Chapter 7: DFT Filter Bank Solutions

■ Problem 7.1

Problem

An FIR Filter has Transfer function $H(z) = 2 + z^{-1} - z^{-2} + 0.5 z^{-3}$ and frequency response $H(\omega)$.

a) Determine the Impulse Response of the filter $F(z)$ with frequency response $F(\omega) = H(\omega - 0.2\pi)$. Is the impulse response going to be real?

b) Determine the impulse response of the filter $G(z)$ with frequency response $H(\omega - 0.2\pi) + H(\omega + 0.2\pi)$. How do you relate the two impulse responses $f[n]$ and $g[n]$?

Solution

a) The transfer function $F(z)$ is determined as $F(z) = H(ze^{-j0.2\pi})$. In fact you can verify that $F(\omega) = F(z)|_{z=e^{j\omega}} = H(e^{j\omega} e^{-j0.2\pi}) = H(\omega - 0.2\pi)$. Substituing for the z-Transform we obtain

$$F(z) = 2 + e^{j0.2\pi} z^{-1} - e^{j0.4\pi} z^{-2} + 0.5 e^{j0.6\pi} z^{-3}$$

This yields an impulse response

$$f[n] = 2\delta[n] + e^{j0.2\pi}\delta[n-1] - e^{j0.4\pi}\delta[n-2] + 0.5 e^{j0.6\pi}\delta[n-3]$$

Notice that it is computed as $f[n] = h[n] e^{j0.2\pi n}$, with $h[n]$ the impulse response of $H(z)$.

b) By the same argument the transfer function can be determined as $G(z) = H(ze^{-j0.2\pi}) + H(ze^{j0.2\pi})$. Therefore the transfer function becomes

$$\begin{aligned} G(z) &= 4 + 2\cos(0.2\pi) z^{-1} - 2\cos(0.4\pi) z^{-2} + \cos(0.6\pi) z^{-3} \\ &= 4 + 1.61803 z^{-1} - 0.6180 z^{-2} - 0.3090 z^{-3} \end{aligned}$$

and the impulse response

$$g[n] = 4\delta[n] + 1.61803\delta[n-1] - 0.6180\delta[n-2] - 0.3090\delta[n-3]$$

It is computed as $g[n] = (e^{j0.2\pi n} + e^{-j0.2\pi n}) h[n] = 2\cos(0.2\pi n) h[n]$. Therefore $g[n] = 2\,\mathrm{Real}\{f[n]\}$.

■ Problem 7.2

Problem

You want to determine the low frequency and high frequency components of a signal $x[n]$. Design an efficient filter bank where the prototype filter has at least 50dB attenuation in the stopband and a transition region of $\Delta\omega = 0.1\,\pi$. Use the window method.

Solution

The prototype filter is a low pass filter with bandwidth $\pi/2$, with impulse response

$$h_0[n] = \frac{\sin\left(\frac{\pi}{2}\,n\right)}{\pi n}\,w[n]$$

with $w[n]$ the non causal window sequence. Since we need 50dB attenuation in the stopband we use a Blackman window, which has a transition region $\Delta\omega = 12\,\pi / N$. Therefore we determine the filter length N from the transition band as

$$\frac{12\,\pi}{N} \leq 0.1\,\pi$$

This yields $N = 121$. The Low Pass Filter $H_0(z)$ has the polyphase decomposition

$$H_0(z) = E_0(z^2) + z^{-1}\,E_1(z^2)$$

where

$$E_0(z) = \sum_n h_0[2\,n]\,z^{-n} = w[0] = 1$$

since $h_0[2\,n] = 0$ for $n \neq 0$. Similarly

$$E_1[n] = \sum_n h_0[2\,n+1]\,z^{-n} = \sum_n \frac{\sin\left(\pi n + \frac{\pi}{2}\right)}{\pi\,(2\,n+1)}\,w[2\,n+1]\,z^{-n}$$

■ Problem 7.3

Solution

The prototype filter has frequency response $H(\omega)$ with bandwith $\frac{\pi}{M} = \frac{\pi}{8}$. Therefore the non causal impulse response of the prototype filter is given by

$$h[n] = \frac{\sin\left(\frac{\pi}{8}\,n\right)}{\pi n}\,w[n]$$

with $w[n]$ being a window of length $N+1 = 21$. For example let $w[n]$ be a hamming window, which has the expression

$$w\left[n - \frac{N}{2}\right] = 0.54 - 0.46\cos\left(\frac{2\,\pi}{N}\,n\right) \quad \text{for } 0 \leq n \leq N$$

where we listed the causal expression generally found in most tables. From this expression it is easy to see that

$$w[n] = 0.54 + 0.46\cos\left(\frac{2\pi}{20}n\right) \quad \text{for } -10 \le n \le 10$$

Finally the expression of the impulse response $h[n]$ becomes

$$h[n] = \left(\frac{\sin\left(\frac{\pi}{8}n\right)}{\pi n}\right)\left(0.54 + 0.46\cos\left(\frac{\pi}{10}n\right)\right) \quad \text{for } -10 \le n \le 10$$

and zero otherwise.

The transfer function of the prototype filter is then given by

$$\begin{aligned} H(z) = {} & -0.0018\, z^{10} - 0.0014\, z^{9} + 0.0000\, z^{8} + \\ & 0.0047\, z^{7} + 0.0149\, z^{6} + 0.0318\, z^{5} + 0.0543\, z^{4} + \\ & 0.0794\, z^{3} + 0.1027\, z^{2} + 0.1191\, z + 0.1250 + \\ & 0.1191\, z^{-1} + 0.1027\, z^{-2} + 0.0794\, z^{-3} + \\ & 0.0543\, z^{-4} + 0.0318\, z^{-5} + 0.0149\, z^{-6} + 0.0047\, z^{-7} + \\ & 0.0000\, z^{-8} - 0.0014\, z^{-9} - 0.0018\, z^{-10} \end{aligned}$$

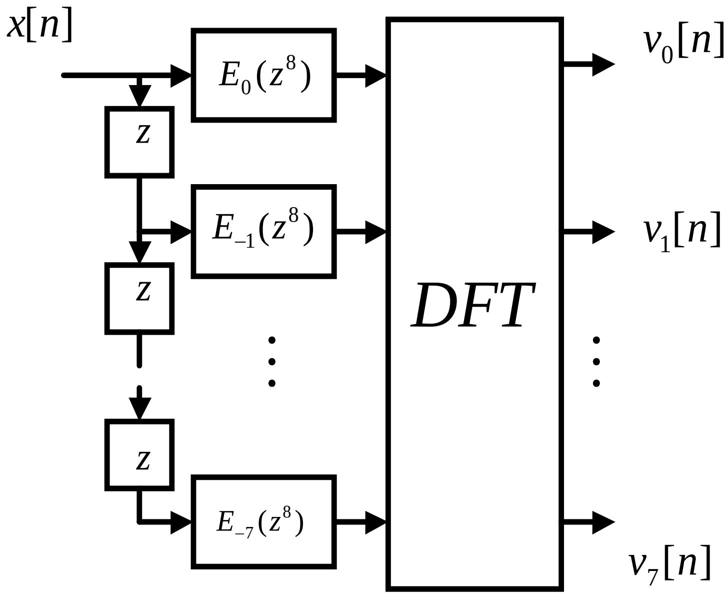
The eight polyphase components of the prototype filter then become as follows:

$$E_{-k}(z^8) = \sum_{n=-\infty}^{+\infty} h[8n-k]\, z^{-8n}, \quad \text{for } k = 0, 1, \ldots, 7$$

which yields

$$\begin{aligned} E_0(z^8) &= h[-8]\, z^8 + h[0]\, z^0 + h[8]\, z^{-8} = 0.1250 \\ E_{-1}(z^8) &= h[-9]\, z^8 + h[-1]\, z^0 + h[7]\, z^{-8} = -0.0014\, z^8 + 0.1191 + 0.0047\, z^{-8} \\ E_{-2}(z^8) &= h[-10]\, z^8 + h[-2]\, z^0 + h[6]\, z^{-8} = -0.0018\, z^8 + 0.1027 + 0.0149\, z^{-8} \\ E_{-3}(z^8) &= h[-3]\, z^0 + h[5]\, z^{-8} = 0.0794 + 0.0318\, z^{-8} \\ E_{-4}(z^8) &= h[-4]\, z^0 + h[4]\, z^{-8} = 0.0543 + 0.0543\, z^{-8} \\ E_{-5}(z^8) &= h[-5]\, z^0 + h[3]\, z^{-8} = 0.0318 + 0.0794\, z^{-8} \\ E_{-6}(z^8) &= h[-6]\, z^0 + h[2]\, z^{-8} + h[10]\, z^{-16} = 0.0149 + 0.1027\, z^{-8} - 0.0018\, z^{-16} \\ E_{-7}(z^8) &= h[-7]\, z^0 + h[1]\, z^{-8} + h[9]\, z^{-16} = 0.0047 + 0.1191\, z^{-8} - 0.0014\, z^{-16} \end{aligned}$$

The implementation is shown below.

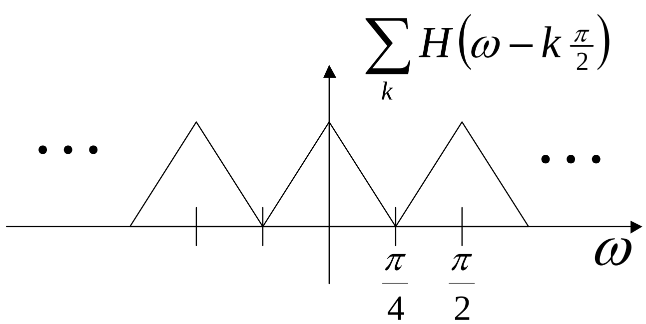


■ Problem 7.4

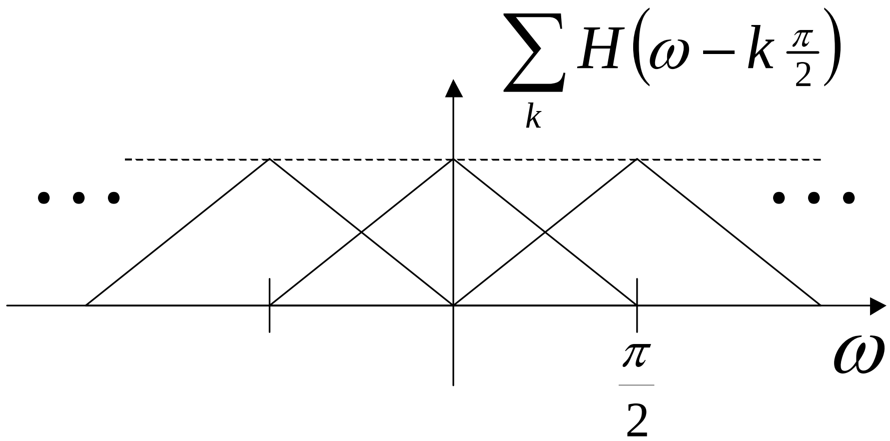
Solution

a) , b), c) $H(z)$ is M – Band, since $h[4n] = 0$ for $n \neq 0$;

d) $H(z)$ is not M – Band since $\sum_k H(\omega - k\frac{\pi}{2})$ is not a constant for all ω, as shown below.



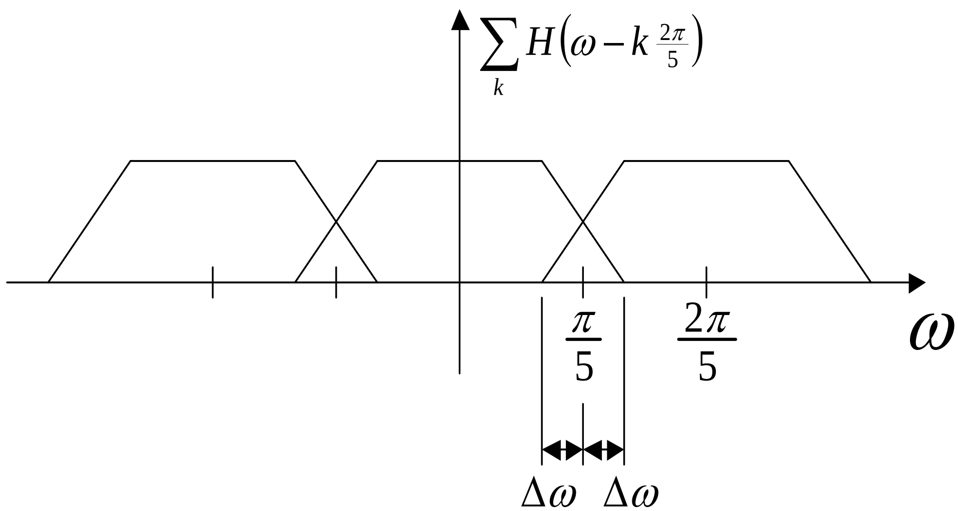
e) $H(z)$ is M – Band since $\sum_k H(\omega - k\frac{\pi}{2})$ is a constant as shown below.



■ Problem 7.5

Solution

Let $\omega_1 = \frac{\pi}{5} - \Delta\omega$ and $\omega_2 = \frac{\pi}{5} + \Delta\omega$ for any $0 \le \Delta\omega \le \frac{\pi}{5}$. Then $H(z)$ is an M –Band filter with $A = \frac{1}{5}$ as shown below.



■ Problem 7.6

Solution

With $M = 16$ the prototype filters for both Analysis and Synthesis have a bandwidth $\omega_c = \pi / 16$. From what we have seen about maximally decimated DFT Filter banks, if we want to use FIR filters, the only possibility for perfect reconstruction is that both filters $h[n]$ and $g[n]$ in the analysis and synthesis network have length $M = 16$. In this way we would have

$$H(z) = h[0] + h[-1]\,z + \ldots + h[-15]\,z^{15}$$
$$G(z) = g[0] + g[1]\,z^{-1} + \ldots + g[15]\,z^{-15}$$

with the Perfect Reconstruction condition

$$h[n]\,g[n] = \frac{1}{16}$$

What makes this problem a bit different from the standard FIR window based design problem is the fact the filter order is odd, ie the total filter length is 16, which is even. In Chapter 4 we have considered only the case where the total filter length is odd as $N = 2\,L + 1$. Although most of the time this is not a major restriction, in this case we have to design a filter with the precise length, and none of the filter coefficients can be zero. In other words we cant use (say) a filter with length 14, and assume $h[-15] = 0$, since this would require $g[15] = \infty$, clearly not feasable.

In order to design a filter with even length, we can call $H_d(\omega)$ the frequency response of an ideal Low Pass Filter with bandwidth ω_c, and compute

$$h_d[n] = \text{IDTFT}\{H(\omega)\,e^{-j\,\frac{\omega}{2}}\} = \frac{1}{2\,\pi}\int_{-\omega_c}^{+\omega_c} e^{-j\,\frac{\omega}{2}}\,e^{\mathrm{j}\omega \mathrm{n}}\,\mathrm{d}\omega$$

This leads to the impulse response

$$h_d[n] = \frac{\sin(\omega_c(n-\frac{1}{2}))}{\pi(n-\frac{1}{2})}$$

Now the goal is to find a linear phase approximation with a finite number of coefficients. In particular let

$$\hat{H}_L(\omega) = \sum_{n=-L+1}^{L} h_d[n]\,e^{-\mathrm{j}\omega \mathrm{n}}$$

which can be written as

$$\hat{H}_L(\omega) = \sum_{n=0}^{L-1} h_d[-n]\,e^{\mathrm{j}\omega \mathrm{n}} + \sum_{n=1}^{L} h_d[n]\,e^{-\mathrm{j}\omega \mathrm{n}}$$

It is easy to see that $h_d[-n] = h_d[n+1]$ and therefore we can write

$$\hat{H}_L(\omega) = e^{-\mathrm{j}\omega}\sum_{n=1}^{L} h_d[n]\,e^{\mathrm{j}\omega \mathrm{n}} + \sum_{n=1}^{L} h_d[n]\,e^{-\mathrm{j}\omega \mathrm{n}}$$

This shows that $e^{j\,\frac{\omega}{2}}\,\hat{H}_L(\omega)$ is real, since

$$e^{j\,\frac{\omega}{2}}\,\hat{H}_L(\omega) = e^{-j\,\frac{\omega}{2}}\sum_{n=1}^{L} h_d[n]\,e^{\mathrm{j}\omega \mathrm{n}} + e^{-j\,\frac{\omega}{2}}\sum_{n=1}^{L} h_d[n]\,e^{-\mathrm{j}\omega \mathrm{n}}$$

and therefore $\hat{H}_L(\omega)$ has linear phase. As a consequence a causal translation has linear phase too, which leads to the linear phase FIR filter with frequency response

$$\hat{H}_L(\omega)\,e^{-\mathrm{j}\omega(L-1)} = \sum_{n=0}^{2\,L-1} h_d[n-L+1]\,e^{-\mathrm{j}\omega \mathrm{n}}$$

In our case, the bandwidth is $\omega_c = \frac{\pi}{16}$, the filter order is $15 = 2L - 1$, which yields $L = 8$, and the FIR filter bacomes

$$h_d[n] = \frac{\sin(\frac{\pi}{16}(n-7-\frac{1}{2}))}{\pi(n-7-\frac{1}{2})}, \text{ for } n = 0, \ldots, 15$$

without including the window. Finally the filters $h[n]$ and $g[n]$ of the analysis and synthesis networks become

$$h[-n] = h_d[n]\,w[n] = \frac{\sin(\frac{\pi}{16}(n-7-\frac{1}{2}))}{\pi(n-7-\frac{1}{2})}\left(0.54 - 0.46\cos\left(\frac{2\pi}{15}n\right)\right) \quad \text{for } 0 \le n \le N$$

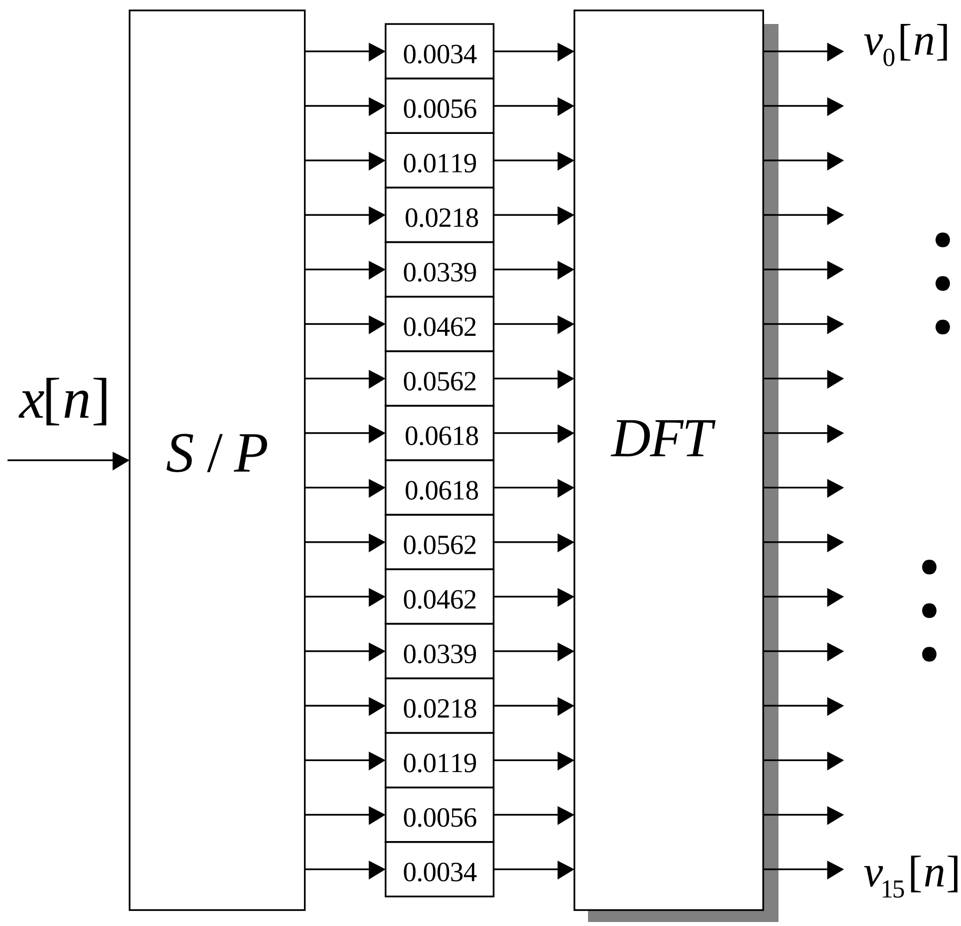
and

$$g[n] = \frac{16}{h[-n]}, \quad \text{for } 0 \le n \le N$$

In terms of the polyphase decomposition, every term is a constant, as

$$E_{-k}(z) = h[-k]$$
$$F_k(z) = g[k]$$

for $k = 0, \ldots, 15$. It is just a matter of computing the coefficients to determine the final result shown in the figure below for the analysis network.

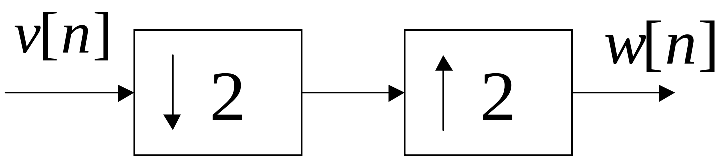


■ Problem 7.7

Solution

First we can verify that, in the the system below

$$w[n] = \begin{cases} v[n] & \text{if } n \text{ even} \\ 0 & \text{if } n \text{ odd} \end{cases}$$



which yields $w[n] = v[n]\,\delta_2[n] = \frac{1}{2}\,(v[n] + (-1)^n\,v[n])$. Therefore, as you recall,

$$W(\omega) = \frac{1}{2}\,V(\omega) + \frac{1}{2}\,V(\omega - \pi)$$

and therefore

$$W(z) = \frac{1}{2}\,V(z) + \frac{1}{2}\,V(-z)$$

Applying this result it is easy to see that

$$\begin{aligned} Y(z) &= G(z)\,(\tfrac{1}{2}\,H(z)\,X(z) + \tfrac{1}{2}\,H(-z)\,X(-z)) \\ &= \tfrac{1}{2}\,G(z)\,H(z)\,X(z) + \tfrac{1}{2}\,G(z)\,H(-z)\,X(-z) \end{aligned}$$

■ Problem 7.8

Solution

In this case we have a filter bank with two filters. Therefore $M = 2$ and forperfect reconstruction the filters have to be

$$\begin{aligned} H(z) &= h[0] + h[-1]\,z \\ G(z) &= g[0] + g[1]\,z^{-1} \end{aligned}$$

with the condition

$$\begin{aligned} h[0]\,g[0] &= \tfrac{1}{2} \\ h[-1]\,g[1] &= \tfrac{1}{2} \end{aligned}$$

Then Let us see how to relate $X(z) = Z\{x[n]\}$ with $Y(z) = Z\{y[n]\}$. Applying the result from the previous problem we have

$$Y(z) = G(z)\,(\tfrac{1}{2}\,H(z)\,X(z) + \tfrac{1}{2}\,H(-z)\,X(-z)) + \\ + G(-z)\,(\tfrac{1}{2}\,H(-z)\,X(z) + \tfrac{1}{2}\,H(z)\,X(-z))$$

which becomes

$$Y(z) = \tfrac{1}{2}\,(G(z)\,H(z) + G(-z)\,H(-z))\,X(z) + \tfrac{1}{2}\,(G(z)\,H(-z) + G(-z)\,H(z))\,X(-z)$$

Now let's see the two transfer functions $X(z) \to Y(z)$ and $X(-z) \to Y(z)$ with the perfect reconstruction conditions above:

$$\tfrac{1}{2}\,(G(z)\,H(z) + G(-z)\,H(-z)) = \\ \tfrac{1}{2}\,((g[0] + g[1]\,z^{-1})\,(h[0] + h[-1]\,z) + ((g[0] - g[1]\,z^{-1})\,(h[0] - h[-1]\,z)) = \\ = \\ \tfrac{1}{2}\,(2\,(g[0]\,h[0] + g[1]\,h[-1]) + (g[0]\,h[-1] - g[0]\,h[-1])\,z + (g[1]\,h[0] - g[1]\,h[0])\,z^{-1}) \\ = 1 \quad \text{for all } z$$

$$\tfrac{1}{2}\,(G(z)\,H(-z) + G(-z)\,H(z)) = \\ \tfrac{1}{2}\,((g[0] + g[1]\,z^{-1})\,(h[0] - h[-1]\,z) + ((g[0] - g[1]\,z^{-1})\,(h[0] + h[-1]\,z)) = \\ = \\ \tfrac{1}{2}\,(2\,(g[0]\,h[0] - g[1]\,h[-1]) + (-g[0]\,h[-1] + g[0]\,h[-1])\,z + (g[1]\,h[0] - g[1]\,h[0])\,z^{-1}) \\ = 0 \quad \text{for all } z$$

Therefore. as expected,

$$Y(z) = X(z)$$

and the filter bank perfectly reconstructs the input signal.

■ Problem 7.9

Solution

a) From the Problem 7.8, we can write $Y(z)$ in terms of the input signal $X(z)$ and its alias $X(-z)$. The aliasing comes from the downsampling operation.

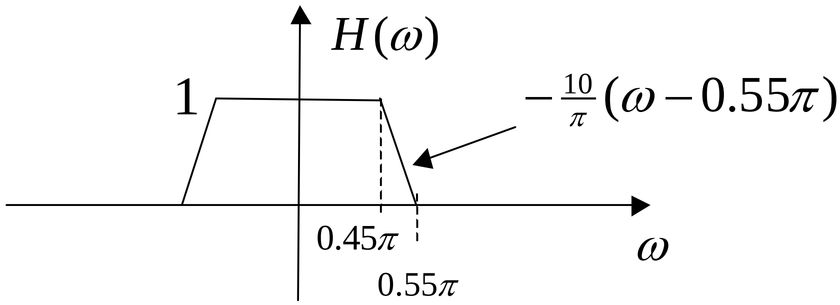
In terms of the DTFT we can write

$$Y(\omega) = A(\omega)\,X(\omega) + B(\omega)\,X(\omega - \pi)$$

where

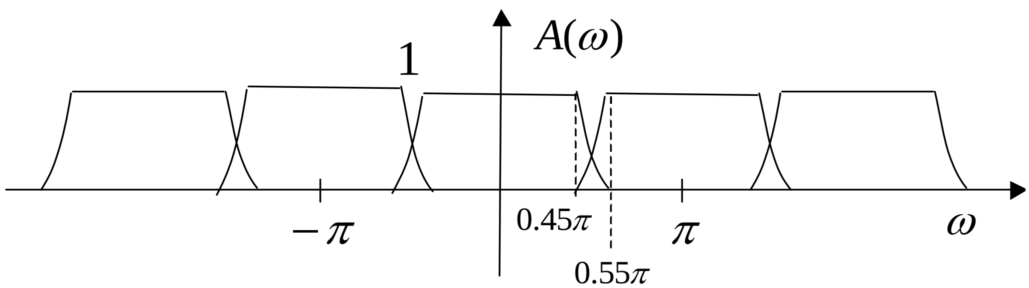
$$A(\omega) = \tfrac{1}{2}\,G(\omega)\,H(\omega) + \tfrac{1}{2}\,G(\omega - \pi)\,H(\omega - \pi) \\ B(\omega) = \tfrac{1}{2}\,G(\omega)\,H(\omega - \pi) + \tfrac{1}{2}\,G(\omega - \pi)\,H(\omega)$$

In our case the two prototype filters have frequency response $H(\omega) = G(\omega)$ as shown below.



Therefore:

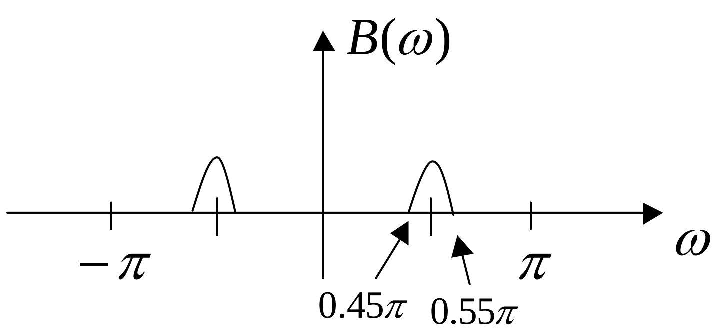
$$A(\omega) = \begin{cases} \left(\frac{10}{\pi}\right)^2 \left|\omega - 0.55\,\pi\right|^2 + \left(\frac{10}{\pi}\right)^2 \left|\omega - 0.45\,\pi\right|^2 & \text{if } 0.45\,\pi < |\,\omega\,| < 0.55\,\pi \\ 1 & \text{otherwise} \end{cases}$$



Analogously:

$$B(\omega) = \begin{cases} -\left|\frac{10}{\pi}\right|^2 (|\,\omega\,| - 0.55\,\pi)(|\,\omega\,| - 0.45\,\pi) & \text{if } 0.45\,\pi < |\,\omega\,| < 0.55\,\pi \\ 0 & \text{otherwise} \end{cases}$$

and the maximum value is at $\omega = \pm\frac{\pi}{2}$ where the maxixmum is $B(\pm\frac{\pi}{2}) = 0.25$, as shown below.



b) For the given signal

$$X(\omega) = 20\,\pi\delta(\omega) + 2\,\pi\delta(\omega - 0.2\,\pi) + 2\,\pi\delta(\omega + 0.2\,\pi) - 3\,\pi\delta(\omega - 0.7\,\pi) - 3\,\pi\delta(\omega + 0.7\,\pi), \text{ for } -\pi \le \omega < \pi$$

Therefore the reconstructed signal becomes

$$Y(\omega) = A(\omega)\,X(\omega) + B(\omega)\,X(\omega - \pi)$$

with $A(\omega)$, $B(\omega)$ as above, and

$$X(\omega - \pi) = 20\,\pi\delta(\omega - \pi) + 2\,\pi\delta(\omega + 0.8\,\pi) + 2\,\pi\delta(\omega - 0.8\,\pi) - 3\,\pi\delta(\omega + 0.3\,\pi) - 3\,\pi\delta(\omega - 0.3\,\pi)$$

From the plot of $A(\omega)$ and $B(\omega)$ we can verify that

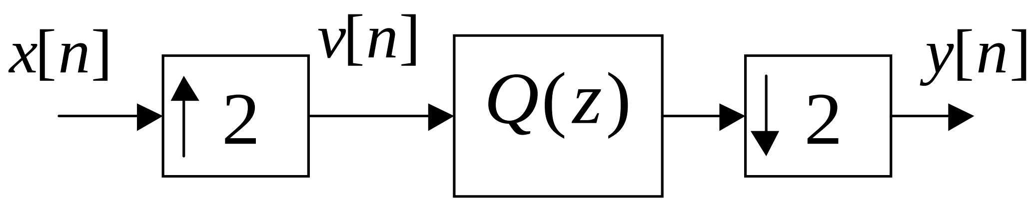
$$A(0) = A(\pm 0.2\,\pi) = A(\pm 0.7\,\pi) = 1$$
$$B(\pm\pi) = B(\pm 0.8\,\pi) = A(\pm 0.3\,\pi) = 0$$

and therefore, for the given signal, $y[n] = x[n]$.

■ Problem 7.10

Solution

First let's see what is the transfer function (or the frequency response) of the system shown below.



Applying standard considerations we can see that

$$Y(\omega) = \frac{1}{2}\,Q(\tfrac{\omega}{2})\,V(\tfrac{\omega}{2}) + \frac{1}{2}\,Q(\tfrac{\omega}{2} - \pi)\,V(\tfrac{\omega}{2} - \pi)$$

Also, from the upsampler, $V(\omega) = X(2\,\omega)$, which implies

$$V(\tfrac{\omega}{2}) = X(\omega)$$
$$V(\tfrac{\omega}{2} - \pi) = X(2\,(\tfrac{\omega}{2} - \pi)) = X(\omega - 2\,\pi) = X(\omega)$$

using the periodicity of the DTFT. Therefore, subsituting into the expression for $Y(\omega)$ we obtain

$$Y(\omega) = \frac{1}{2}\,(Q(\tfrac{\omega}{2}) + Q(\tfrac{\omega}{2} - \pi))\,X(\omega)$$
$$= Q_0(\omega)\,X(\omega)$$

Therefore the impulse response $q_0[n] = \text{IDTFT}\{Q_0(\omega)\}$ is the impulse response $q[n]$ downsampled by two, ie

$$q_0[n] = q[2\,n]$$

In other words from the polyphase decomposition

$$Q(z) = Q_0(z^2) + z^{-1}\,Q_1(z^2)$$

where

$$Q_k(z) = Z\,\{q[2\,n + k]\}$$

we can determine the transfer function $Q_0(z)$.

a) We want to determine the four transfer functions $Y_i(z)\,/\,X_j(z)$ for $i,\ j = 0,\ 1$. See each one separately:

$\frac{Y_0(z)}{X_1(z)} = 0:$ since, in this case

$$Q(z) = G_1(z)\,H_0(z) = \frac{1}{4}\,(1 - z^{-1} + z^{-2} - z^{-3})\,\frac{1}{4}\,(1 + z^1 + z^2 + z^3)$$
$$= \frac{1}{16}\,(-z^{-3} - z^{-1} + z + z^3)$$

and therefore $Q_0(z) = 0$ since there are no even powers of z in $Q(z)$.

$\frac{Y_1(z)}{X_0(z)} = 0:$ since, in this case

$$Q(z) = G_0(z)\,H_1(z) = \frac{1}{4}\,(1 + z^{-1} + z^{-2} + z^{-3})\,\frac{1}{4}\,(1 - z^1 + z^2 - z^3)$$
$$= \frac{1}{16}\,(z^{-3} + z^{-1} - z - z^3)$$

and therefore $Q_0(z) = 0$ since there are no even powers of z in $Q(z)$

$\frac{Y_0(z)}{X_0(z)} = \frac{1}{16}\,(2\,z^{-1} + 4 + 2\,z):$ since, in this case

$$Q(z) = G_0(z)\,H_0(z) = \frac{1}{4}\,(1 + z^{-1} + z^{-2} + z^{-3})\,\frac{1}{4}\,(1 + z^1 + z^2 + z^3)$$
$$= \frac{1}{16}\,(z^{-3} + 2\,z^{-2} + 3\,z^{-1} + 4 + 3\,z + 2\,z^2 + z^3)$$

and the polyphase decomposition

$$Q(z) = \frac{1}{16}\,(2\,z^{-2} + 4 + 2\,z^2) + z^{-1}\,\frac{1}{16}\,(z^{-2} + 3 + 3\,z^2 + z^4)$$

which yields

$$Q_0(z) = \frac{1}{16}\,(2\,z^{-1} + 4 + 2\,z).$$

$\frac{Y_1(z)}{X_1(z)} = \frac{1}{16}\,(2\,z^{-1} + 4 + 2\,z):$ since, in this case

$$Q(z) = G_1(z)\,H_1(z) = \frac{1}{4}\,(1 - z^{-1} + z^{-2} - z^{-3})\,\frac{1}{4}\,(1 - z^1 + z^2 - z^3)$$
$$= \frac{1}{16}\,(-z^{-3} + 2\,z^{-2} - 3\,z^{-1} + 4 - 3\,z + 2\,z^2 - z^3)$$

and the polyphase decomposition

$$Q(z) = \frac{1}{16}\,(2\,z^{-2} + 4 + 2\,z^2) + z^{-1}\,\frac{1}{16}\,(-z^{-2} - 3 - 3\,z^2 - z^4)$$

which yields

$$Q_0(z) = \frac{1}{16}\,(2\,z^{-1} + 4 + 2\,z).$$

b)