Chapter 5: Problem Solutions

Digital Filter Implementation

State Space Realizations

■ Problem 5.1.

Problem

Given the system with transfer function

$$H(z) = \frac{2z+1}{z^3+2z^2+z+1}$$

determine:

a) the difference equation relating the input $x[n]$ to the output $y[n]$;

b) a block diagram realization, together with its state space equations;

c) Its diagonal state space realization;

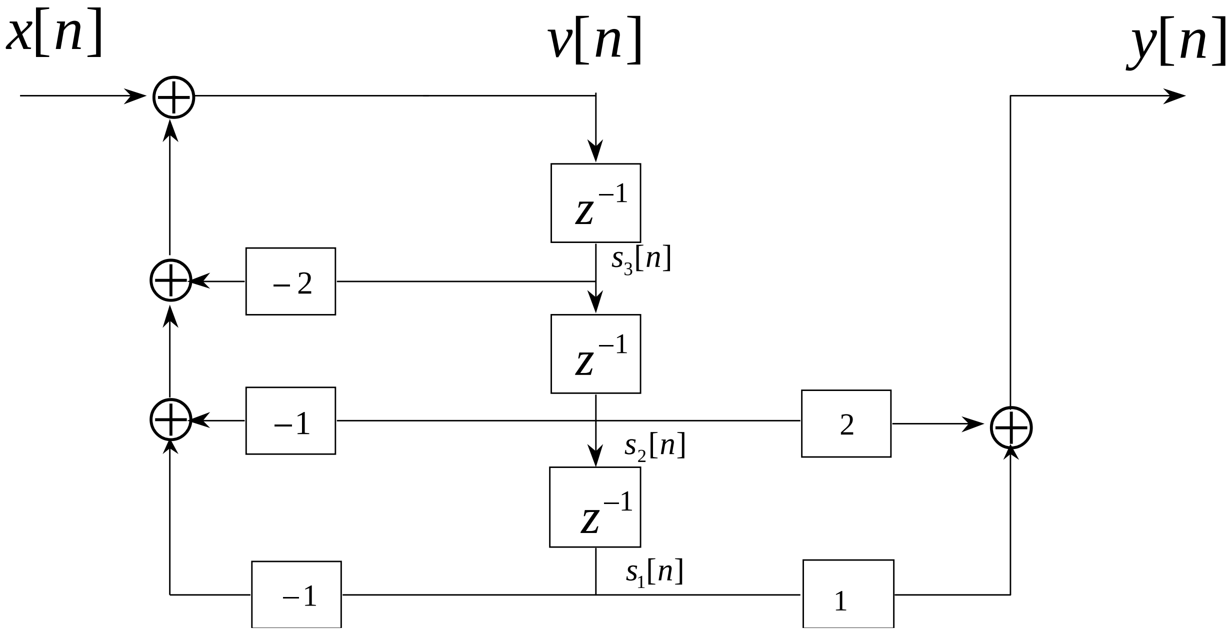
A realization made of blocks of first and second order only.

Solution

a) Call $x[n]$ and $y[n]$ the input and output sequences respectively. Then the difference equation is

$$y[n] + 2y[n-1] + y[n-2] + y[n-3] = 2x[n-2] + x[n-3]$$

b) Block Diagram representation for a Type I realization:



From the diagram we can determine the state space equations as

$$s_1[n+1] = s_2[n]$$
$$s_2[n+1] = s_3[n]$$
$$s_3[n+1] = -s_1[n] - s_2[n] - 2\,s_3[n] + x[n]$$
$$y[n] = s_1[n] + 2\,s_2[n]$$

In matrix form these become

$$s[n+1] = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & -2 \end{pmatrix} s[n] + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} x[n]$$

$$y[n] = \begin{pmatrix} 1 & 2 & 0 \end{pmatrix} s[n]$$

where $s[n] = \begin{pmatrix} s_1[n] & s_2[n] & s_3[n] \end{pmatrix}'$ being the state vector.

c) The eigenvalues and eigenvectors of the matrix A are the following:

$$\lambda_1 = -0.1226 + j\,0.7449, \quad q_1 = [-0.6626 - j\,0.2981,\ 0.3032 - j\,0.4570,\ 0.3032 + j\,0.2819]^T$$
$$\lambda_2 = -0.1226 - j\,0.7449, \quad q_2 = [-0.6626 + j\,0.2981,\ 0.3032 + j\,0.4570,\ 0.3032 - j\,0.2819]^T$$
$$\lambda_3 = -1.7549, \qquad q_3 = [-0.2715,\ \ 0.4765,\ \ -0.8362]$$

Calling $Q = [q_1,\ q_2,\ q_3]$ the 3×3 matrix of eigenvenvectors, the diagonal realization is given by the matrices

$$\overline{A} = Q^{-1} A\, Q = \begin{pmatrix} -0.1226 + j\,0.7449 & 0 & 0 \\ 0 & -0.1226 - j\,0.7449 & 0 \\ 0 & 0 & -1.7549 \end{pmatrix}$$

$$\overline{B} = Q^{-1} B = (0.3872 + j\,0.3395,\ \ 0.3872 - j\,0.3395,\ \ -1.1440)^T$$

$$\overline{C} = (\,-0.0561 - j\,1.2120,\quad -0.0561 + j\,1.2120,\quad 0.6815)$$

As you notice the eigenvalues are complex and in general we want to avoid using complex operations if we do not need to. A better realization would be to the define the transformation matrix Q using the real and imaginary parts of the eigenvectors. In this way define

$$Q = [\mathrm{Re}\{q_1\},\ \mathrm{Im}\{q_1\},\ q_3] = \begin{pmatrix} -0.6626 & -0.2981 & -0.2715 \\ 0.3032 & -0.4570 & 0.4765 \\ 0.3032 & 0.2819 & -0.8362 \end{pmatrix}$$

and therefore we obtain

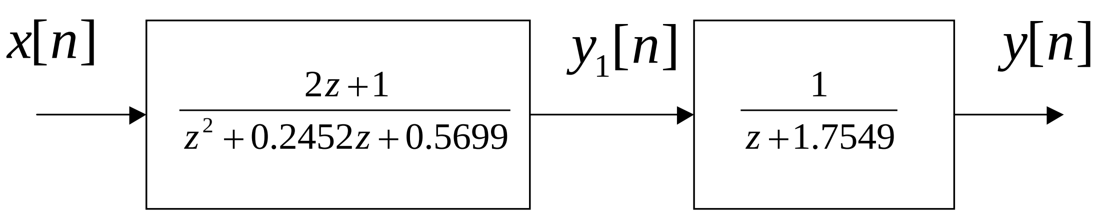
$$\overline{A} = Q^{-1} A\, Q = \begin{pmatrix} -0.1226 & 0.7449 & 0 \\ 0.7449 & -0.1226 & 0 \\ 0 & 0 & -1.7549 \end{pmatrix}$$

$$\overline{B} = Q^{-1}\, B\ =\ (0.7743,\quad -0.6790,\quad -1.1440)^T$$

$$\overline{C} = C\, Q\ =\ (-0.0561,\ -1.2120,\ 0.6815)$$

d) Factor the transfer function in terms of zeros and poles

$$H(z) = \frac{2\,(z+\frac{1}{2})}{(z^2+0.2452\,z+0.5699)\,(z+1.7549)} = \left(\frac{2\,z+1}{z^2+0.2452\,z+0.5699}\right)\left(\frac{1}{(z+1.7549)}\right)$$



■ Problem 5.2

Problem

Repeat problem 5.1 for the transfer function

$$\mathrm{H}\,(\mathrm{z}) = \frac{5\,\mathrm{z}-2}{\mathrm{z}^2+\mathrm{z}+1}$$

Solution

a) Difference equation

$$y[n] = -y[n-1] - y[n-2] + 5\,x[n-1] - 2\,x[n-2]$$

b) The matrices for the state space realization

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$
$$B = (0, \quad 1)^T$$
$$C = (-2, \quad 5)$$
$$D = 0$$

c) Eigenvalues and eigenvectors

$$\lambda_1 = -0.5 + j\,0.866, \quad q_1 = [0.6124 - j\,0.3536, \quad j\,0.7071]^T$$
$$\lambda_2 = -0.5 - j\,0.866 \quad q_2 = [0.6124 + j\,0.3536, \quad -j\,0.7071]^T$$

Then the diagonal realization

$$\overline{A} = Q^{-1}\,A\,Q = \begin{pmatrix} -0.5 + j\,0.866 & 0 \\ 0 & -0.5 - j\,0.866 \end{pmatrix}$$
$$\overline{B} = Q^{-1}\,B = (0.4082 - j\,0.7071, \quad 0.4082 + j\,0.7071)^T$$
$$\overline{C} = C\,Q = (-1.2247 + j\,4.2426, \ -1.2247 - j\,4.2426)$$

d) The filter is already a second order system with complex poles, and it cannot be reduced any further.

■ Problem 5.3

Problem

Repeat problem 5.1 for the transfer function

$$H(z) = \frac{2\,z^2 + 2\,z + 2}{z^4 - 1.6\,z^3\ 0.64\,z^2\ 1.024\,z - 0.8192}$$

Solution

a) The difference equation

$$y[n] = 1.6\,y[n-1] - 0.64\,y[n-2] - 1.24\,y[n-3] + 0.8192\,y[n-4] + 2\,x[n-2] + 2\,x[n-3] + 2\,x[n-4]$$

b) Matrices for the State Space equations:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0.8192 & -1.024 & -0.64 & 1.6 \end{pmatrix}$$

$$B = (0,\ 0,\ 0,\ 1)^T$$

$$C = (\ 2,\ 2,\ 2,\ 0)$$

c) Eigenvalues and eigenvectors:

$$\lambda_1 = 0.8 + \mathrm{j}0\,.8,$$

$$q_1 = [0.3487 - j\,0.2113,\quad 0.4480 + j\,0.1099,\quad 0.2704 + j\,0.4463,\quad -0.1407 + j\,0.5734]$$

$$\lambda_2 = 0.8 - \mathrm{j}0\,.8,$$

$$q_2 = [0.3487 + j\,0.2113,\quad 0.4480 - j\,0.1099,\quad 0.2704 - j\,0.4463,\quad -0.1407 - j\,0.5734]$$

$$\lambda_3 = 0.8, \qquad q_3 = [0.6577,\quad 0.5262,\quad 0.4209,\quad 0.3367]$$

$$\lambda_4 = -0.8, \qquad q_4 = [0.6577,\quad -0.5262\,,\quad 0.4209,\quad -0.3367]$$

A real diagonal realization (with all real entries) is obtained from the transformation matrix

$$Q = [\mathrm{Re}\,\{q_1\},\ \ \mathrm{Im}\,\{q_1\},\ \ q_3,\ \ q_4]$$

which yields

$$\overline{A} = Q^{-1}\ A\,Q\ = \begin{pmatrix} 0.8 & 0.8 & 0 & 0 \\ -0.8 & 0.8 & 0 & 0 \\ 0 & 0 & 0.8 & 0 \\ 0 & 0 & 0 & -0.8 \end{pmatrix}$$

$$\overline{B} = Q^{-1}\,B\ =\ (-2.1353\,,\quad 0.1735,\quad 1.4848,\quad -0.2970\,)^T$$

$$\overline{C} = C\,Q\ =\ (2.1343,\quad 0.6901,\quad 3.2096,\quad 1.1049)$$

d) Factorize numerator and denominator:

$$H(z) = \frac{2\,(z^2+z+1)}{(z^2-1.6\,z+1.28)\,(z-0.8)\,(z+0.8)} = \left(\frac{2\,(z^2+z+1)}{z^2-1.6\,z+1.28}\right)\left(\frac{1}{z-0.8}\right)\left(\frac{1}{z+0.8}\right)$$

■ Problem 5.4

Problem

You want design a Low Pass Filter by appropriately placing zeros and poles. You come up with the transfer function

$$\mathrm{H}\,(\mathrm{z}) = \mathrm{K}\,\frac{(\mathrm{z}+1)^2}{(\mathrm{z}-0.95)^3}$$

a) Determine the value of the gain K so that $|\,\mathrm{H}\,(\omega)\,|_{\omega=0} = 1$;

b) Determine the difference equation associated to this system;

c) Determine a Type I state space realization;

d) In the transfer function, perturb the denominator coeffient of z^0 by any value of your choice, smaller in magnitude than 10^{-3}. Is the system still stable? Would you trust this filter implement on fixed point arythmetic?

e) Implement the filter as the cascade of low order sections (first or second order). How can you guarantee stability in the presence of numerical errors?

Solution

a) $H(\omega)|_{\omega=0} = H(z)|_{z=1} = K\,2^2 / 0.05^3 = 1$. Solve for K to find the gain $K = 3.125 \times 10^5$

b) Expanding the transfer function as

$$H(z) = 10^{-5}\,\frac{3.125 + 6.25\,z + 3.125\,z^2}{-0.857375 + 2.7075\,z - 2.85\,z^2 + z^3}$$

we obtain the difference equation

$$y[n] - 2.85\,y[n-1] + 2.7075\,y[n-2] - 0.857375\,y[n-3] = 10^{-5}\,(3.125\,x[n-1] + 6.25\,x[n-2] + 3.125\,x[n-3])$$

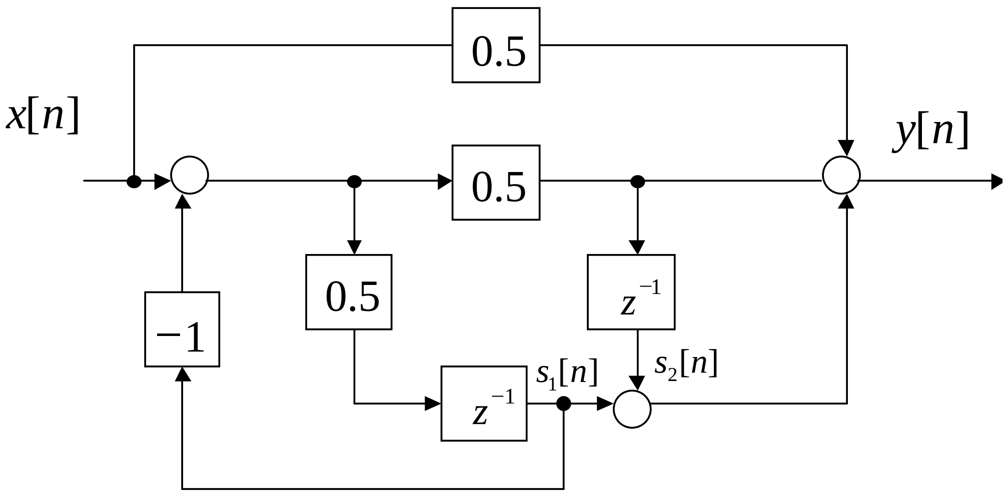
d) If we perturb the denominator polynomial to be $\hat{D}(z) = z^3 - 2.85\,z^2 + 2.7075\,z - 0.856$ you can verify that one of the roots is outside the unit circle, and therefore the system is unstable.

e) By simple factorization, we can write $H(z) = K\left(\frac{z+1}{z-0.96}\right)\left(\frac{z+1}{z-0.96}\right)\left(\frac{1}{z-0.95}\right)$. Each section of the filter is stable and it is easy to guarantee stability of the whole system.

■ Problem 5.5

Problem

Given the system shown



Determine the State Space equations and the Transfer Function.

Solution

In this class of problems, it is easy to go from the block diagram to a state space realization. The transfer function then comes easy.

To determine a state space realization call (say) $s_1[n]$ and $s_2[n]$ the output of each time delay as shown in the figure. Then we can write:

$$s_1[n+1] = -0.5\, s_1[n] + 0.5\, x[n]$$
$$s_2[n+1] = -0.5\, s_1[n] + 0.5\, x[n]$$
$$y[n] = -0.5\, s_1[n] + s_1[n] + s_2[n] + 0.5\, x[n]$$

In matrix form it becomes:

$$s[n+1] = \begin{pmatrix} -0.5 & 0 \\ -0.5 & 0 \end{pmatrix} s[n] + \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix} x[n]$$
$$y[n] = \begin{pmatrix} 0.5 & 1 \end{pmatrix} s[n] + 0.5\, x[n]$$

The Transfer function therefore becomes

$$H(z) = (0.5 \quad 1) \begin{pmatrix} z+0.5 & 0 \\ 0.5 & z \end{pmatrix}^{-1} \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix} + 0.5 =$$
$$= \frac{1}{z\,(z+0.5)} (0.5 \quad 1) \begin{pmatrix} z & 0 \\ -0.5 & z+0.5 \end{pmatrix} \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix} + 0.5$$
$$= \frac{0.75\, z}{z\,(z+0.5)} + 0.5 = \frac{0.5\, z+1}{z+0.5}$$

■ Problem 5.6.

Problem

You have given the following program:

```
begin loop
    read input x[n]
    y[n]=v1+x[n];
    output y[n]
    v1_new=v1+v2-0.5v3;
    v2_new=v1-v2-v3+x[n];
    v3_new=v1+v2+v3;
    v1=v1_new
    v2=v2_new
    v3=v3_new
end loop
```

a) Determine a Block Diagram realization;

b) Determine the state space equations;

c) Determine the transfer function and the difference equation;

d) Is this system stable?

Solution

a) First notice that there are three state variables. The labels v1, v2, v3 refer to the current values at index n, while v1_new, v2_new, v3_new refer to the values for the next iteration, ie at index n+1. Therefore we have the follwoing state space equations:

$$v_1[n+1] = v_1[n] + v_2[n] - 0.5\, v_3[n]$$
$$v_2[n+1] = v_1[n] - v_2[n] - v_3[n] + x[n]$$
$$v_3[n+1] = v_1[n] + v_2[n] + v_3[n]$$
$$y[n] = v_1[n] + x[n]$$

Therefore the state space matrices

$$A = \begin{pmatrix} 1 & 1 & -0.5 \\ 1 & -1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$B = (0, \quad 1, \quad 0)^T$$

$$C = (1, \quad 0, \quad 0)$$

$$D = 1$$

This yields a transfer function

$$H(z) = \frac{z^3 - z^2 + 0.5\,z + 1.5}{z^3 - z^2 - 0.5\,z + 3}$$

which is not stable, since there is at least one pole outside the unit circle (all of them as a matter of fact!).

The difference equation is

$$y[n] = y[n-1] + 0.5\,y[n-2] - 3\,y[n-3] + x[n] - x[n-1] + 0.5\,x[n-2] + 1.5\,x[n-3]$$

■ Problem 5.7

Problem

Repeat Problem 5.6 for the following program

```
begin loop
    read input x[n]
    y[n]=v1+x[n];
    output y[n]
    v1=v1+v2-0.5v3;
    v2=v1-v2-v3+x[n];
    v3=v1+v2+v3;
end loop
```

How is this different from the program in Problem 5.6?

Solution

This problem is slightly different from the previous, but it gives a totally different answer. Since we do not assign different variables to the "new" values, things get a bit mixed up. In this case the state space equations become

$$v_1[n+1] = v_1[n] + v_2[n] - 0.5\, v_3[n]$$
$$v_2[n+1] = v_1[n+1] - v_2[n] - v_3[n] + x[n]$$
$$v_3[n+1] = v_1[n+1] + v_2[n+1] + v_3[n]$$
$$y[n] = v_1[n] + x[n]$$

After simple substitutions we obtain

$$v_1[n+1] = v_1[n] + v_2[n] - 0.5\, v_3[n]$$
$$v_2[n+1] = v_1[n] - 1.5\, v_3[n] + x[n]$$
$$v_3[n+1] = 2\, v_1[n] + v_2[n] - 2\, v_3[n]$$
$$y[n] = v_1[n] + x[n]$$

This yields a transfer function

$$H(z) = \frac{z^3+z^2+0.5\, z+1.5}{z^3+z^2-0.5\, z}$$

■ Problem 5.8

Problem

Repeat Problem 5.6 for the following program

```
begin loop
    read input x[n]
    v1_new=v1+v2-0.5v3;
    v2_new=v1-v2-v3+x[n];
    v3_new=v1+v2+v3;
    v1=v1_new
    v2=v2_new
    v3=v3_new
    y[n]=v1+x[n];
    output y[n]
end loop
```

How is this different from the program in Problem 5.6?

Solution

a) First notice that there are three state variables. The labels v1, v2, v3 refer to the current values at index n, while v1_new, v2_new, v3_new refer to the values for the next iteration, ie at index n+1. Therefore we have the following state space equations:

$$v_1[n+1] = v_1[n] + v_2[n] - 0.5\, v_3[n]$$
$$v_2[n+1] = v_1[n] - v_2[n] - v_3[n] + x[n]$$
$$v_3[n+1] = v_1[n] + v_2[n] + v_3[n]$$

The output equation becomes

$$y[n] = v_1[n+1] + x[n] = v_1[n] + v_2[n] - 0.5\, v_3[n] + x[n]$$

where we substituted for $v_1[n+1]$.

Therefore the matrices for the state space equations become

$$A = \begin{pmatrix} 1 & 1 & -0.5 \\ 1 & -1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$
$$B = (0, \quad 1, \quad 0)^T$$
$$C = (1,\ 1,\ -0.5)$$
$$D = 1$$

This yields a transfer function

$$H(z) = \frac{z^3 - 2\, z + 3}{z^3 - z^2 - 0.5\, z + 3}$$

which again is not stable.

The difference equation is

$$y[n] = y[n-1] + 0.5\, y[n-2] - 3\, y[n-3] + x[n] - 2\, x[n-2] + 3\, x[n-3]$$

■ Problem 5.9

Problem

Repeat Problem 5.6 for the following program

```
begin loop
    read input x[n]
    y[n]=v1+v2+3x[n];
```

```
        v1_new=v1+v2;
        v2_new=v1-v2+x[n];
        v1=v1_new
        v2=v2_new
    end loop
```

Solution

There are two state variables $v_1[n]$ and $v_2[n]$. From the program we can write

$$v_1[n+1] = v_1[n] + v_2[n]$$
$$v_2[n+1] = v_1[n] - v_2[n] + x[n]$$
$$y[n] = v_1[n] + v_2[n] + 3\,x[n]$$

The state space equations have matrices

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$
$$B = (0,\ \ 1)^T$$
$$C = (1,\ \ 1)$$
$$D = 3$$

Thereofre the transfer function is given by

$$H(z) = \frac{3\,z^2+z-6}{z^2-2}$$

and the difference equation

$$y[n] = 2\,y[n-2] + 3\,x[n] + x[n-1] - 6\,x[n-2]$$

Lattice Implementation

■ Problem 5.11

Problem

Given the transfer function

$$A(z) = (1 - 0.8\, z^{-1})^2$$

a) Determine its reverse $\tilde{A}(z)$. Also compute its zeros and poles. Compare them to the zeros and poles of $A(z)$;

b) Determine the reflection coefficients of $A(z)$, and the lattice realization.

Solution

a) By expansion we determine $A(z) = 1 - 1.6\, z^{-1} + 0.64\, z^{-2}$ and therefore
$\tilde{A}(z) = 0.64 - 1.6\, z^{-1} + z^{-2}$

The transfer function $A(z)$ has two zeros at $z = 0.8$ and two poles at $z = 0$, while the transfer function $\tilde{A}(z)$ has two zeros at $z = 1 / 0.8 = 1.25$ and two poles at $z = 0$;

b) The reflection coefficents are determined by the recursive formula

$$\begin{bmatrix} A_m(z) \\ z^{-1}\tilde{A}_m(z) \end{bmatrix} = \frac{1}{1-K_m^2} \begin{bmatrix} 1 & -K_m \\ -K_m & 1 \end{bmatrix} \begin{bmatrix} A_{m+1}(z) \\ \tilde{A}_{m+1}(z) \end{bmatrix}$$

$$K_m = \frac{a_{m+1}[m+1]}{a_{m+1}[0]}$$

Therefore

$$K_1 = 0.64$$
$$A_1(z) = \frac{1}{(1-0.64^2)} (A(z) - 0.64\, \tilde{A}(z)) = 1 - .97561\, z^{-1}$$
$$K_0 = -0.97561$$
$$A_0(z) = 1$$

■ Problem 5.12

Problem

Repeat problem 5.11 for the following transfer function

$$A(z) = 1 - 1.8856\, z^{-1} + 0.7728\, z^{-2} + 0.8610\, z^{-3} - 1.1221\, z^{-4} + 0.5398\, z^{-5} - 0.1296\, z^{-6}$$

Solution

Applying the same reasoning as in Problem 5.11 we start with

$$A_6(z) = 1 - 1.8856\, z^{-1} + 0.7728\, z^{-2} + 0.8610\, z^{-3} - 1.1221\, z^{-4} + 0.5398\, z^{-5} - 0.1296\, z^{-6}$$
$$\tilde{A}_6(z) = -0.1296 + 0.5398\, z^{-1} - 1.1221\, z^{-2} + 0.8610\, z^{-3} + 0.7728\, z^{-4} - 1.8856\, z^{-5} + z^{-6}$$

Therefore

$$K_5 = \frac{a_6[6]}{a_6[0]} = -0.1296$$

is the first reflection coefficient. This yields

$$A_5(z) = \frac{1}{1-K_5^2}\,(A_6(z) - K_5\,\tilde{A}_6(z)) = 1 - 1.8467\,z^{-1} + 0.6381\,z^{-2} + 0.9892\,z^{-3} - 1.0394\,z^{-4} + 0.3005\,z^{-5}$$

Continuing the same arguments we obtain

$$K_4 = 0.3005$$

$$A_4(z) = 1 - 1.6866\,z^{-1} + 0.3747\,z^{-2} + 0.8766\,z^{-3} - 0.5326\,z^{-4}$$
$$K_3 = -0.5326$$

$$A_3(z) = 1 - 1.7028\,z^{-1} + 0.8017\,z^{-2} - 0.0303\,z^{-3}$$
$$K_2 = -0.0303$$

$$A_2(z) = 1 - 1.6800\,z^{-1} + 0.7508\,z^{-2}$$
$$K_1 = 0.7508$$

$$A_1(z) = 1 - 0.9596\,z^{-1}$$
$$K_0 = -0.9596$$
$$A_0(z) = 1$$

■ Problem 5.13

Problem

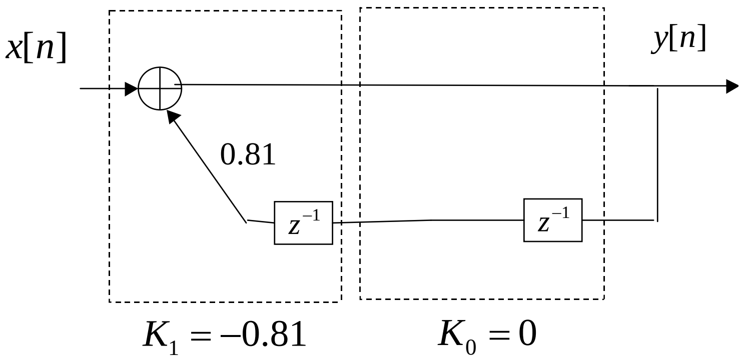
Given the transfer function $H(z) = \frac{1}{1-0.81\,z^{-2}}$ determine a lattice realization.

Solution

From the denominator $A(z) = 1 - 0.81\,z^{-2}$ we obtain

$$K_1 = -0.81$$
$$A_1(z) = 1$$
$$K_0 = 0$$
$$A_0(z) = 1$$

Therefore the IIR lattice implementation becomes as shown below.



■ Problem 5.14

Problem

Repeat Problem 5.13 for the Transfer Function

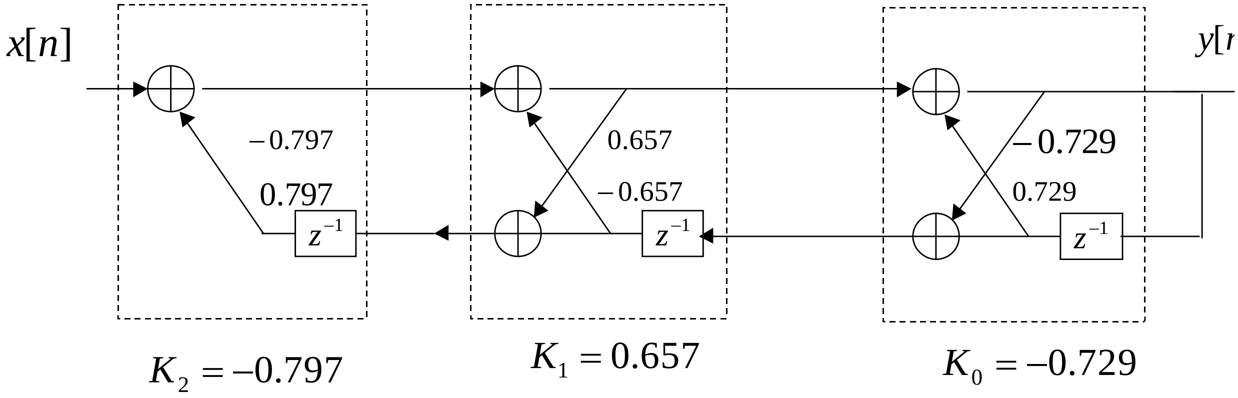
$$H(z) = \frac{1}{1-1.8\,z^{-1}+1.62\,z^{-2}-0.729\,z^{-3}}$$

Solution

From the denominator $A(z) = 1 - 1.8\,z^{-1} + 1.62\,z^{-2} - 0.729\,z^{-3}$ we obtain

$$K_2 = -0.729$$
$$A_2(z) = 1 - 1.32111\,z^{-1} + 0.656908\,z^{-2}$$
$$K_1 = 0.656908$$
$$A_1(z) = 1 - 0.797337\,z^{-1}$$
$$K_0 = -0.797337$$
$$A_0(z) = 1$$

The implementation is shown below.



■ Problem 5.15

Problem

Repeat problem 5.13 for the transfer function

$$H(z) = \frac{1}{(1-0.98\,z^{-1})^3}$$

Solution

Starting from the polynomial

$$A_3(z) = (1 - 0.98\,z^{-1})^3 = 1 - 2.9400\,z^{-1} + 2.8812\,z^{-2} - 0.941192\,z^{-3}$$

we obtain

$$K_2 = -0.941192$$
$$A_2(z) = 1 - 1.99932\,z^{-1} + 0.999456\,z^{-2}$$
$$K_1 = 0.999456$$
$$A_1(z) = 1 - 0.999932\,z^{-1}$$
$$K_0 = -0.999932$$

Notice that the filter is very close to instability, and the above computations require double precision arithmetic. The filter can be kept stable even with round off errors, by making sure that the reflection coefficients in magnitude do not exceed one.

■ Problem 5.16

Problem

Determine the Lattice Realization of the Transfer Function of the system in Problem 5.4.

Solution

The transfer function is

$$H(z) = \frac{(z+1)^2}{(z-0.95)^3} = \frac{z^{-1}(1+z^{-1})^2}{(1-0.95\,z)^3}$$

First we need to determine the Lattice expansion of the denominator. We obtain:

$$A_3(z) = (z - 0.95)^3 = 1 - 2.85\,z^{-1} + 2.7075\,z^{-2} - 0.857375\,z^{-3}$$

This yields

$$K_3 = -0.857375$$
$$A_2(z) = 1 - 1.99562\,z^{-1} + 0.996501\,z^{-2}$$

$$K_2 = 0.996501$$
$$A_1(z) = 1 - 0.999561\,z^{-1}$$

$$K_0 = -0.999561$$

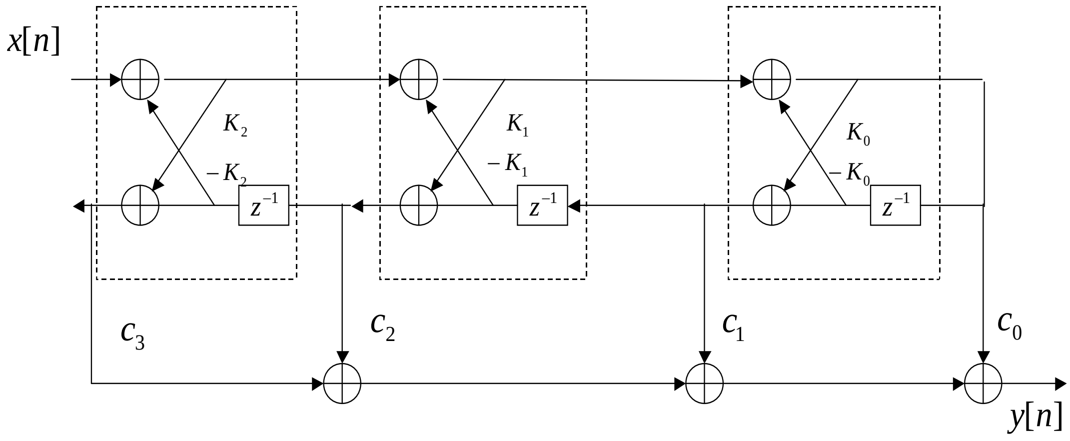
Then we need to determine the coefficients c_0, c_1, c_2, c_3 to solve the polynomial equation

$$z^{-1}(1 + z^{-1})^2 = c_0 + c_1\tilde{A}_1(z) + c_2\tilde{A}_2(z) + c_3\tilde{A}_3(z)$$

This yields

$$c_0 = 3.99213,\ c_1 = 7.97128,\ c_2 = 4.85,\ c_3 = 1$$

The Lattice implementation is shown below.



■ Problem 5.17

Problem

Given the transfer function

$$H(z) = \frac{z^2+1}{(z-0.9)^2 (z+0.9)}$$

a) Determine a lattice realization;

b) Determine the state space equations that implement the lattice realization.

Solution

a) Again we write the transfer function as

$$H(z) = \frac{z^{-1}+z^{-3}}{(1-0.9\,z^{-1})^2 (1+0.9\,z^{-1})}$$

Then we expand the denominator in terms of reflection coefficients:

$$A_3(z) = 1 - 0.9\,z^{-1} - 0.81\,z^{-2} + 0.729\,z^{-3}$$
$$K_2 = 0.729$$

$$A_2(z) = 1 - 0.660557\,z^{-1} - 0.328454\,z^{-2}$$
$$K_1 = -0.328454$$

$$A_1(z) = 1 - 0.983636\,z^{-1}$$
$$K_0 = -0.983636$$

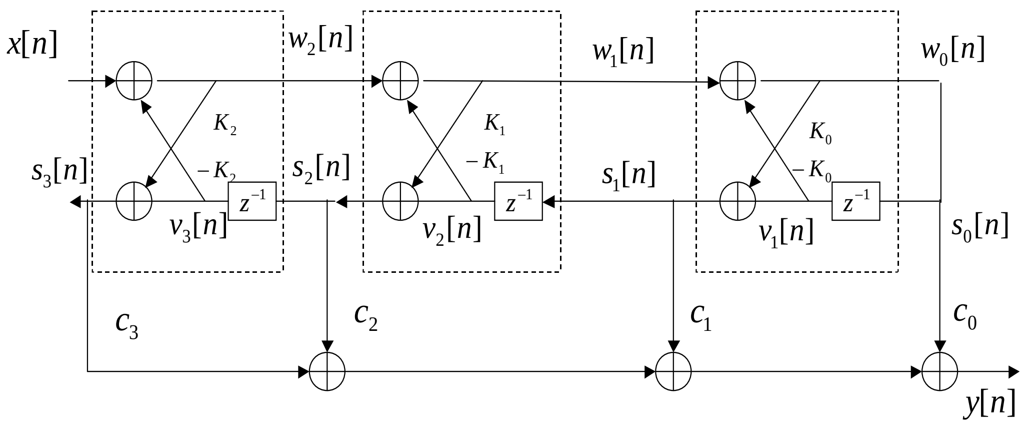
Then we need to determine the coefficients c_0, c_1, c_2, c_3 to solve the polynomial equation

$$z^{-1} + z^{-3} = c_0 + c_1\,\tilde{A}_1(z) + c_2\,\tilde{A}_2(z) + c_3\,\tilde{A}_3(z)$$

This yields

$$c_0 = 1.93176,\ c_1 = 2.4045,\ c_2 = 0.9,\ c_3 = 1$$

The realization is shown below.



b) For the state space equations, define the state vector as

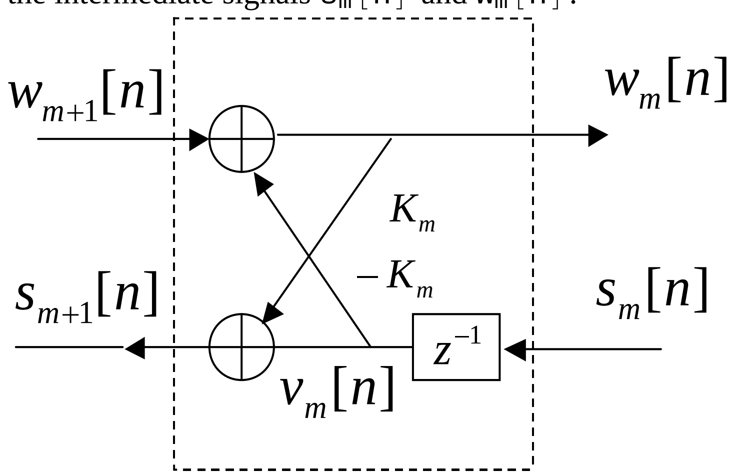
$$v[n] = (v_1[n],\ v_2[n],\ v_3[n])'$$

Then we obtain the following equations:

$$v_1[n+1] = -K_0\, v_1[n] - K_1\, v_2[n] - K_2\, v_3[n] + x[n]$$
$$v_2[n+1] = v_1[n] + K_0\, (-K_0\, v_1[n] - K_1\, v_2[n] - K_2\, v_3[n] + x[n])$$
$$v_3[n+1] = v_2[n] + K_1\, (-K_2\, v_3[n] + x[n])$$
$$\begin{aligned} y[n] = {} & c_0\, (-K_0\, v_1[n] - K_1\, v_2[n] - K_2\, v_3[n] + x[n]) + \\ & + c_1\, (v_1[n] + K_0\, (-K_0\, v_1[n] - K_1\, v_2[n] - K_2\, v_3[n] + x[n])) \\ & + c_2\, (v_2[n] + K_1\, (-K_2\, v_3[n] + x[n])) \end{aligned}$$

from which we can derive the usual state space equations.

However to implement this class of filters it might turn usefull to use the recursive structure shown below, and write the intermediate signals $s_m[n]$ and $w_m[n]$.



In this case we obtain the input-ouput equations of all sections:

$$w_2[n] = -K_2 v_2[n] + x[n]$$
$$s_3[n] = v_2[n] + K_2 w_2[n]$$

$$w_1[n] = -K_1 v_1[n] + w_2[n]$$
$$s_2[n] = v_1[n] + K_1 w_1[n]$$

$$w_0[n] = -K_0 v_0[n] + w_1[n]$$
$$s_1[n] = v_0[n] + K_0 w_0[n]$$
$$s_0[n] = w_0[n]$$

then we compute the output:

$$y[n] = c_0 s_0[n] + c_1 s_1[n] + c_2 s_2[n] + c_3 s_3[n]$$

and finally we update the states

$$v_2[n+1] = s_2[n]$$
$$v_1[n+1] = s_1[n]$$
$$v_0[m+1] = s_0[n]$$

■ Problem 5.18

Problem

Repeat Problem 5.16 for the transfer function $H(z) = (z-1)/(z+0.9)^2$.

Solution

Again we write the transfer function as

$$H(z) = \frac{z^{-1}-z^{-2}}{(1+0.9\,z^{-1})^2}$$

Then we expand the denominator in terms of reflection coefficients:

$$A_2(z) = 1 + 1.8\,z^{-1} + 0.81\,z^{-2}$$
$$K_1 = 0.81$$

$$A_1(z) = 1 + 0.994475\,z^{-1}$$
$$K_0 = 0.994475$$

Then we need to determine the coefficients c_0, c_1, c_2, c_3 to solve the polynomial equation

$$z^{-1} - z^{-2} = c_0 + c_1 \tilde{A}_1(z) + c_2 \tilde{A}_2(z)$$

This yields

$$c_0 = -1.97453,\ c_1 = 2.8,\ c_2 = -1.$$

■ Problem 5.19

Problem

Repeat Problem 5.16 for the transfer function $H(z) = 1/(z^4 - 0.4096)$.

Solution

Again we write the transfer function as

$$H(z) = \frac{z^{-4}}{1-0.4096\,z^{-4}}$$

We then expand the denominator:

$$A_4(z) = 1 - 0.4096\,z^{-4}$$
$$K_3 = -0.4096$$

At this point we obtain

$$A_3(z) = A_1(z) = 1$$
$$K_2 = K_1 = K_0 = 0$$

Finally for the rest of the coefficients to solve the polynomial equation

$$z^{-4} = c_0 + c_1 \tilde{A}_1(z) + c_2 \tilde{A}_2(z) + c_3 \tilde{A}_3(z) + c_4 \tilde{A}_4(z)$$

we obtain

$$c_0 = 0.4096,\ c_1 = 0,\ c_2 = 0,\ c_3 = 0,\ c_4 = 1$$