Note Title 4/5/2011

# Recall: Z-TRANFORM AND FREQ. RESP.

$$x[n] \longrightarrow \boxed{S} \longrightarrow y[n]$$

**Assume:**

$S$ is Linear, Time Invariant (LTI)

**Typical:** LINEAR DIFFERENCE EQ.

$$y[n] + a_1 y[n-1] + \dots + a_N y[n-N] =$$

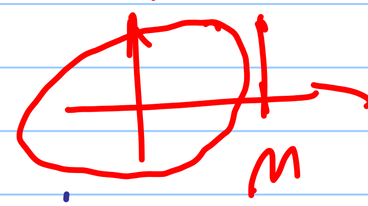
$$= b_0 x[n] + b_1 x[n-1] + \dots + b_N x[n-N]$$

Example: $N$ = order, $N = 1$

$$y[n] + 0.8\,y[n-1] = 2\,x[n-1]$$

PAST

$a_1 = 0.8, \quad b_0 = 0, \quad b_1 = 2$




Implementation by recursion:

EFFECT INPUT

$$y[n] = -0.8\,y[n-1] + 2\,x[n-1]$$

CAUSAL

CURRENT OUTPUT

PAST OUTPUT

PAST INPUT

Observation: discrete time sequence

$x[n]$ = file with numbers

$\boxed{F_s}$

$n=0 \rightarrow$ 0.12

$n=1 \rightarrow$ −0.37

⋮

$n \rightarrow$ pointer in the file.

No time information

no time inf.

$x[n] \rightarrow X(\omega)$

index

$\omega = \frac{2\pi f}{F_s}$ no dim.

## Observation on causality:

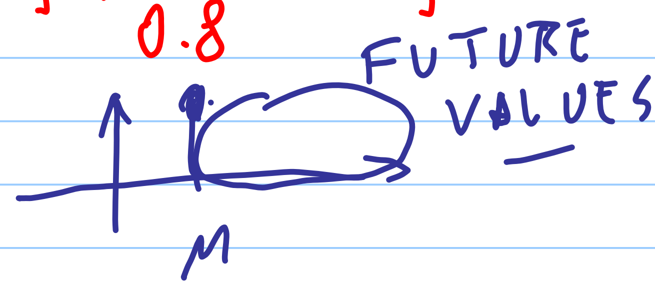
same LDE

$$y[n] + 0.8\,y[n-1] = 2\,x[n-1]$$

① $\check{y}[n] = -0.8\,y[n-1] + 2\,x[n-1]$ CAUSAL

② $\check{y}[n-1] = -\dfrac{1}{0.8}\,y[n] + \dfrac{2}{0.8}\,x[n-1]$

NOT CAUSAL



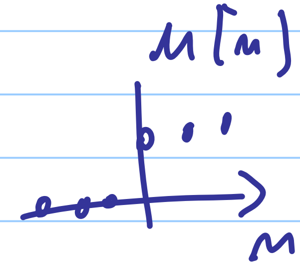
Definition: z-TRANSFORM

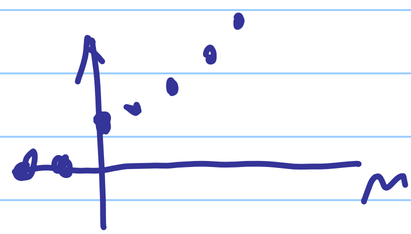
$$X(z) = \mathcal{Z}\{x[n]\} = \underbrace{\sum_{n=-\infty}^{+\infty} x[n] z^{-n}}$$

$z$ = complex variable

Ex: $x[n] = 2^n u[n]$

$$u[n] = \begin{cases} 1, & \text{if } n \geq 0 \\ 0 & n < 0 \end{cases}$$



$$X(z) = \sum_{n=-\infty}^{+\infty} 2^n u[n] z^{-n} = \sum_{n=0}^{+\infty} 2^n z^{-n} =$$

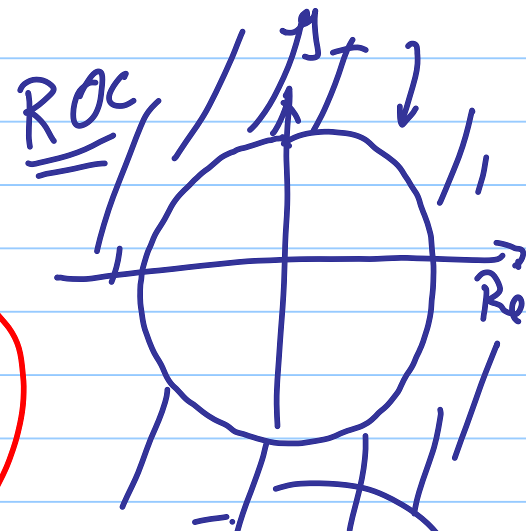
$$= \sum_{n=0}^{+\infty} (2z^{-1})^n = ? \qquad (1)$$

<u>Recall</u>: geometric series

$$\sum_{n=0}^{+\infty} \alpha^n = \begin{cases} \frac{1}{1-\alpha} & \text{if } \underbrace{|\alpha| < 1} \\ \text{not conv. (NaN)} & \text{if } |\alpha| \geq 1 \end{cases}$$

Apply geometric series to (1) : $|z| = 2$

$\alpha = 2z^{-1}$,




the

$$X(z) = \frac{1}{1-2z^{-1}} = \frac{z}{z-2}$$

$$|\alpha| < 1 \Rightarrow |2z^{-1}| < 1 \Rightarrow \underbrace{|z| > 2}_{\text{ROC}}$$

Compare z-Transform and DTFT

Recall:

$$X(w) = \sum_{n=-\infty}^{+\infty} x[n] \underbrace{e^{-jwn}}$$

same as $X(z)$

$z = e^{jw}$

FACT: $X(w) = X(z)\Big|_{z=e^{jw}}$

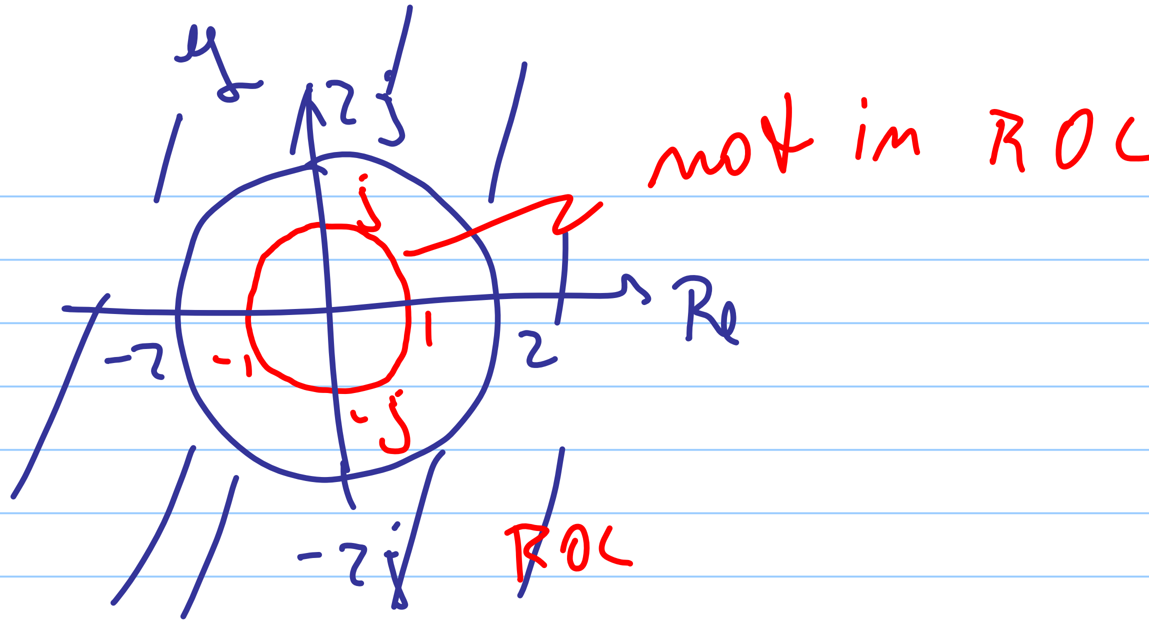
provide $e^{jw} \in ROC$

"belongs to"

Ex: $x[n] = 2^n u[n]$

$X(z) = \frac{z}{z-2}, \underbrace{ROC \;\; |z| > 2}$

$z = e^{jw} \rightarrow |z| = 1 \notin ROC$




Meaning: for $x[n] = 2^n u[n]$

$$X(z) = \frac{z}{z-2} \quad |z| > 2$$

$X(\omega)$ does not exist

Ex: $x[n] = 0.8^n u[n]$

$$X(z) = \sum_{n=0}^{+\infty} 0.8^n z^{-n}$$

$$= \sum_{n=0}^{+\infty} \underbrace{\left(0.8 z^{-1}\right)}_{\alpha = 0.8 z^{-1}}{}^{n} = \frac{z}{z - 0.8}$$

ROC $\quad |0.8 z^{-1}| < 1 \Rightarrow |z| > 0.8$

$|z| = 0.8$

$$X(w) = \sum_{n=0}^{+\infty} 0.8^n e^{-jwn}$$

$z \to e^{jw}$ , $|z| = 1$

ROC: $|z| > 0.8$

unit circle $(|z|=1) \in$ ROC

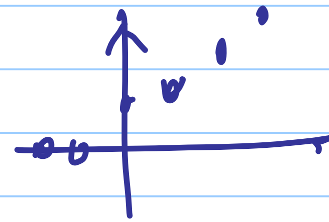
since $X(z)$ converges in $|z| = 1$

$X(w) = X(z)\Big|_{z=e^{jw}}$ $\longrightarrow$

$$X(w) = \frac{e^{jw}}{e^{jw} - 0.8}$$

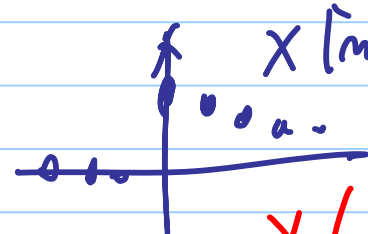
Compare the two cases :

① $x[n] = 2^n u[n]$




no $X(w)$

② $x[n] = 0.8^n u[n]$




$X(w)$

① no $X(w)$

② $X(w) = X(z)\big|_{z=e^{jw}}$

FACT: condition for existence of $X(\omega)$:

$$\underbrace{\sum_{n=-\infty}^{+\infty} |x[n]|}_{\text{convergent}} < +\infty$$

Ex: ① $\sum_{n=0}^{+\infty} 2^n \longrightarrow +\infty$ not conv.

② $\sum_{n=0}^{+\infty} 0.8^n = \frac{1}{1-0.8} = 5 < +\infty$ conv.

Back to LDE:

Properties of Z-TRANSFORM

① $\mathcal{Z}\{x[n-L]\} = z^{-L} X(z)$

$L$ = integer shift

② $\mathcal{Z}\{h[n] * x[n]\} = H(z)\,X(z)$

Proof: in the book.

$$x[n] \longrightarrow \underbrace{\boxed{h[n]}}_{\text{LTI}} \longrightarrow y[n]$$

LDE:

$$y[n] + a_1 y[n-1] + \dots + a_N y[n-N] =$$

$$= b_0 x[n] + \dots + b_N x[n-N]$$

apply z-Transform on both sides:

$$Y(z) + a_1 z^{-1} Y(z) + \dots + a_N z^{-N} Y(z) =$$

$$= b_0 X(z) + b_1 z^{-1} X(z) + \dots + b_N z^{-N} X(z)$$

some algebra

$$Y(z) = H(z)\,X(z)$$

$$H(z) = \frac{b_0 + b_1 z^{-1} + \dots + b_N z^{-N}}{1 + a_1 z^{-1} + \dots + a_N z^{-N}} =$$

$$= \frac{b_0 z^N + b_1 z^{N-1} + \dots b_N}{z^N + a_1 z^{N-1} + \dots + a_N}$$

① $H(z)$ = transfer function

② $H(z) = Z\{h[n]\}$, $h[n]$ impulse response

Example :

$$y[n] + 0.8\,y[n-1] = 2\,x[n-1]$$ ✓

$$H(z) = \frac{2z^{-1}}{1 + 0.8\,z^{-1}} =$$

$$= \frac{2}{z + 0.8}$$

TRANSFER FUNCTION

# FREQUENCY RESPONSE ?

Recall: $H(\omega) = \text{DTFT}\{h[n]\}$

definition

$x[n] = e^{j\omega n} \longrightarrow \boxed{H(\omega)} \longrightarrow y[n] = \underbrace{H(\omega)}_{\text{effect of system on mag. and phase}} e^{j\omega n}$

Freq. Response from Transfer function

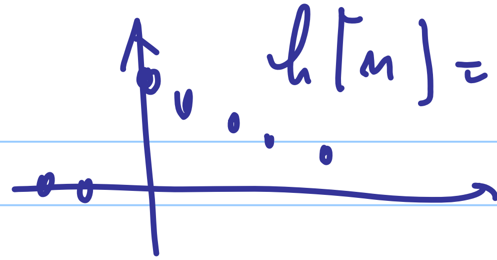
what we have seen:

$$H(\omega) = H(z)\Big|_{z=e^{j\omega}}$$

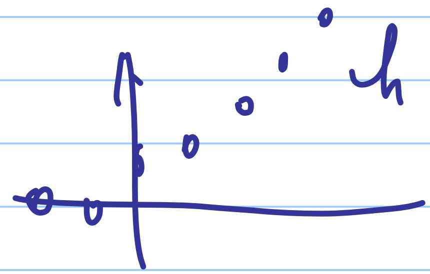
provided $\displaystyle\sum_{n=-\infty}^{+\infty} |h[n]| < \infty$

if not $H(\omega)$ does not exist

Ex :

(1)  $h[n] = 0.8^n u[n]$ ✓

STABLE

(2)  $h[n] = 2^n u[n]$

UNSTABLE

FACT : if system stable

$$H(w) = H(z)\Big|_{z=e^{jw}} ;$$

if system unstable

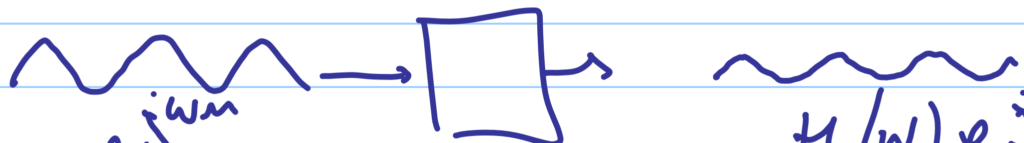
no $H(w)$ freq. response.

DTFT vs Z-Transform

( FT vs. Laplace )

Freq. response : steady state

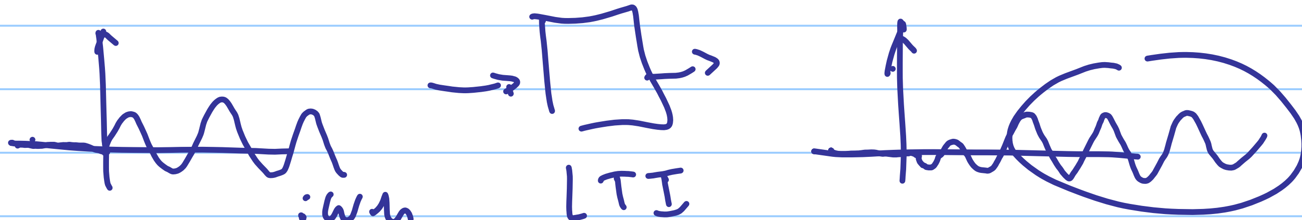
<u>Theory</u>:



$$e^{j\omega n} \rightarrow \boxed{\phantom{H}} \rightarrow H(\omega)\, e^{j\omega n}$$

$-\infty < n < +\infty$

"In practice: if system stable



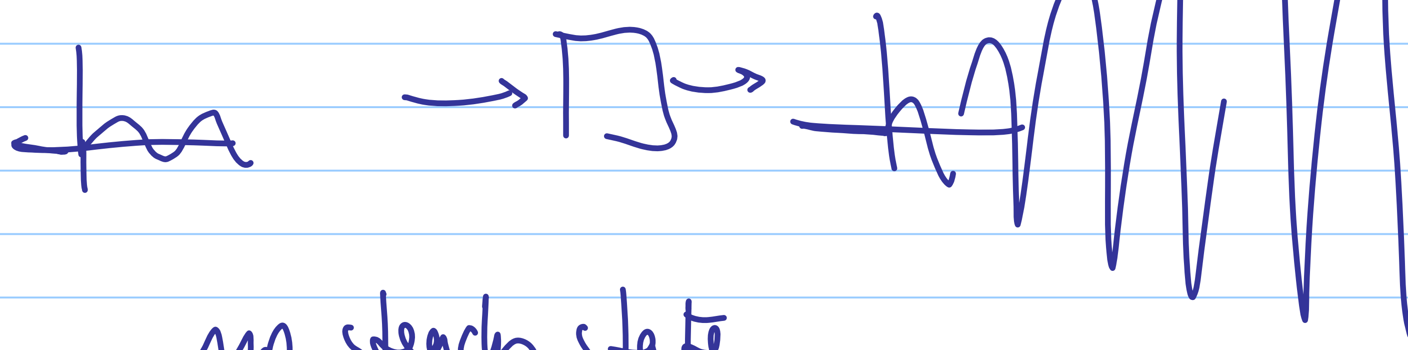
$x[n] = e^{j\omega n}$ LTI STEADY STATE

$$y[n] = H(\omega)\, e^{j\omega n}$$

STEADY STATE EXISTS $\Longrightarrow$

$\Longrightarrow$ FREQ. RESP. EXISTS

If system unstable



no steady state

no Freq. Resp.

no $H(w)$

FACT: stability of causal systems

$$H(z) = \frac{B(z)}{A(z)}, \quad B(z), A(z) \text{ polynomials}$$

FACTOR $B(z)$, $A(z)$ into roots:

$$H(z) = K \frac{(z - z_1) \cdots (z - z_M)}{(z - p_1) \cdots (z - p_N)}$$

$z_i$ = ZEROS

$p_i$ = poles.

Ex:

$$y[n] + y[n-1] + y[n-2] = 2x[n-1] - x[n-2]$$

$$H(z) = \frac{2z^{-1} - z^{-2}}{1 + z^{-1} + z^{-2}} =$$

$$= \frac{2z - 1}{z^2 + z + 1} = \frac{B(z)}{A(z)}$$

$z_i \Rightarrow 2z - 1 = 0 \Rightarrow z_1 = \frac{1}{2}$ $\underline{\underline{zero}}$

$p_i \rightarrow z^2 + z + 1 = 0 \Rightarrow p_{1,2} = \frac{-1 \pm \sqrt{1-4}}{2} =$

$= \left(-0.5\right) \pm j\left(\frac{\sqrt{3}}{2}\right)$ $\underline{\underline{poles}}$

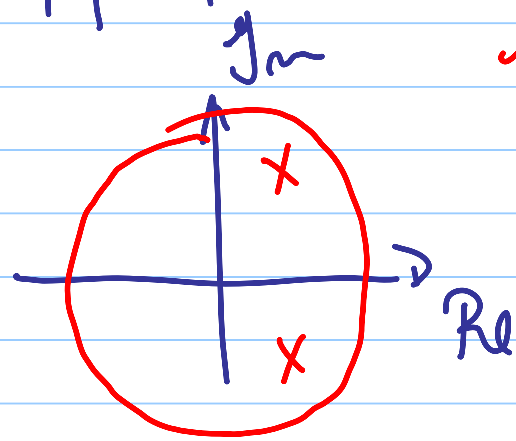
$= -0.5 \pm j\,0.866$

Then

$$H(z) = 2\,\frac{z - 0.5}{(z + 0.5 - j0.866)(z + 0.5 + j0.866)}$$

FACT: a causal system is stable if and only if

$$|p_i| < 1$$ , i.e. inside unit circle.



For example:

$$|p_i| = \sqrt{0.5^2 + \frac{3}{4}} = 1$$

not inside the unit circle

i.e. system unstable

Ex:

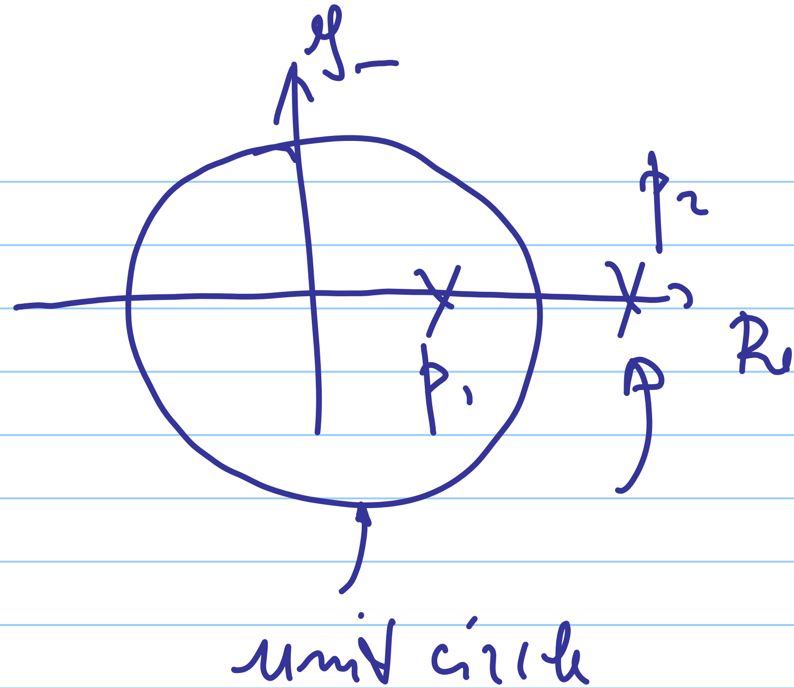
$$y[n] + 2y[n-1] + 0.8y[n-2] = x[n-1]$$

$$H(z) = \frac{z^{-1}}{1+2z^{-1}+0.8z^{-2}} =$$

$$= \frac{z}{z^2+2z+0.8}$$

$p_i$: $z^2+2z+0.8 = 0$, $p_{1,2} = \dfrac{-2 \pm \sqrt{4-0.64}}{2}$

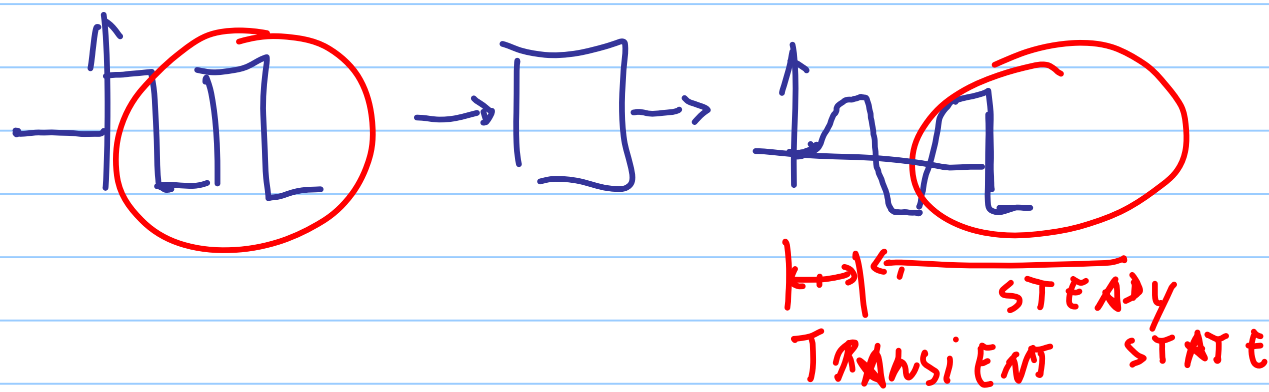
$= -1 \pm (\dots)$




since at least one pole outside unit circle $\Rightarrow$ system unstable

For stable systems: Z-Transform vs DTFT
(LT vs FT)

① steady state: DTFT



(7) Transfer function $H(z)$

transient and steady state