n}$.

6. Generate Sinusoidal Signals using Matlab

VIDEO: Sinusoids in Matlab (10:01)

http://faculty.nps.edu/rcristi/eo3404/a-signals/videos/section2-seg3_media/section2-seg3-0.wmv
http://faculty.nps.edu/rcristi/eo3404/a-signals/videos/sinusoidsMatlab.mp4

In this section, let us see how we can generate a discrete time sinuosoid of a given frequency F_0, sampling frequency F_S, amplitude A, phase α and length N. In this example we use numbers for illustration.

First, we need to specify the parameters and determine the digital frequency:

```
F0=2500;             % Frequency in Hz
Fs=22000;            % Sampling Frequency in Hz
w0=2*pi*F0/Fs;       % Digital Frequency in radians
```

Then we need to generate a vector of indeces

```
N=1000;              % total number of data points
n=0:N-1;             % vector of indeces n=[0,1,2,...,N-1]
```

Now we can generate a sinusoidal signal

```
A=2.5;                % Amplitude
Alpha=0.18;           % phase in radians
x=A*cos(w0*n+alpha);% vector of samples
```

Now we can plot the signal. The simplest plot:

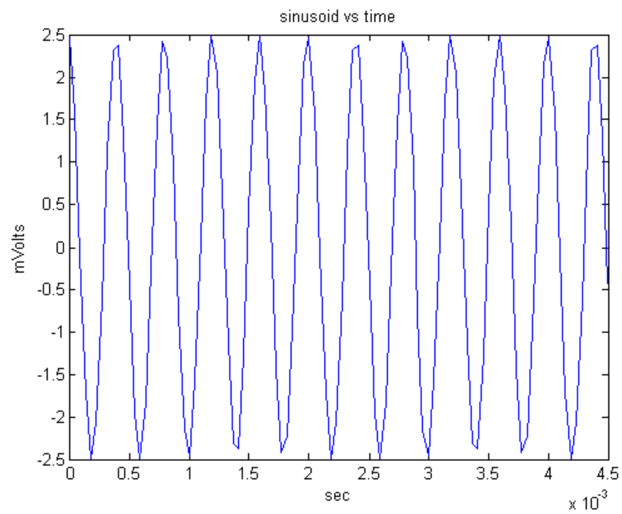
```
plot(x)
```

But if we want to plot it as a function of time, we need a vector for the horizontal axis:

```
Ts=1/Fs;              % sampling interval in seconds
t=n*Ts;               % vector of sampling instants t=[0, Ts,
                      2Ts,...,(N-1)Ts]

plot(t,x), title('sinusoid vs time'), xlabel('sec'),
ylabel('mVolts')
```

The plot is shown below.



7. Matlab Videos

The following material is covered in these video clips:

- M-Files (13:58)
- Functions (17:31)
- Aliasing (04:42)

http://faculty.nps.edu/rcristi/eo3404/a-signals/videos/section2-seg3_media/section2-seg3-1.wmv
http://faculty.nps.edu/rcristi/eo3404/a-signals/videos/MFiles.mp4

http://faculty.nps.edu/rcristi/eo3404/a-signals/videos/section2-seg3_media/section2-seg3-2.wmv
http://faculty.nps.edu/rcristi/eo3404/a-signals/videos/Functions.mp4

http://faculty.nps.edu/rcristi/eo3404/a-signals/videos/section2-seg3_media/section2-seg3-3.wmv http://faculty.nps.edu/rcristi/eo3404/a-signals/videos/Aliasing.mp4