Chapter 6: Problem Solutions

Multirate Digital Signal Processing: Fundamentals

Sampling, Upsampling and Downsampling

■ Problem 6.1

Solution

From the definiton of downsampling,

$$y[n] = x[2n]$$

a) $y[n] = \delta[2n] = \delta[n]$

b) $y[n] = \delta[2n-1] = 0$

c) $y[n] = (-1)^{2n} u[2n] = u[n]$

d) $y[n] = e^{j0.2\pi n}$

e) $y[n] = e^{j0.2\pi n} u[2n] = y[n] = e^{j0.2\pi n} u[n]$

f) $y[n] = 2\cos(0.4\pi n)$

g) $y[n] = 2\cos(0.5\pi 2n) = 2\cos(\pi n) = (-1)^n$

h) $y[n] = 2\sin(\pi n) = 0$

i) $y[n] = \cos(2\pi n) = 1$

j) $y[n] = 2\sin(2\pi n) = 0$

PROBLEM 6.2

Note Title 11/30/2015

a) $X(w) = 1$. Then $Y(w) = \frac{1}{2}\left(X\left(\frac{w}{2}\right) + X\left(\frac{w}{2} - \pi\right)\right) = \frac{1}{2}(2) = 1$

Then $y[n] = \delta[n]$.

b) $X(w) = e^{-jw}$. Then $Y(w) = \frac{1}{2}\left(e^{-j\frac{w}{2}} + e^{-j\left(\frac{w}{2} - \pi\right)}\right) =$

$= \frac{1}{2}\left(e^{-j\frac{w}{2}} - e^{-j\frac{w}{2}}\right) = 0 \Rightarrow y[n] = 0$

c) $x[n] = (-1)^n u[n]$. Then $X(w) = \dfrac{e^{jw}}{e^{jw} + 1}$ and

$Y(w) = \frac{1}{2} \dfrac{e^{j\frac{w}{2}}}{e^{j\frac{w}{2}} + 1} + \frac{1}{2} \dfrac{e^{j\left(\frac{w}{2} - \pi\right)}}{e^{j\left(\frac{w}{2} - \pi\right)} + 1} = \frac{1}{2} \dfrac{e^{j\frac{w}{2}}}{e^{j\frac{w}{2}} + 1} + \frac{1}{2} \dfrac{-e^{j\frac{w}{2}}}{-e^{j\frac{w}{2}} + 1} = \ldots$

$$= \frac{1}{2} \frac{e^{j\frac{w}{2}}\left(e^{j\frac{w}{2}} \cancel{-1} + e^{j\frac{w}{2}} \cancel{+1}\right)}{\left(e^{j\frac{w}{2}}+1\right)\left(e^{j\frac{w}{2}}-1\right)} = \frac{e^{jw}}{e^{jw}-1} = \text{DTFT}\{x[n]\}.$$

d) $X(w) = 2\pi\,\delta(w - 0.1\pi)$, $\quad -\pi < w \leq \pi$

No aliasing, since there are no frequencies above $w = \frac{\pi}{2}$. Then

$$Y(w) = \frac{1}{2} X\left(\frac{w}{2}\right) = \frac{1}{2} 2\pi\,\delta\left(\frac{w}{2} - 0.1\pi\right) = 2\pi\,\delta(w - 0.2\pi)$$

$$-\pi < w \leq \pi \qquad //$$

e) $X(w) = \dfrac{e^{jw}}{e^{jw} - e^{j0.1\pi}}$. Then

$$Y(w) = \frac{1}{2}\,\frac{e^{j\frac{w}{2}}}{e^{j\frac{w}{2}} - e^{j0.1\pi}} + \frac{1}{2}\,\frac{e^{j\left(\frac{w}{2}-\pi\right)}}{e^{j\left(\frac{w}{2}-\pi\right)} - e^{j0.1\pi}} =$$

$$= \frac{1}{2}\,\frac{e^{j\frac{w}{2}}}{e^{j\frac{w}{2}} - e^{j0.1\pi}} + \frac{1}{2}\,\frac{-e^{j\frac{w}{2}}}{-e^{j\frac{w}{2}} - e^{j0.1\pi}} = \frac{e^{jw}}{e^{jw} - e^{j0.2\pi}}$$

which is the IDTFT of $y[n] = e^{j0.2\pi n}\,u[n]$ //

f) $X(w) = 2\pi\delta(w-0.2\pi) + 2\pi\delta(w+0.2\pi), \quad -\pi < w \le \pi$

Since no aliasing,

$$Y(w) = \frac{1}{2}X\left(\frac{w}{2}\right) = \frac{1}{2}2\pi\,\delta\left(\frac{w}{2}-0.2\pi\right) + \frac{1}{2}2\pi\,\delta\left(\frac{w}{2}+0.2\pi\right) =$$

$$= 2\pi\delta(w-0.4\pi) + 2\pi\delta(w+0.4\pi).$$

g) $X(w) = 2\pi\delta\left(w-\frac{\pi}{2}\right) + 2\pi\delta\left(w+\frac{\pi}{2}\right) \quad -\pi < w \le \pi$

Then

$$X(w-\pi) = 2\pi\underbrace{\delta\left(w-\frac{\pi}{2}+\pi\right)}_{\delta\left(w-\frac{\pi}{2}-\pi+2\pi\right) \text{ since periodic}} + 2\pi\delta\left(w+\frac{\pi}{2}-\pi\right), \quad -\pi < w \le \pi$$

This implies $Y(w) = \frac{1}{2}\, 2\pi \left(2\pi\, \delta\left(\frac{w}{2} - \frac{\pi}{2}\right) + \delta\left(\frac{w}{2} + \frac{\pi}{2}\right)\right) =$

$$= 4\pi\, \delta(w-\pi) + 4\pi\, \delta(w+\pi), \quad -\pi \leq w \leq \pi$$

and $$Y(w) = 4\pi\, \delta(w-\pi), \qquad -\pi < w \leq \pi$$

since $\delta(w+\pi) = \delta(w-\pi+2\pi)$ is the repetition of $\delta(w-\pi)$. Then $y[n] = 2(-1)^n$ //

h) $X(w) = \frac{1}{2j}\left(\delta\left(w-\frac{\pi}{2}\right) - \delta\left(w+\frac{\pi}{2}\right)\right), \quad -\pi < w \leq \pi$

$$X(w-\pi) = \frac{1}{2j}\left(\delta\left(w-\frac{\pi}{2}+\pi\right) - \delta\left(w+\frac{\pi}{2}-\pi\right)\right) =$$

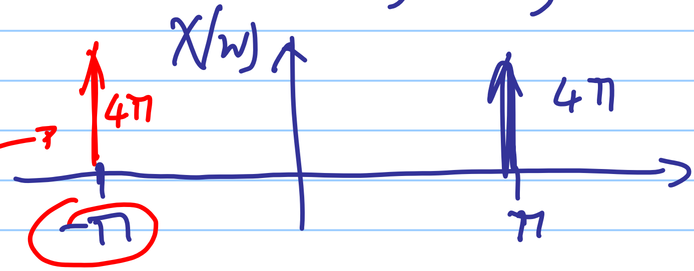
$$= \frac{1}{2j}\left(\delta\left(w+\frac{\pi}{2}\right) - \delta\left(w-\frac{\pi}{2}\right)\right) = -X(w)$$

Then $Y(w) = \frac{1}{2}\left(X\left(\frac{w}{2}\right) + X\left(\frac{w}{2}-\pi\right)\right) =$

$$= \frac{1}{2}\left(X\left(\frac{w}{2}\right) - X\left(\frac{w}{2}\right)\right) = 0 \Rightarrow y[n] = 0.$$

i) Recall that $2\cos(\pi n) = 2(-1)^n = 2e^{-j\pi n}$.

Then $X(w) = 4\pi\delta(w-\pi)$, $\quad -\pi < w \leq \pi$



this is the periodic repetition

not in the interval $-\pi < w \leq \pi$

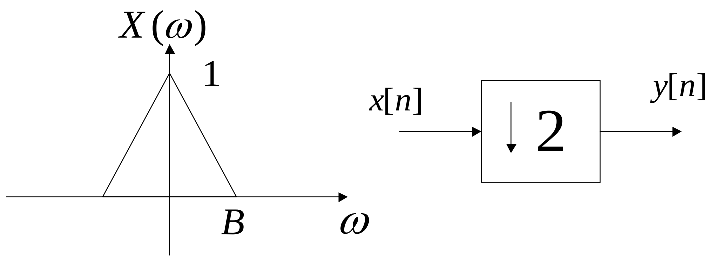
Therefore: $\quad X(w-\pi) = 4\pi\delta(w-\pi+\pi) = 4\pi\delta(w)$

and $Y(w) = \frac{1}{2}\left(4\pi\delta\left(\frac{w}{2}\right) + 4\pi\delta\left(\frac{w}{2}-\pi\right)\right) = 4\pi\delta(w), \quad -\pi < w \leq \pi$

ij) $x[n] = 2\sin(\pi n) = 0$ for all n. Then $y[n] = 0$ for all n //.

■ Problem 6.3

Solution



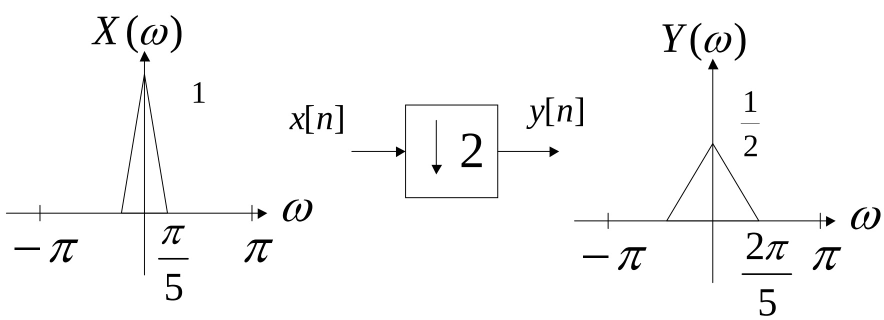
In all cases

$$Y(\omega) = \frac{1}{2} X\left(\frac{\omega}{2}\right) + \frac{1}{2} X\left(\frac{\omega}{2} - \pi\right) \quad \text{for all } \omega$$

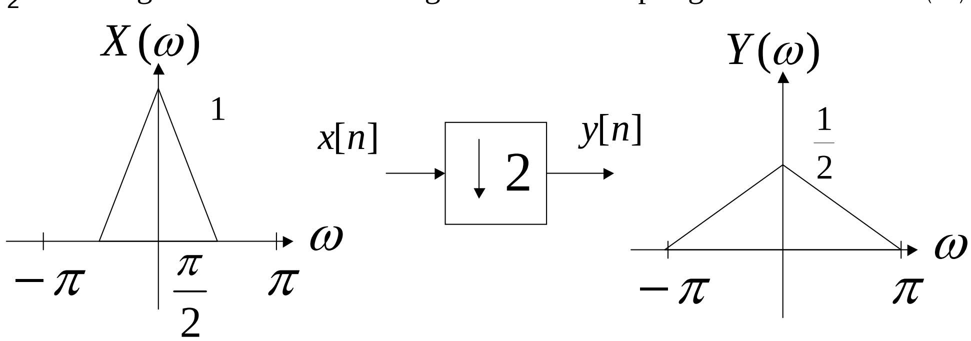
Whene there is no aliasing, ie $X(\omega) = 0$ for $\frac{\pi}{2} \leq |\omega| \leq \pi$ then this relation simpilfies to

$$Y(\omega) = \frac{1}{2} X\left(\frac{\omega}{2}\right)$$

a) $B = \frac{\pi}{5} < \frac{\pi}{2}$. Then there is no aliasing after downsampling and therefore $Y(\omega)$ is as shown below



b) $B = \frac{\pi}{2}$. Then again there is no aliasing after downsampling and therefore $Y(\omega)$ is as shown below

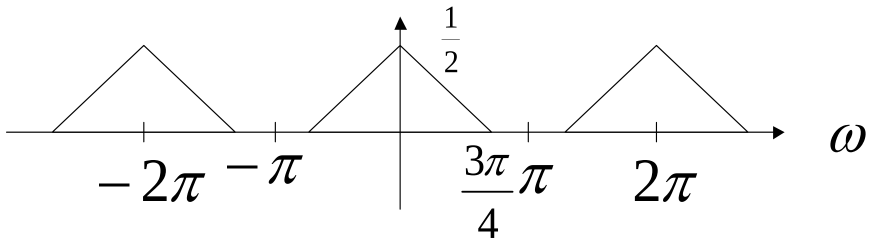


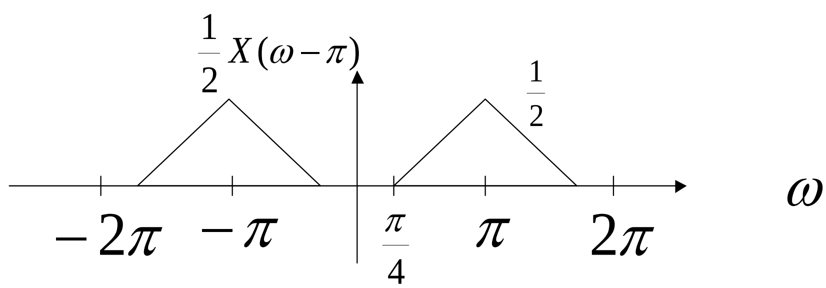
c) $B = \frac{3\pi}{4}$. In this cases there is aliasing and we have to account for it. Best way to do it is proceed in two steps: sampling and then downsampling.

The sampling operation yields

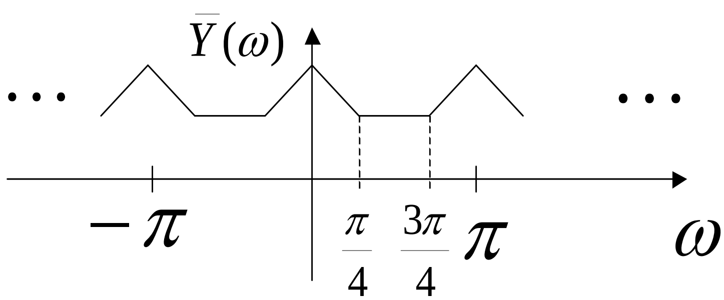
$$\bar{Y}(\omega) = \frac{1}{2} X(\omega) + \frac{1}{2} X(\omega - \pi)$$

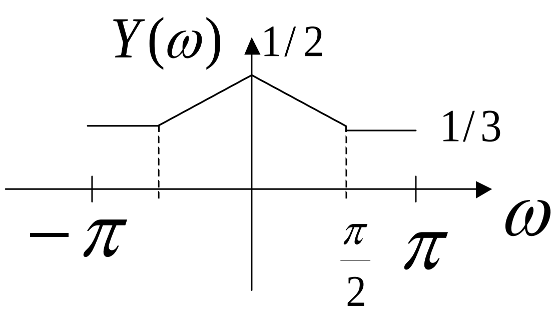
The figure below shows both $\frac{1}{2} X(\omega)$ and $\frac{1}{2} X(\omega - \pi)$.





Then downsampling yields $Y(\omega) = \bar{Y}\left(\frac{\omega}{2}\right)$, just a rescaling of the frequency axis. The final results is shown below.





d) $B = \pi$. Same reasoning as in c). This time it is easy to see that

$$\tilde{Y}(\omega) = \frac{1}{2} X(\omega) + \frac{1}{2} X(\omega - \pi) = \frac{1}{2} \text{ for all } \omega$$

and therefore $Y(\omega) = \tilde{Y}\left(\frac{\omega}{2}\right) = \frac{1}{2}$ for all ω.

■ Problem 6.4

Solution

Recall that $y[n] = x\left[\frac{n}{2}\right] \delta_2[n] = \frac{1}{2} x\left[\frac{n}{2}\right] (1 + (-1)^n) = \frac{1}{2} x\left[\frac{n}{2}\right] (1 + e^{-j\pi n})$

a) $y[n] = \delta[n]$;

b) $y[n] = \frac{1}{2} ((-1)^n + 1) u[n]$

c) $y[n] = \frac{1}{2} e^{j0.05\pi n} + \frac{1}{2} e^{j(0.05\pi - \pi)n}$

d) $y[n] = \left(\frac{1}{2} e^{j0.05\pi n} + \frac{1}{2} e^{j(0.05\pi - \pi)n}\right) u[n]$

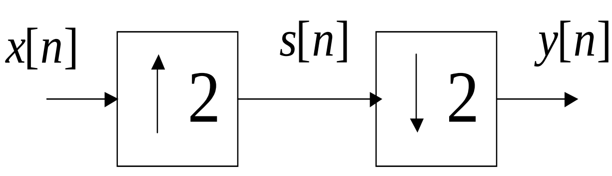
e)
$y[n] = \left(\frac{1}{4} e^{j0.05\pi n} + \frac{1}{4} e^{j(0.05\pi - \pi)n}\right) u[n] + \left(\frac{1}{4} e^{-j0.05\pi n} + \frac{1}{4} e^{-j(0.05\pi - \pi)n}\right) u[n]$
which becomes

$$y[n] = \frac{1}{2} \cos(0.05\pi n) u[n] + \frac{1}{2} \cos(0.95\pi n) u[n]$$

f) similarly $y[n] = \frac{1}{2} \cos(0.05\pi n) + \frac{1}{2} \cos(0.95\pi n)$

■ Problem 6.5

Solution



Using the DTFT. For upsampling

$$S(\omega) = X(2\omega)$$

and downsampling

$$Y(\omega) = \frac{1}{2} S\left(\frac{\omega}{2}\right) + \frac{1}{2} S\left(\frac{\omega}{2} - \pi\right)$$

Substitute for $S(\omega)$ to obtain

$$S\left(\frac{\omega}{2}\right) = X\left(2\,\frac{\omega}{2}\right) = X(\omega)$$
$$S\left(\frac{\omega}{2} - \pi\right) = X\left(2\left(\frac{\omega}{2} - \pi\right)\right) = X(\omega - 2\pi)$$

Therefore

$$Y(\omega) = \frac{1}{2}X(\omega) + \frac{1}{2}X(\omega - 2\pi) = X(\omega)$$

from the periodicity of the DTFT.

■ Problem 6.6

Solution

From the diagram it is easy to verify that

$$y[n] = \begin{cases} x[n] & \text{if } n \text{ even} \\ 0 & \text{if } n \text{ odd} \end{cases}$$

Therefore $y[n] = x[n]\,\delta_2[n]$, and $Y(z) = \frac{1}{2}(X(z) + X(-z))$, or equivalently, $Y(\omega) = \frac{1}{2}(X(\omega) + X(\omega - \pi))$.

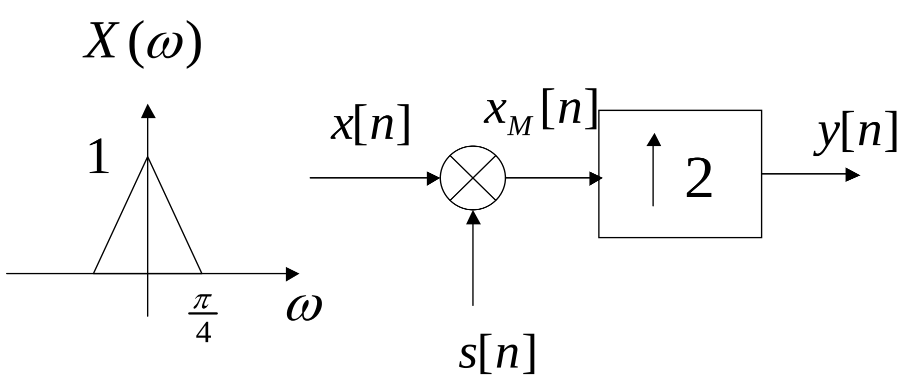
One way we can verify this is the following: call $v[n]$ the output of the downsampler. Then

$$V(\omega) = \frac{1}{2}\left(X\left(\frac{\omega}{2}\right) + X\left(\frac{\omega}{2} - \pi\right)\right)$$

Since $y[n]$ is the output of the upsampler then $Y(\omega)\,V(2\omega) = \frac{1}{2}(X(\omega) + X(\omega - \pi))$ as we expect.

■ Problem 6.7

Solution



The effect of modulation on the frequency spectrum is as follows

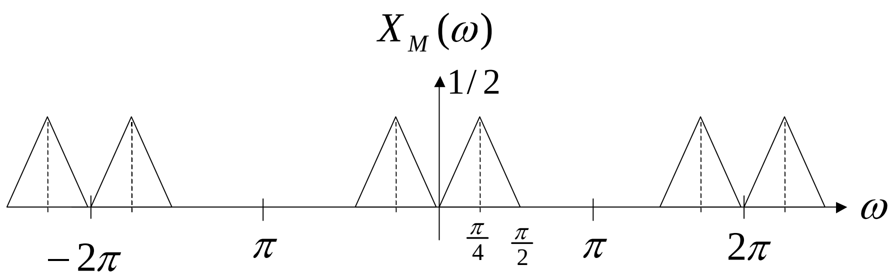
$$\text{DTFT}\{x_M[n]\} = \frac{1}{2}X(\omega - \omega_0) + \frac{1}{2}X(\omega + \omega_0)$$

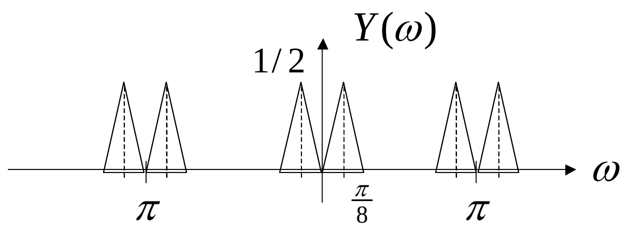
where $x_M[n] = x[n]\cos(\omega_0 n)$.

After upsampling by 2 the signal $y[n]$ has DTFT

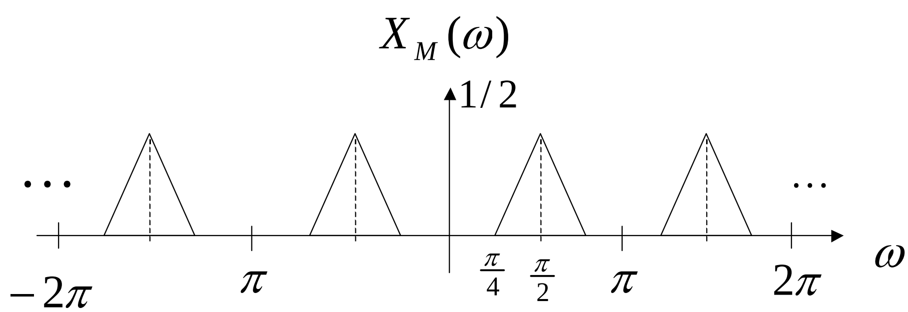
$$Y(\omega) = X_M(2\omega) = \frac{1}{2} X(2\omega - \omega_0) + \frac{1}{2} X(2\omega + \omega_0)$$

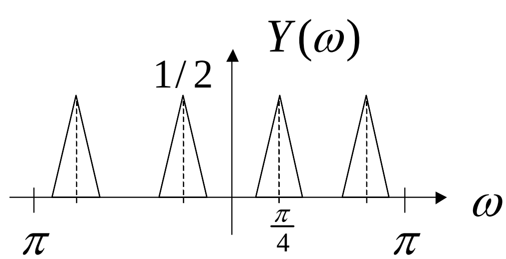
a) $\omega_0 = \frac{\pi}{4}$. Then $X_M(\omega)$ and $Y(\omega)$ are shown below.



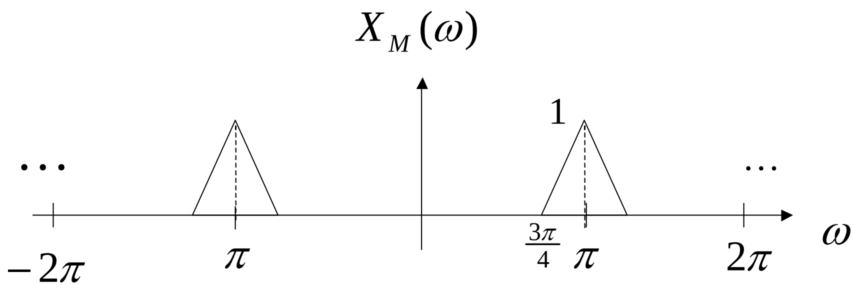


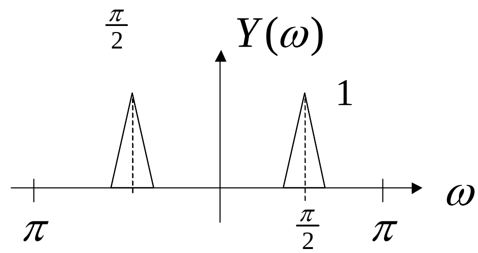
b) $\omega_0 = \frac{\pi}{2}$. Then $X_M(\omega)$ and $Y(\omega)$ are shown below.





c) $\omega_0 = \pi$. Then $X_M(\omega)$ and $Y(\omega)$ are shown below.



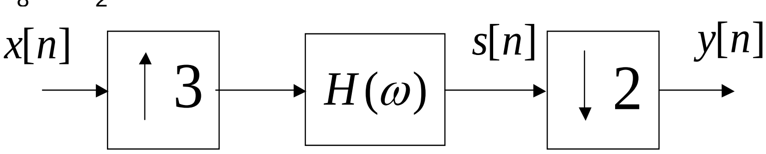


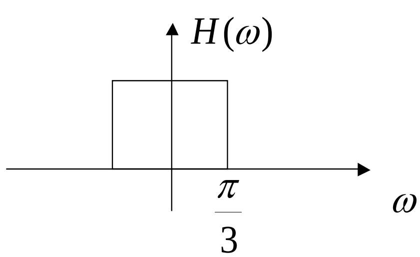
In this case notice the maximum amplitude of the DTFT being "one" (rather then 1/2 as in the previous cases). This is due to the fact that $X(\omega - \pi) = X(\omega + \pi)$.

■ Problem 6.8

Solution

In this problem we need to increase the sampling frequency from $F_s = 8$ kHz to $F_s = 12$ kHz, ie by a factor $\frac{12}{8} = \frac{3}{2}$. Therefore with ideal filters the scheme is as shown below.





■ Problem 6.9

Solution

In this case the Low Pass Filter $H(\omega)$ is a non ideal FIR filter. The whole problem is to choose the correct specifications for the filter.

The stopband has to be $\omega_S = \frac{\pi}{3}$, since the purpose of this filter is to stop the frequency artifacts generated by the upsampling operation. The passband has to be decided on the basis of the bandwirdth of the signal we want to pass and the desired complexity of the filter. For a window based, recall that we need a hamming window (from the desired attenuation) with transition region $\triangle\omega = 8\pi/N$. Therefore an FIR filter of length N will have a passband $\omega_p = \frac{\pi}{3} - \frac{8\pi}{N}$.

■ Problem 6.10

Solution

The digital signal has frequencies at $\omega_1 = \pm 2\pi/6 = \pm\pi/3$ rad and $\omega_2 = 2\pi 2\ /6 = \pm 2\pi/3$ rad. Therefore the output signal has frequencies at $\pm\pi/3$, $\pm 2\pi/3$ and also at $\pi/3 - \pi = -2\pi/3$, $-\pi/3 + \pi = 2\pi/3$, $2\pi/3 - \pi = -\pi/3$ and $-2\pi/3 + \pi = \pi/3$. All components at $\pm\pi/3$ and $\pm\ 2\pi/3$ are going to sum with each other.

■ Problem 6.11

Solution

a) $H(z) = (2 - 4z^{-3} - 6z^{-6}) + z^{-1}(-3 - 5z^{-3} + 2z^{-6}) + z^{-2}(2 + 2z^{-3})$

b) $H(z) = (2 + 2z^{-2} - 5z^{-4} - 6z^{-6}) + z^{-1}(-3 - 4z^{-2} + 2z^{-4} + 2z^{-6})$

c) $H(z) = (2z^2 + 1 + 2z^{-2} - 5z^{-4} - 6z^{-6}) + z^{-1}(-z^2 - 3 - 4z^{-2} + 2z^{-4} + 2z^{-6})$

d) $H(z) = (-z^3 - 4z^{-3} - 6z^{-6}) + z^{-1}(z^3 + 2 - 5z^{-3} + 2z^{-6}) + z^{-2}(2z^6 - 3z^3 + 2z^{-3})$

e) $H(z) = \sum_{n=0}^{+\infty} 0.5^n z^{-n} = \sum_{n=0}^{+\infty} 0.5^{2n} z^{-2n} + z^{-1}\sum_{n=0}^{+\infty} 0.5^{2n+1} z^{-2n}$ whixh yields

$$H(z) = \frac{1}{1-0.25z^{-2}} + z^{-1}\frac{0.5}{1-0.25z^{-2}}$$

f) $H(z) = \frac{z}{z-0.8}$ which can be written as
$H(z) = \sum_{n=0}^{+\infty} 0.8^n z^{-n} = \sum_{n=0}^{+\infty} 0.8^{2n} z^{-2n} + z^{-1}\sum_{n=0}^{+\infty} 0.8^{2n+1} z^{-2n}$. This becomes

$$H(z) = \frac{1}{1-0.64z^{-2}} + z^{-1}\frac{0.8}{1-0.64z^{-2}}$$

The same result can be obtained by an alternative way:

$$H(z) = \frac{z}{z-0.8}\ \frac{z+0.8}{z+0.8} = \frac{z^2}{z^2-0.64} + z^{-1}\ \frac{0.8\,z^2}{z^2-0.64}$$

amd you can verify that the two answers are the same.

g) You can verify that the general exprezzion for the polyphase terms is

$$H_k(z) = \sum_{n=-\infty}^{+\infty} h[nM-k]\ z^{-n}$$

for $k = 0, \ldots, M-1$. Applying this formula we obtain

$$H_k(z) = \sum_{n=-\infty}^{+\infty} \frac{\sin\left(\frac{\pi}{5}(3n-k)\right)}{\frac{\pi}{5}(3n-k)}\ z^{-n} \text{ for } k = 0, 1, 2$$

■ Problem 6.12

Solution

a) From the transfer function of the filter $H(z) = 1 + z^{-1} + 2z^{-2} - z^{-3} + z^{-4} - z^{-5} + z^{-6}$

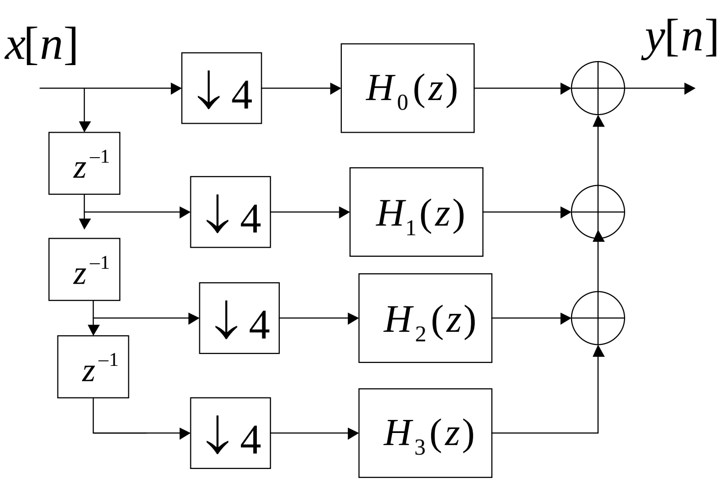
and $M = 4$ we obtain the decomposition

$$H(z) = H_0(z^4) + z^{-1} H_1(z^4) + z^{-2} H_2(z^4) + z^{-3} H_3(z^4)$$

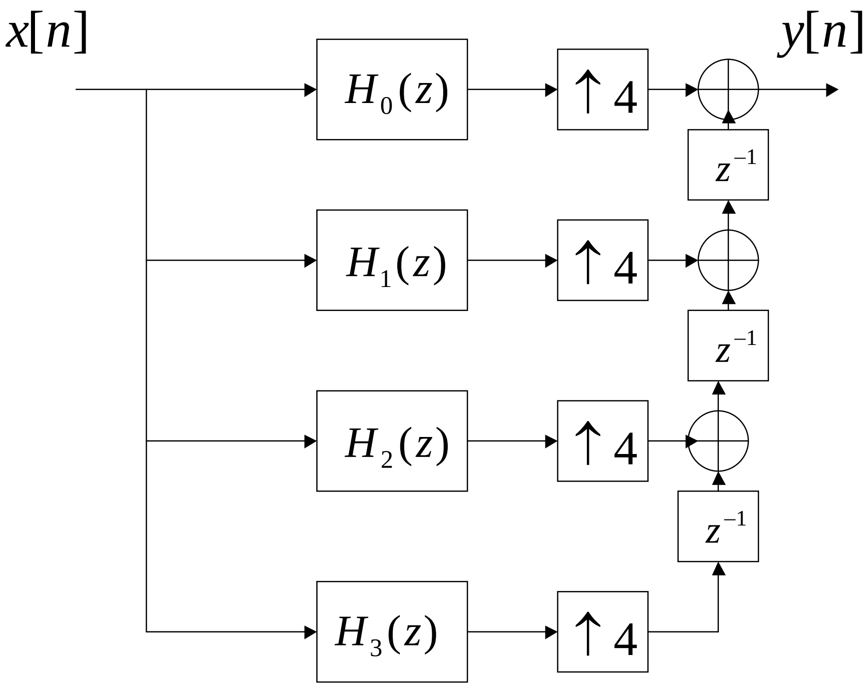
with

$$H_0(z) = 1 + z^{-1}$$
$$H_1(z) = 1 - z^{-1}$$
$$H_2(z) = 2 + z^{-1}$$
$$H_3(z) = -1$$

The block diagram of the system is shown below:

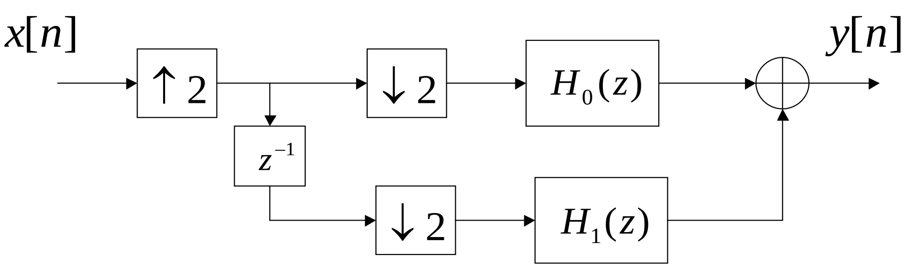


b) Using the same filters, the realization is as follows:

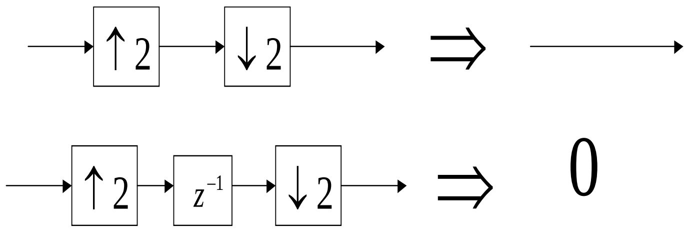


c) When $M = 2$ the polyphase decomposition becomes $H_0(z) = 1 + 2z^{-1} + z^{-2} + z^{-3}$ and $H(z) = 1 - z^{-1} - z^{-2}$

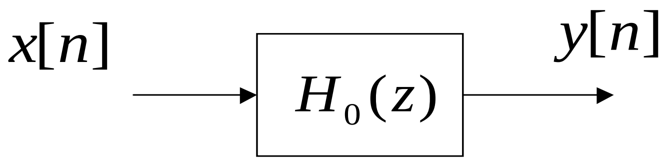
Therefore the system becomes as shown below:



Now notice that the cascade upsampler - downsampler (both by 2) is just an identity. Also the cascede upsampler - time delay - downsampler as shown gives an output of zero, no matter what the input is (easy to verify). This is shown below:



Therefore the overall system looks like this one:



Applications of MultiRate

■ Problem 6.13

Solution

a) In digital frequency, the passband is $\omega_P = 2\pi \times 25 / 6000 = \pi / 120$ radians, and the stopband is $\omega_S = 2\pi \times 30 / 6000 = \pi / 100$ radians. As a consequence the transition region is $\Delta\omega = \omega_S - \omega_P = 0.0017\pi$. For a 60 dB attenuation we can use (say) a Kaiser window with parameters

$$\beta = 0.1102\,(60 - 8.7) = 5.6533$$
$$N = \frac{60-8}{2.285\,\Delta\omega} = 4,261.1$$

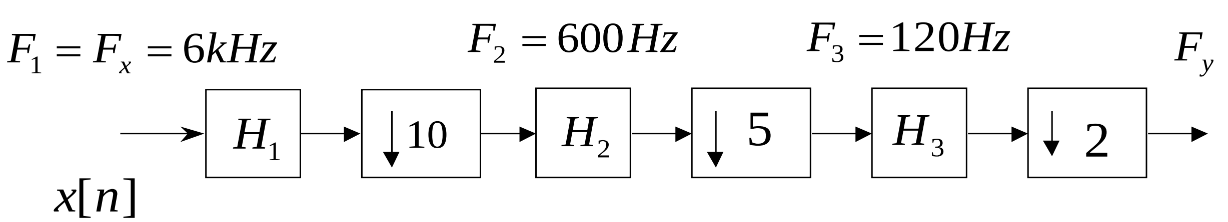
and therefore the order is $4,262$. This yields a total number of

$$4,262 \times 6,000 = 25.572 \times 10^6 \text{ multiplications and additions per second}$$

b) Since we want to reject all frequencies above 30Hz, we can downsample from the orginal sampling frequency (6kHz) down to 60Hz, ie by a factor $D = 6,000 / 60 = 100$. As seen in class, this can be done in three stages by factoring $D = 100$ as (say)

$$100 = 10 \times 5 \times 2$$

as shown below



Now the specifications of the filters become as follows:

$H_1 : F_p = 25\,\text{Hz}, \quad F_S = 600 - 30 = 570\,\text{Hz}, \quad \Delta\omega = 2\pi \times (570 - 25) / 6000 = 0.1817\pi$

$H_2 : F_p = 25\,\text{Hz}, \quad F_S = 120 - 30 = 90\,\text{Hz}, \quad \Delta\omega = 2\pi \times (90 - 25) / 600 = 0.2167\pi$

$H_3 : F_p = 25\,\text{Hz}, \quad F_S = 60 - 30 = 30\,\text{Hz}, \quad \Delta\omega = 2\pi \times (30 - 25) / 120 = 0.0833\pi$

From the transition bands we can determine the orders of the filters. using the Kaiser window again we have the orders N_1, N_2, N_3 of the threee filters to be. respectively,

$N_1 = \frac{60-8}{2.285\,\Delta\omega} = 40$

$N_2 = \frac{60-8}{2.285\,\Delta\omega} = 34$

$N_3 = \frac{60-8}{2.285\,\Delta\omega} = 87$

Finally the total number of operations per second becomes:

$$\text{ops} / \text{sec} = 40 \times 6000 + 34 * 600 + 87 * 120 = 270,\ 840 = 0.28 \times 10^6$$

which is reduction of a factor of more than 90. Also notice that we are not even attempting to save even more in computation using the polyphase decomposition!

■ Problem 6.14

Solution

Q1) Recall the Butterworth filter frequency response:

$$|H(\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}} = \frac{1}{1+\epsilon^2\left(\frac{\Omega}{\Omega_p}\right)^{2N}}$$

Then for the reconstruction filter, the passband is $F_p = 22\,\text{kHz} = 2\pi \times 22,\ 000$, that is to say $\Omega_p = 44,\ 000\,\pi\,\text{rad} / \text{sec}$. The stopband has to be at half the sampling frequency, ie $F_S = 44.1 / 2 = 22.05\,\text{kHz}$, that si to say $\Omega_S = 44,\ 100\,\pi\,\text{rad} / \text{sec}$.

From the passband ripple we determine the factor ϵ from

$$\frac{1}{1+\epsilon^2} = (1 - 0.01)^2 = 0.9801 \;\rightarrow\; \epsilon^2 = 0.0203 \text{ and therefore } \epsilon = 0.1425$$

We determine the order from the requirement

$$|H(\Omega_S)|^2 = \frac{1}{1+0.0203\left(\frac{44.1}{44}\right)^{2N}} = 0.01^2 = 10^{-4}$$

Solving for N gives a very large number, and the analog filter cannot be implemented. Furthermore there will be a distortion in phase from the reconstruction filter itself.

Q2) Now the filter has still the same expression, but with $N = 4$, since we are restricted to a 4 pole filer. Therefore the filter is given by

$$|H(\Omega)|^2 = \frac{1}{1+0.0203\left(\frac{\Omega}{44,000\,\pi}\right)^8}$$

Since we want the attenuation to be 40dB in the stopband, we can solve for the stopband, as

$$\frac{1}{1+0.0203\left(\frac{\Omega}{44,000\,\pi}\right)^8} = 0.01^2 = 10^{-4}$$

which yields

$$\frac{\Omega_S}{44,000\,\pi} = \left(\frac{10^4-1}{0.0203}\right)^{1/8} = 5.1470$$

Therefore the stopband of this filter is at $F_S = 5.1470 \times 22\ \text{kHz} = 113.24\ \text{kHz}$. This has to coincide with half the sampling frequency, and therefore the new sampling frequency is $F_s = 2 \times 113.24\ \text{kHz} = 226.5\ \text{kHz}$. In other words we have to upsample at least by 226.5/44.1=5.1 times before the Digital to Analog conversion.