Note Title 5/31/2011

Ch 4: IIR Filter

1. $s \longrightarrow z$

$\frac{d}{dt} \longrightarrow$ time delay

Two transformations

1. Euler $s = \frac{1 - z^{-1}}{T_s}$

(+) it preserves stability

(−) it does not preserve freq. resp.

2. Bilinear transformation

$$s = \frac{2}{T_s} \frac{z-1}{z+1}$$

⊕ preserves stability

⊕ preserves freq. response

$$H_{digital}(z) = H_{analog}(s) \Big|_{s = \frac{2}{T_s}\frac{z-1}{z+1}}$$

Design with Bilinear transformation

LPF

Given:
F_p passband Hz

F_{stop} stopband Hz

passband Ripple dB

stopband attenuation dB

sampling freq. F_s Hz

Step 1: specs in digital freq.

$$\omega_p = \frac{2\pi F_p}{F_s} \quad \text{pass rad.}$$

$$\omega_{STOP} = 2\pi \frac{F_{STOP}}{F_s} \quad \text{stop rad.}$$

Ripple: same

Attenuation: same

step 2: determine the prototype filter in continuous time $H_{analog}(s)$.

<u>Recall:</u> $\omega \longleftrightarrow \Omega$ for BT

$$\omega = 2\arctan\left(\frac{\Omega T_s}{2}\right)$$

$$\Omega = \frac{2}{T_s}\tan\left(\frac{\omega}{2}\right) \leftarrow \begin{cases} \Omega_p \checkmark \\ \Omega_{stop} \checkmark \end{cases}$$

Step 3: design analog filter with specs in step 2.

$H_{analog}(s)$

Most likely: Butterworth Filter

Step 4 transform into $H(z)$ by Bilinear transformation

$$s = \frac{2}{T}\frac{z-1}{z+1}$$

cumbersome by brute force

Better way: map zeros and poles

$$H_{analog}(s) = K \frac{(s-z_1)\cdots(s-z_M)}{(s-p_1)\cdots(s-p_N)}$$

$N > M$,

Zeros: ① $\frac{2}{T_s}\frac{z-1}{z+1} - z_k = 0$, $k=1,\dots,M$ ✓

② $z=-1$ multiplicity $N-M$ ✓

Poles: $\frac{2}{T_s}\frac{z-1}{z+1} - p_k = 0$, $k=1,\dots,N$ ✓

Typical: Butterworth Filter

$$|H(\Omega)|^2 = \frac{1}{1 + \left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

determines PB Ripple

$$= \frac{1}{1 + \varepsilon^2 \left(\frac{\Omega}{\Omega_p}\right)^{2N}}$$ ✓

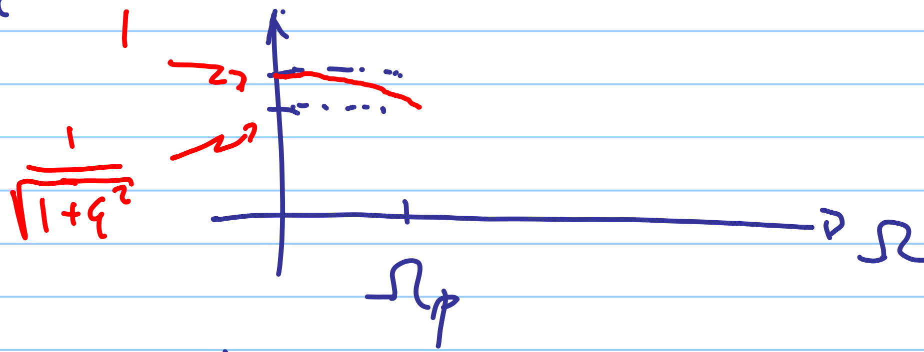
$D \equiv$

Ω_c = cut off (or 3dB) freq.

Ω_p = passband

When $0 \le \Omega \le \Omega_p$,

$$\frac{1}{\sqrt{1+\epsilon^2}} \le |H(\Omega)| \le 1$$



$$R_p = 20\log_{10}\frac{|H_{MAX}|}{|H_{min}|} = 20\log_{10}\frac{1}{\frac{1}{\sqrt{1+\epsilon^2}}} \Rightarrow$$

$$R_p = 10 \log_{10}(1+\varepsilon^2) \quad \text{in dB}$$

Ex: let $R_p = 0.5$ dB

$\varepsilon = ?$ solve

$$0.5 = 10 \log_{10}(1+\varepsilon^2)$$

$$\Rightarrow 1+\varepsilon^2 = 10^{\frac{0.5}{10}} \Rightarrow \varepsilon = \ldots\ldots$$

Determine order N from attenuation in stopband

$$|H(\Omega)| < \delta_{STOP}, \quad \forall\, \Omega > \Omega_{STOP}$$

$$\frac{1}{1+\varepsilon^2\left(\frac{\Omega_{STOP}}{\Omega_P}\right)^{2N}} < \delta_{STOP}^2$$

solve for N (integer) //-

Transformations from LPF to other filters on page 191 - 194

$$s \to g(s)$$

map zeros and poles like in the Bilinear Transformations.

Note: formula for N page LT?:

"log" any base, since you

take the ratio.

Recall: $\log_a b = (\log_a c)(\log_c b)$ //

PROBLEM 4.17

$$H(s) = \frac{2s+1}{s^2+s+1}$$

use BT, $F_s = 10$ HZ

$$s = 20\,\frac{z-1}{z+1}$$

$\underline{H(s)}$: $\underline{\text{ZEROS}}$: $s = -\frac{1}{2}$

$s = \infty$ (one zero)

poles: $s^2+s+1=0$

$$p_{1,2}=\frac{-1\pm\sqrt{-3}}{2}=-\frac{1}{2}\pm j\frac{\sqrt{3}}{2}$$

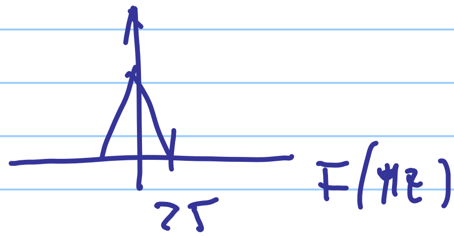
zeros: $20\frac{z-1}{z+1}=-0.5$

$z=-1$

poles: $20\frac{z-1}{z+1}=-\frac{1}{2}\pm j\frac{\sqrt{3}}{2} \Rightarrow$ solve

Ch 6:

6.13



$F_s = 6.0\ kHz$

Problem:

$x[n] \to \boxed{LPF} \to \boxed{\downarrow 100} \to y[n]$

$F_{s_1} = 6000\ Hz$ $\qquad$ $F_{s_2} = 60\ Hz$

1. Brute Force

LPF: $F_p = 25\ Hz$

$F_{STOP} = 30\ Hz$

$F_{s_1} = 6000\ Hz$

$\Rightarrow$ $\omega_p = 2\pi \frac{25}{6000}$

$\omega_{STOP} = 2\pi \frac{30}{6000}$

$\Delta\omega = \omega_{STOP} - \omega_p$ very small

$\Rightarrow N$ very large

② do it in stages

$$x[n] \rightarrow \boxed{H_1(z)} \rightarrow \boxed{\downarrow 10} \rightarrow \boxed{H_2(z)} \rightarrow \boxed{\downarrow 5} \rightarrow \boxed{H_3(z)} \rightarrow \boxed{\downarrow 2} \rightarrow y[m]$$

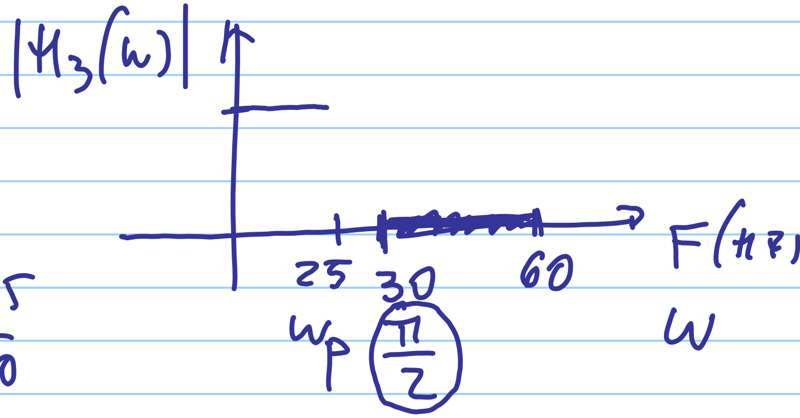
$F_x = 6000$ Hz	$F_2 = 600$ Hz	$F_3 = 120$ Hz	$F_y = 60$ Hz
PASS: $F_p = 25$ $\omega_p = 2\pi \frac{25}{6000}$	$F_p = 25$ Hz	$F_p = 25$ Hz	
STOP: $F_{STOP} = 600 - 30$ $= 570$ Hz	$F_{STOP} = 120 - 30$ $= 90$ Hz	$F_{STOP} = 30$ Hz	

Recall:

Last stage

$$x_3[n] \rightarrow \boxed{H_3(z)} \rightarrow \boxed{\downarrow 2} \rightarrow y[n]$$

$F_s = 120\,Hz$ $\qquad$ $F_y = 60\,Hz$

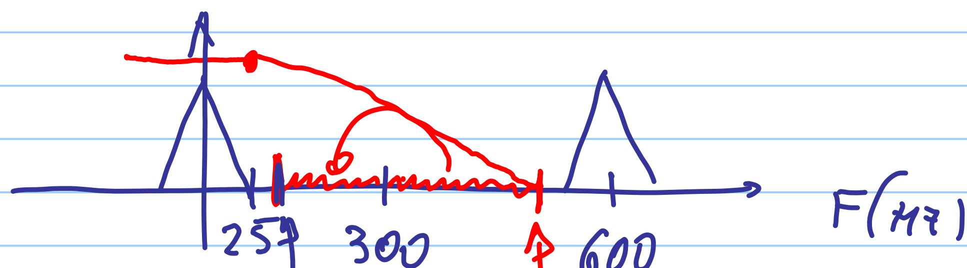


$$w_p = \frac{2\pi\, 25}{120}$$

i-th stage: $(i=1)$

$$x[n] \rightarrow \boxed{H_1(z)} \rightarrow \boxed{\downarrow 10} \rightarrow x_2[n]$$

$F_s = 6000\,Hz$ $\qquad$ $F_2 = 600\,Hz$



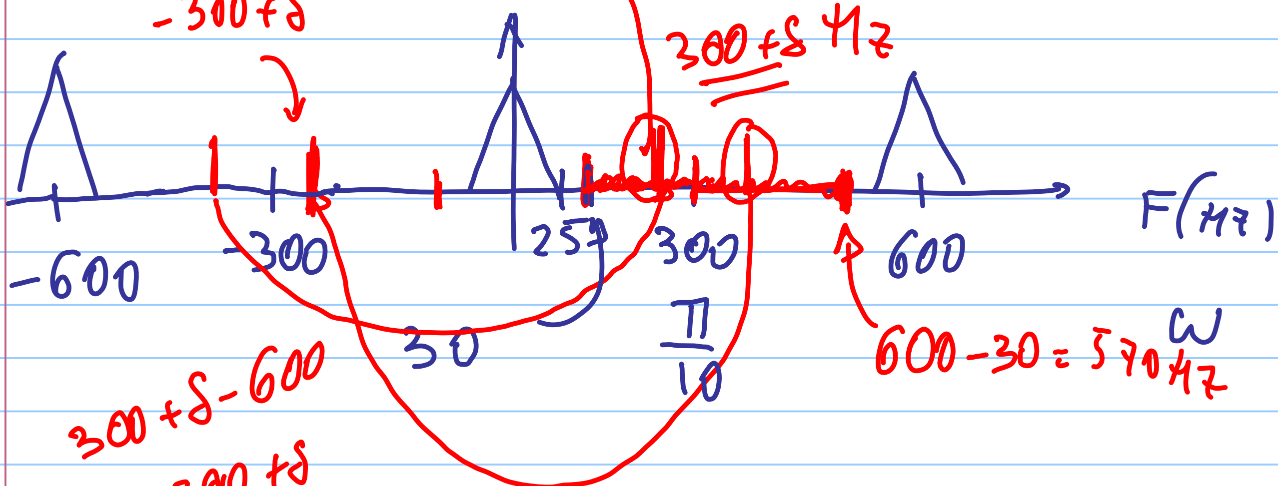
30 $\qquad$ $\frac{\pi}{10}$ $\qquad$ $600-30 = 570\,Hz$ $\quad$ ω

Reason:

$300 - \delta = -300 - \delta + 600$

$-300 + \delta$

$300 + \delta$ Hz



$600 - 30 = 570$ Hz

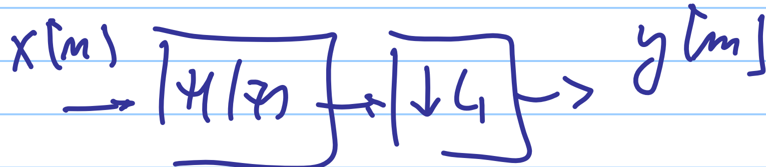
$300 + \delta - 600$

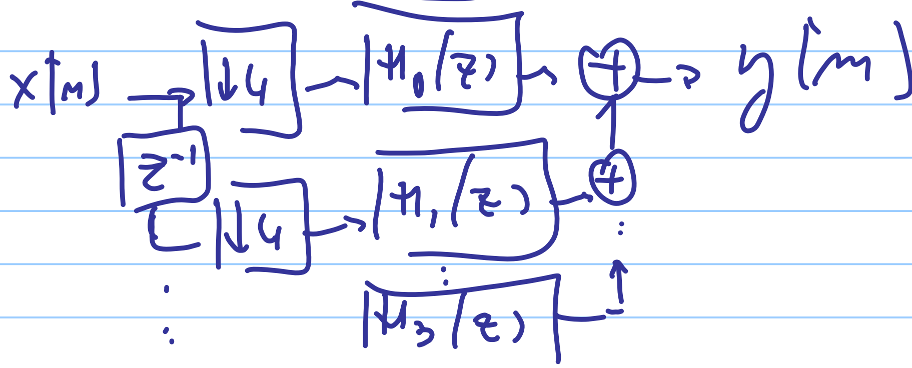
$= -300 + \delta$

PROBLEM 6.12

Use polyphase decomposition

$$H(z) = 1 + z^{-1} + 2z^{-2} + \dots \quad z^{-6}$$

a) 



$$H(z) = 1 + z^{-1} + 2z^{-2} - z^{-3} + z^{-4} - z^{-5} + z^{-6}$$ ✓

$\underline{\underline{N = 4}}$

$$\underline{\underline{H(z)}} = H_0(z^4) + z^{-1} H_1(z^4) + z^{-2} H_2(z^4) + z^{-3} H_3(z^4)$$

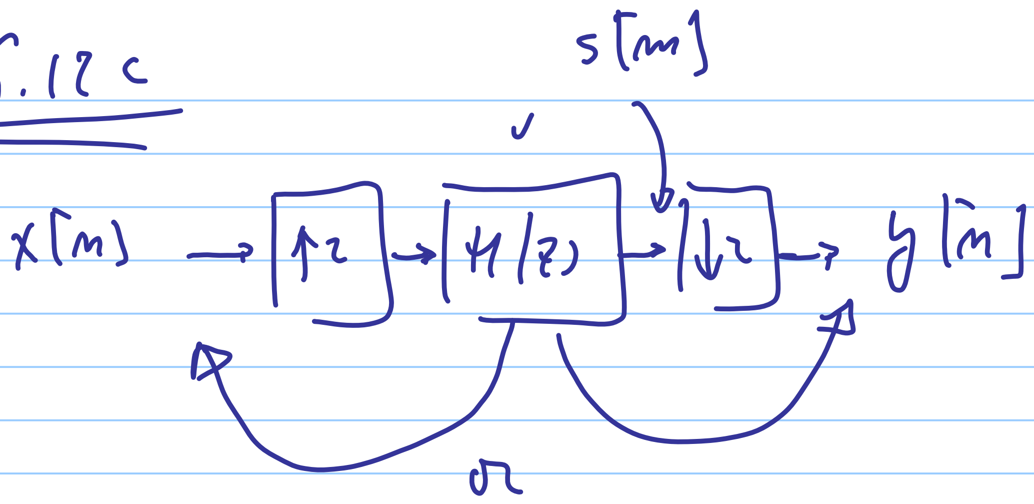
$$H_0(z^4) = 1 + z^{-4} \Rightarrow H_0(z) = 1 + z^{-1}$$

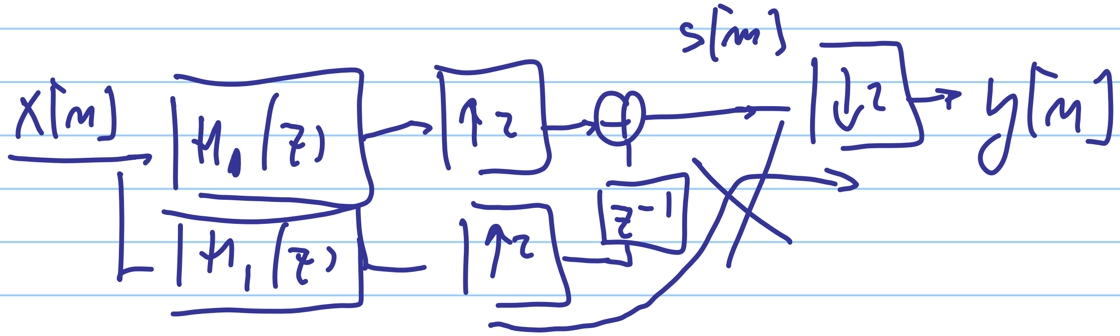
$$H_1(z^4) = 1 - z^{-4} \qquad H_1(z) = 1 - z^{-1}$$

$$H_2(z^4) = 2 + z^{-4} \qquad H_2(z) = 2 + z^{-1}$$

$$H_3(z^4) = -1 \qquad H_3(z) = -1$$

6.12 c

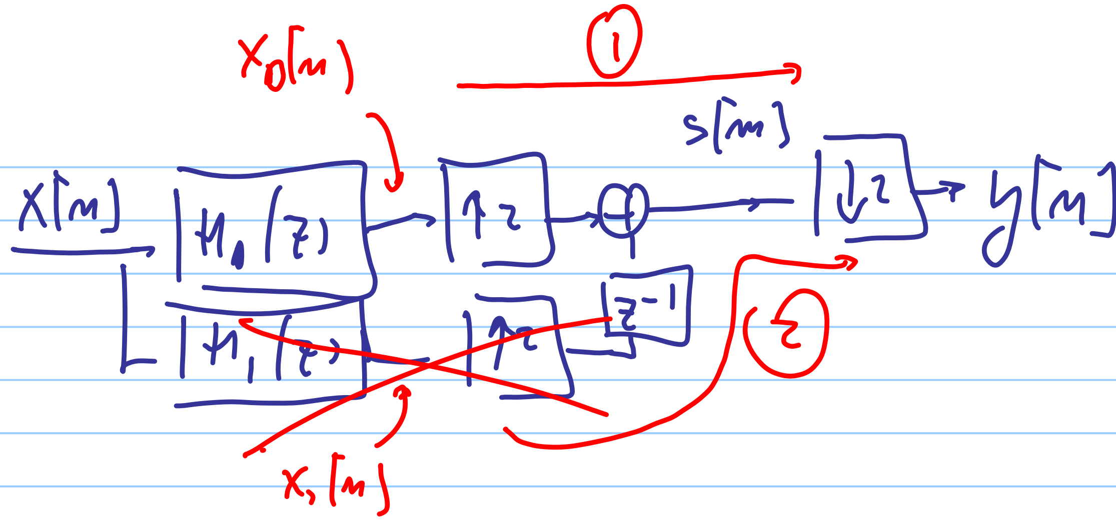




$$H(z) = 1 + z^{-1} + 2z^{-2} - z^{-3} + z^{-4} - z^{-5} + z^{-6}$$

$$H_0(z^2) = 1 + 2z^{-2} + z^{-4} + z^{-6}$$

$$H_1(z^2) = 1 - z^{-2} - z^{-4}$$



① $x_0[n] \to \cancel{\boxed{\uparrow 2}} \to \cancel{\boxed{\downarrow 2}} \to y_0[n] = x_0[n]$

② $x_1[n] \to \boxed{\uparrow 2} \to \boxed{z^{-1}} \to \boxed{\downarrow 2} \to y_1[n] = 0$

even

$$x[n] \to \boxed{H_0(z)} \to y[n]$$

$\Updownarrow$ **SAME!**

$$x[n] \to \boxed{\uparrow 2} \to \boxed{H(z)} \to \boxed{\downarrow 2} \to y[n]$$

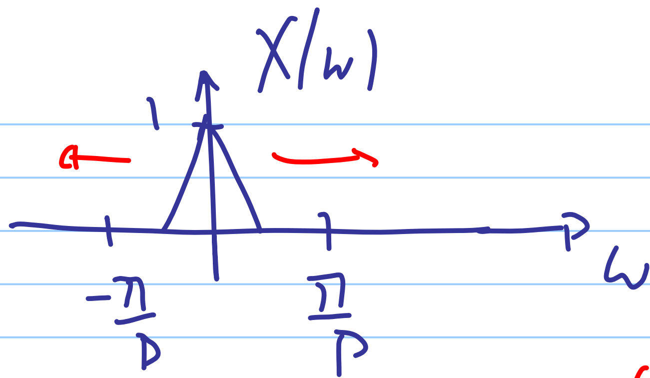
Recall:

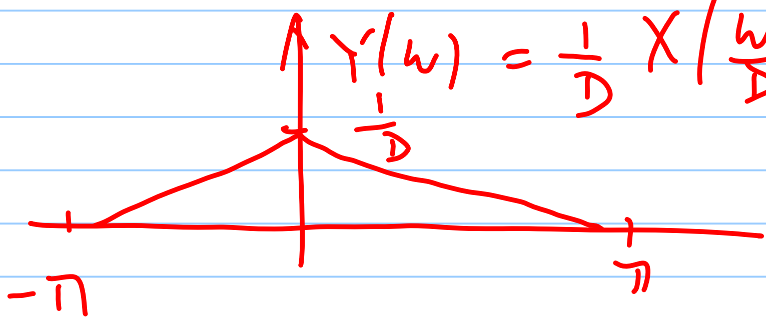
$$x[n] \longrightarrow \boxed{\downarrow D} \longrightarrow y[m]$$

$$Y(w) = \frac{1}{D} \sum_{k=0}^{D-1} X\left(\frac{w}{D} - k\frac{2\pi}{D}\right) \quad \text{general}$$

If no aliasing (i.e. $X(w) = 0, \ \forall |w| > \frac{\pi}{D}$)

$$Y(w) = \frac{1}{D} X\left(\frac{w}{D}\right) \qquad -\pi \leq w < \pi$$





$$Y(w) = \frac{1}{D} X\left(\frac{w}{D}\right)$$

Example:

$$x[n] = 2\cos\left(\frac{\pi}{10} n + \frac{\pi}{5}\right)$$

$$x[n] \longrightarrow \boxed{\downarrow 4} \longrightarrow y[n]$$

$$Y(w) = ?$$

$$X(w) = 2\pi e^{j\frac{\pi}{5}} \delta\left(w - \frac{\pi}{10}\right) + 2\pi e^{-j\frac{\pi}{5}} \delta\left(w + \frac{\pi}{10}\right)$$

$$-\pi \le w < \pi$$

no aliasing -

The

$$Y(w) = \frac{1}{4} X\left(\frac{w}{4}\right) =$$

$$= \frac{1}{4} 2\pi e^{j\frac{\pi}{5}} \underbrace{\delta\left(\frac{w}{4} - \frac{\pi}{10}\right)}_{4\,\delta\left(\frac{1}{4}\left(w - \frac{4\pi}{10}\right)\right)} + \frac{1}{4} 2\pi e^{-j\frac{\pi}{5}} \underbrace{\delta\left(\frac{w}{4} + \frac{\pi}{10}\right)}_{4\,\delta\left(\frac{1}{4}\left(w + \frac{4\pi}{10}\right)\right)}$$

$$Y(w) = \frac{1}{\cancel{4}} 2\pi e^{j\frac{\pi}{5}} \cancel{4}\, \delta\left(w - \frac{2\pi}{5}\right) +$$

$$+ \frac{1}{\cancel{4}} 2\pi e^{-j\frac{\pi}{5}} \cancel{4}\, \delta\left(w + \frac{2\pi}{5}\right)$$

$$y[n] = 2\cos\left(\frac{2\pi}{5} n + \frac{\pi}{5}\right) \quad //$$