Chapter 4: Problem Solutions

Digital Filters

Problems on Non Ideal Filters

■ Problem 4.1

We want to design a Discrete Time Low Pass Filter for a voice signal. The specifications are:

Passband F_p = 4 kHz, with 0.8 dB ripple;

Stopband F_S = 4.5 kHz, with 50dB attenuation;

Sampling Frequency F_s = 22 kHz.

Determine a) the discrete time Passband and Stopband frequencies, b) the maximum and minimum values of $|H(\omega)|$ in the Passband and the Stopband, where $H(\omega)$ is the filter frequency response.

Solution

a) Recall the mapping from analog to digital frequency $\omega = 2\pi F / F_s$, with F_s the sampling frequency. Then the passband and stopband frequencies become $\omega_p = 2\pi 4 / 22$ rad $= 0.36\pi$ rad, $\omega_s = 2\pi 4.5 / 22$ rad $= 0.41\pi$ rad;

b) A 0.8 dB ripple means that the frequency response in the passband is within the interval $1 \pm \delta$ where δ is such that $20\log_{10}(1 + \delta) = 0.8$ This yields $\delta = 10^{0.04} - 1 = 0.096$. Therefore the frequency response within the passband is within the interval $0.9035 \le |H(\omega)| < 1.096$. Similarly in the stopband the maximum value is $|H(\omega)| < 10^{-50/20} = 0.0031$

■ Problem 4.2

A Digital Filter has frequency response $H(\omega)$ such that

$$0.95 \le |H(\omega)| \le 1.05 \quad \text{for} \quad 0 \le \omega \le 0.3\pi$$
$$0 \le \left|H(\omega)\right| \le 0.005 \quad \text{for} \quad 0.4\pi \le \omega \le \pi$$

Also let the sampling frequency be $F_s = 8\,kHz$. Determine the Passband and Stopband frequencies in kHz, the Passband ripple and the Stopband attenuation in dB.

Solution

The passband ripple is given by $20\,\log_{10}\,(1.05) = 0.42\,dB$, and the attenuation in the stopband $-20\,\log_{10}\,0.005 = 46\,dB$. The analog passband frequency is $0.3\,\pi\,F_s\,/\,2\,\pi = 1.2\,kHz$ and the stopband $0.4\,\pi\,F_s\,/\,2\,\pi = 1.6\,kHz$

■ Problem 4.3

A continuous time filter has frequency response

$$H\,(F)\;=\;\frac{1}{1+\frac{j2\pi F}{1000}}$$

Determine the passband and stopband frequencies in Hz, assuming a passband ripple of 1dB and attenuation of 40dB in the stopband. Also determine the half power frequency F_c.

Solution.

A passband ripple of 1dB means that the frequency response is within the interval $1 - \delta \le |\,H\,(F)\,| \le 1 + \delta$ with $20\,\log_{10}\,1 + \delta = 1$, which yields $\delta = 0.12$. Since $|\,H\,(F)\,| = \frac{1}{\sqrt{1+\left(\frac{2\,\pi F}{1000}\right)^2}}$ then we determine the passband from the equation

$$\frac{1}{\sqrt{1+\left(\frac{2\,\pi F}{1000}\right)^2}}\;=\;1-0.12 = 0.88$$

which yields $F = 85.9Hz$. Similarly for the stopband, we need to determine the frequency where $|\,H\,(F)\,| = 10^{-40/20} = 0.01$ which yields $F = 15,\,914\,Hz$ Notice that this filter has a very long transition region, as we can see from the plot of its magnitude:
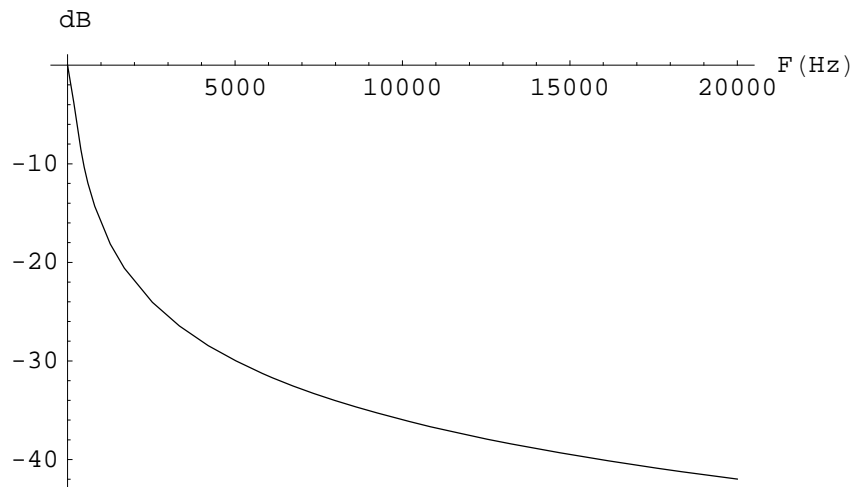
■ Problem 4.4

A Digital Filter is defined by the difference equation

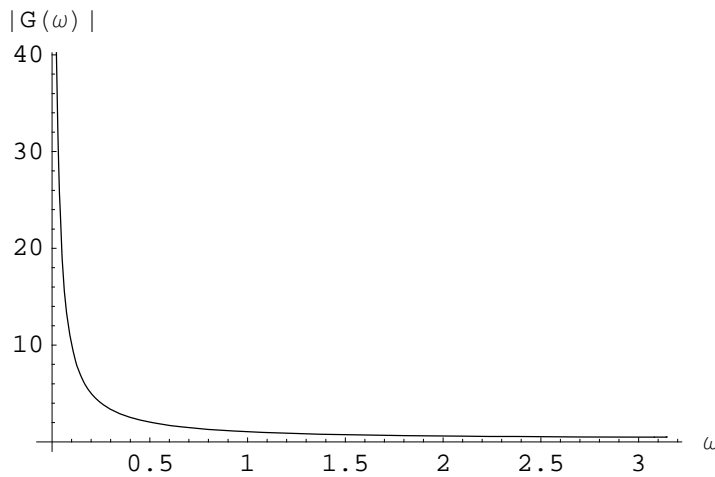$$y[n] = 0.99 \, y[n - 1] + x[n]$$

The filter is clearly recursive. Determine the impulse response $h[n]$.

a) Is the filter stable?

b) Would you classify it as Low Pass, Band Pass ... or what?

c) Would you feel comfortable in implementing this on a digital machine?

Solution

a) The filter is stable since its transfer function $H(z) = \frac{1}{1 - 0.99 \, z^{-1}} = \frac{z}{z - 0.99}$ has one pole at $z = 0.99$;

b) It is a low pass filter since it has one pole close to $z = 1$, ie $\omega = 0$. This makes the frequency response "large" at small frequencies. A plot of its magnitude is as follows:

$|G(\omega)|$

(plot showing $|G(\omega)|$ versus ω, with vertical axis marked 10, 20, 30, 40 and horizontal axis marked 0.5, 1, 1.5, 2, 2.5, 3)

■ Problem 4.5

A simple averaging filter is defined as

$$y[n] = \frac{1}{N} (x[n-1] + \ldots + x[n-N])$$

This is clearly an FIR Filter.

a) Let $N = 4$. Determine the transfer function, its zeros and poles;

b) Determine a general form for zeros and poles for any N;

c) By comparing $y[n]$ and $y[n-1]$ determine a recursive implementation. Also the transfer function, together with its zeros and poles of the recursive implementation. Looking at this example, can we say that "any" recursive filter is IIR?
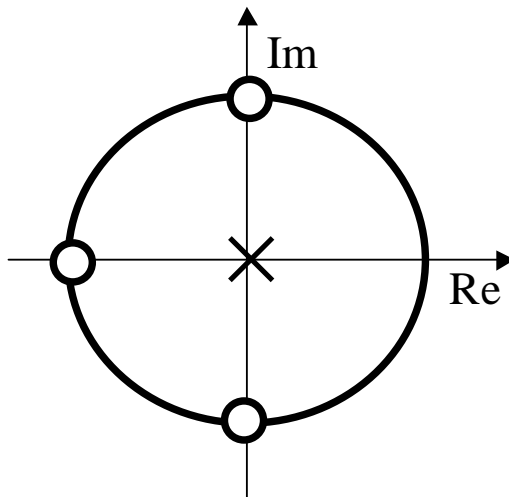
Solution

a) With $N = 4$ we obtain the transfer function $H(z) = \frac{1}{4} (z^{-1} + z^{-2} + z^{-3} + z^{-4})$. After normalization this becomes

$$H(z) = \frac{1}{4} \frac{z^3 + z^2 + z + 1}{z^4}$$

The are four poles at $z = 0$ and three zeros from the solution

$$z^3 + z^2 + z + 1 = \frac{1 - z^4}{1 - z} = 0$$

Therefore the zeros must be such that $z^4 = 1$, with the exclusion of $z = 1$. That is to say $z^4 = e^{jk2\pi}$ for $k = 1, 2, 3$, and therefore the zeros are $z = j^k$ with $k = 1, 2, 3$, ie $z = j, -1, -j$. This is shown in the z-plane below.

b) Since the transfer function is of the form

$$H(z) = \frac{1}{N} \frac{1-z^N}{z^N (1-z)}$$

the zeros are of the form $z = e^{jk\frac{2\pi}{N}}, k = 1, \ldots, N-1$ and the poles are all at $z = 0$.

c) Since $y[n] = \frac{1}{N} (x[n-1] + \ldots + x[n-N])$ and
$y[n-1] = \frac{1}{N} (x[n-2] + \ldots + x[n-N-1])$ by comparing $y[n]$ and $y[n-1]$ we see that

$$y[n] = y[n-1] + \frac{1}{N} x[n-1] - \frac{1}{N} x[n-N-1]$$

This yields the transfer function

$$H(z) = \frac{1}{N} \frac{z^{-1}-z^{-N-1}}{1-z^{-1}} = \frac{1}{N} \frac{1-z^N}{z^N (1-z)}$$

as we saw before. This is an example of a recursive filter with finite impulse response (FIR).

Problems on FIR Filters

■ Problem 4.6

We want to design a Low Pass FIR Filter with the following characteristics:

Passband 10kHz,

Stopband 11kHz, with attenuation of 50dB,

Sampling frequency 44kHz

Determine the causal impulse response $h[n]$, and an expression for the phase within the passband. Use one of the standard windows listed in section 4.3.

Solution

First we have to determine the specifications in the digital freq. domain.

Passband: $\omega_p = 2\pi\, 10 / 44 = 0.4545\,\pi\,\text{rad}$

Stopband: $\omega_S = 2\pi\, 11 / 44 = 0.5\,\pi\,\text{rad}$

Therefore we choose the passband of the ideal filter as $\omega_C = \frac{1}{2}\ (\omega_p + \omega_S)\ =\ \frac{21}{44}\,\pi\ =\ 0.477\,\pi$. We need a Blackman window to satisfy the 50dB attenuation in the stopband. With this window the transition region has a width of $12\,\pi / N$. Since we want a transition region $\omega_S - \omega_p = 2\pi / 44$ we determine the filter length N as

$$\frac{2\pi}{44} \geq 12\,\frac{\pi}{N}$$

which yields $N \geq 12 \times 22 = 264$. Therefore we choose $N = 265$ and a shift $L = 132$. Finally the impulse response is

$$h[n] = h_d[n - 132]\ w_{\text{Blackman}}[n]$$
$$= \frac{\sin\,(0.4545\,\pi\,(n-132))}{\pi\,(n-132)}\ w_{\text{Blackman}}[n]$$

which is shown below.
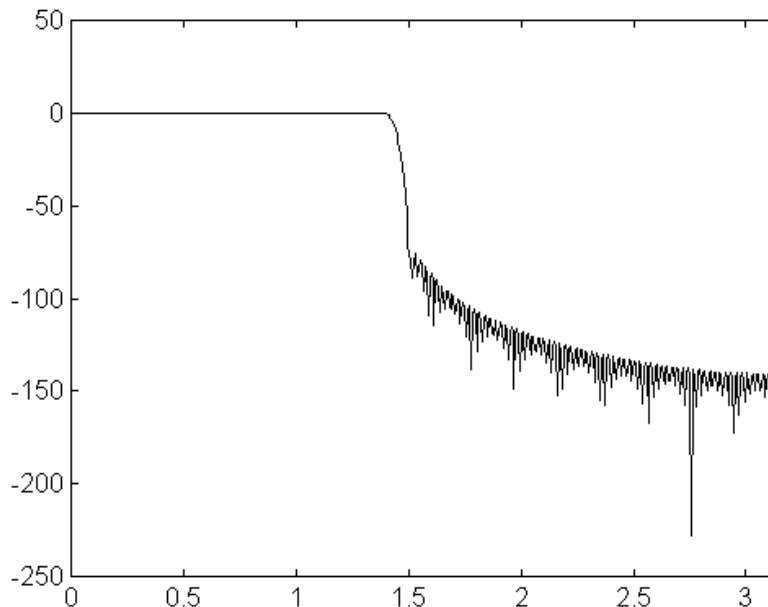
Within the passband the phase is linear and it is given by the expression

$$\angle H\ (\omega)\ =\ -\omega\, L\ =\ -132\ \omega$$

■ Problem 4.7

Repeat Problem 2.1 with an equiripple filter using the "remez" function in Matlab. Plot the two frequency responses and compare the two filters in terms of performance and complexity.

Solution

With Matlab we need first to determine the order of the filter. Use the function "remezord" as follows:

```
[N, fo, mo, k] = remezord ([10000, 11000], [1, 0], [delta, delta], 44000);
```
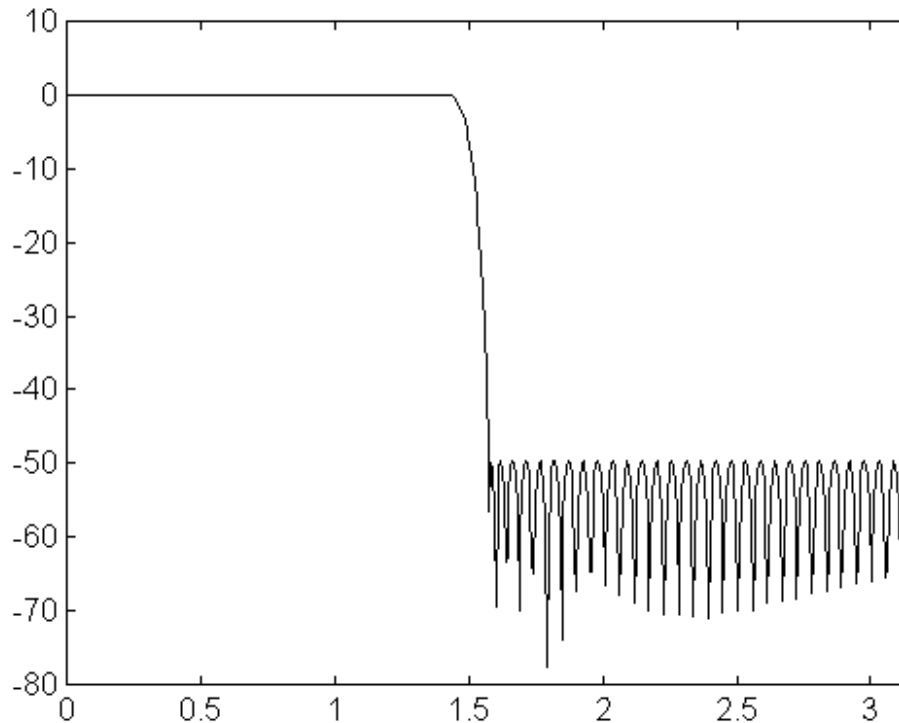
with `delta = 10 ^ (-50 / 20)` the maximum deviation corresponding to 50dB's. This yields an order $N = 114$, in the sense that the transfer function is of the form

$$H\ (z)\ =\ h[0]\ +\ h[1]\ z^{-1}\ +\ \ldots\ +\ h[114]\ z^{-114}$$

The impulse response `h[n]` is obtained as

```
h = remez (N, fo, mo, k)
```

where `fo,` `mo` and `k` are from remezord.

Notice that the order of the equiripple filter N = 114 is considerably smaller than the order of the filter designed with the Blackman window in Problem 4.6.

■ Problem 4.8

Repeat Problem 4.6 using the Kaiser window.

Solution

Wi the Kaiser window we have to determine the parameters N and β from the specifications. In particular we want an attenuation A = 50 dB which yields a factor β from the expression

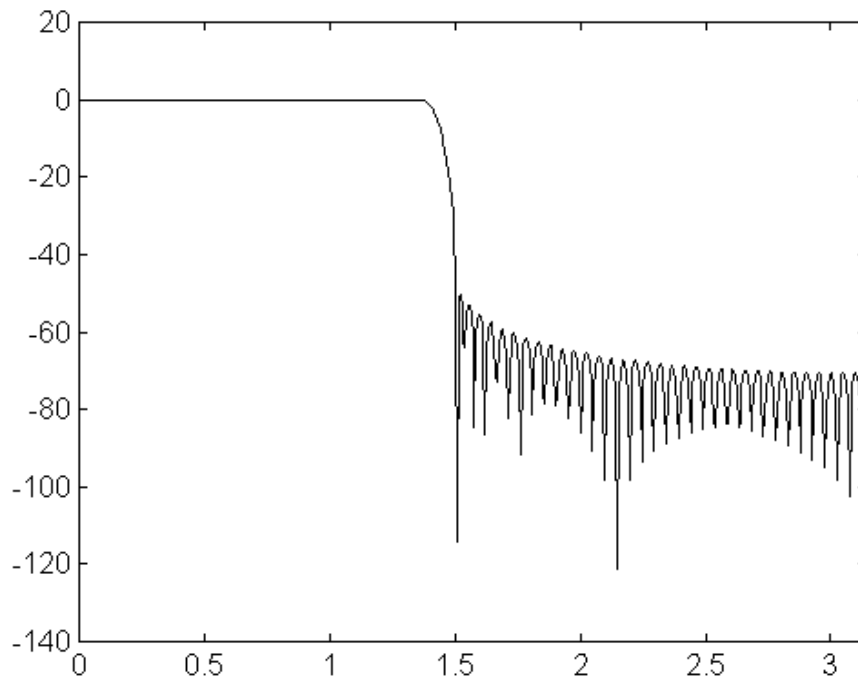$$\beta = 0.5842 \, (A - 21)^{0.4} + .07886 \, (A - 21) = 4.53$$

Also the filter length is determine from the expression

$$N \geq \frac{A-8}{2.285 \, \Delta\omega} = \frac{42}{2.285 \, (\pi \, / \, 22)} = 128.717$$

So we can choose N = 129 and L = 64. The frequency response of the filter therefore becomes

$$\frac{\sin \, (0.4545 \, \pi \, (n-64))}{\pi \, (n-64)} \, W_{Kaiser}[n]$$

Its magnitude is shown below.

■ Problem 4.9

We want to approximate a filter with frequency response

$$H(F) = \begin{cases} e^{-0.1|F|} & \text{if} \quad |F| < 10\ \text{Hz} \\ 0 & \text{if} \quad |F| > 11\ \text{Hz} \end{cases}$$

Let the sampling frequency be $F_s = 50\ \text{Hz}$, and the attenuation in the stopband be 40dB. Determine the impulse response of a FIR filter which approximates this frequency response. Plot the frequency response in terms of magnitude and phase to verify that the approximation holds.

Solution

In the digital domain, let $\omega = 2\pi F / F_s$ and therefore $F = \omega F_s / 2\pi$. Therefore the filter's desired frequency response becomes

$$H(\omega) = \begin{cases} e^{-5|\omega|/2\pi} & \text{if} \quad |\omega| < 2\pi/5\ \text{rad} \\ 0 & \text{if} \quad |F| > 2.2\pi/5\ \text{rad} \end{cases}$$

The ideal filter therefore is going to have a frequency response $H_d(\omega)$ given by

$$H_d(\omega) = \begin{cases} e^{-5|\omega|/2\pi} & \text{if} \quad |\omega| < 2.1\pi/5 \\ 0 & \text{otherwise} \end{cases}$$

and the impulse response

$$h_d[n] \ = \ \frac{1}{2\pi} \int\limits_{-2.1\,\pi/5}^{+2.1\,\pi/5} e^{-5\,|\omega|/2\,\pi} \ e^{j\omega n} \ d\omega \ = \ \frac{2.27405 - 0.795775\,\text{Cos}[1.31947\,n] + 1.\,n\,\text{Sin}[1.31947\,n]}{5.68512 + 8.97758\,n^2}$$

Since we want 40dB attenuation in the stopband we can use a hamming window, which has a transition region of width $8\,\pi\,/\,N$. The desired width is $\triangle\omega \ = \ 2\,\pi\,/\,50$ and therefore N is determined from the equation

$$\frac{2\,\pi}{50} \ \geq \ \frac{8\,\pi}{N}$$

and $N \geq 50 \times 4 = 200$. Choose $N = 201$ and $L = 100$. This yields the impulse response

$$h[n] \ = \ h_d[n - 100]\ w_{\text{hamming}}[n]$$

■ Problem 4.10

A bandpass filter needs to be designed, to pass a signal within frequencies $4\ \text{kHz}$ and $8\ \text{kHz}$, with two transition regions not exceeding $\triangle F \ = \ 0.5\ \text{kHz}$. Also we want the attenuation in the stopband not exceeding 50dB, and the same error within the passband. Finally let the sampling frequency be $F_s \ = \ 44\ \text{kHz}$.

a) Determine the impulse response of the ideal filter;

b) Design the filter using the Kaiser window;

c) Design the filter using the equiripple method.

Compare the two frequency responses.

Solution

a) In the digital frequency domain we want to design a bandpass filter which passes the frequencies between $2\,\pi\,4\,/\,44 \ = \ 2\,\pi\,/\,11\ \text{rad}$ and $2\,\pi\,8\,/\,44 \ = \ 4\,\pi\,/\,11\ \text{rad}$. Therefore the impulse response of the ideal filter is

$$h_d[n] \ = \ \frac{1}{2\pi} \int\limits_{2\,\pi/11}^{4\,\pi/11} e^{j\omega n}\ d\omega \ + \ \frac{1}{2\pi} \int\limits_{-4\,\pi/11}^{-2\,\pi/11} e^{j\omega n}\ d\omega \ = \ \begin{cases} \frac{-\text{Sin}[\frac{2\,n\,\pi}{11}] + \text{Sin}[\frac{4\,n\,\pi}{11}]}{n\,\pi} & \text{if } n \neq 0 \\ 0 & \text{if } n = 0 \end{cases}$$

b) From the formulas of the Kaiser window we determine the parameters β and N as follows from the attenuation $A \ = \ 50\ \text{dB}$ and the width of the stopband $\triangle\omega \ = \ 2\,\pi\,0.5\,/\,44 \ = \ \pi\,/\,44\ \text{rad}.$ Recall the formulas:

$$0.5842 \ (A - 21)^{0.4} + .07886 \ (A - 21) = 4.53$$

$$N \geq \frac{A-8}{2.285 \triangle \omega} \ 257.434$$

which yields a window of length $N = 259$ and a time delay $L = 129$. Finally thr impulse response becomes

$$h[n] = h_d[n - 129] \ w_{kaiser}[n]$$

■ Problem 4.11

You want to design a low pass filter with passband $F_p = 2 \ kHz$ and stopband $F_S = 2.5 \ kHz$, with attenuation of at least 40dB. Let the sampling frequency be $F_s = 10 \ kHz$. Using the techniques you know, determine the design with the least number of coefficients.

Solution

First we translate the specifications into the digital frequency domain:

Passband $\omega_p = 2 \ \pi \times \frac{2}{10} = \frac{2 \pi}{5}$

Stopband $\omega_S = 2 \ \pi \times \frac{2.5}{10} = \frac{\pi}{2}$

$\triangle \omega = \frac{\pi}{2} - \frac{2 \pi}{5} = \frac{\pi}{10}$

We know three techniques:

a) Window based: from the desired attenuation we need a hamming window. From the transition region

$$\triangle \omega = \frac{8 \pi}{N} \geq \frac{\pi}{10}$$

we obtain the length of the filter $N = 81$;

b) Kaiser window: applying the formulas with $A = 40$ and $\triangle \omega = 0.1 \ \pi$ we obtain

$$N = \frac{40-8}{2.285 \times 0.1 \times \pi} = 45$$

c) Equiripple Filter: using the matlab function "remezord" we obtain the order $N = 39$ which yields the lowest complexity. The corresponding frequency response is shown below (magnitude only in dB's).
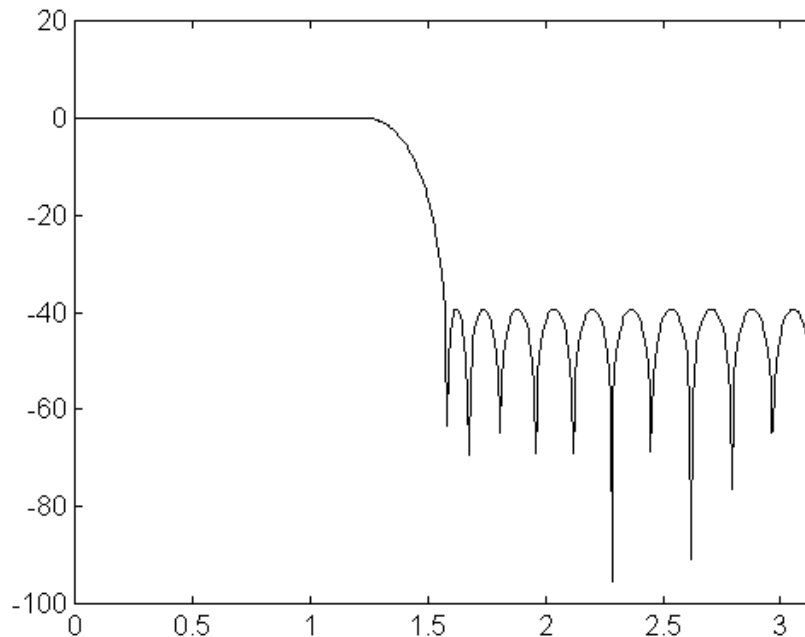
■ Problem 4.12

In the section on equiripple filters we have taken a few things for granted. In this problem try to prove the following simple facts:

a) when the order of the filter N increases, the maximum error decreases;

b) the solution is unique, in the sense that, for any given order N, there is only one impulse response h[0], ..., h[N] which minimizes the maximum error.

Solution

a) Call $\hat{h}_N$ = [$\hat{h}_N$[0], ..., $\hat{h}_N$[N]] the optimal solution (in the minmax sense) of order N. Then the vector

$$h_{N+1} = [\hat{h}_N[0], \ldots, \hat{h}_N[n], 0] \neq \hat{h}_{N+1}$$

represents an impulse response of order N + 1, not necessarily optimal. Therefore

$$e\,(\hat{h}_{N+1}) \leq e\,(h_{N+1}) = e\,(\hat{h}_N)$$

The leftmost inequality is due to the fact that h_{N+1} is not the optimal solution of order N + 1, and the rightmost equality is due to the fact

$$H_{N+1}\,(\omega) = \hat{H}_N\,(\omega) \text{ for all } \omega$$

where H_{N+1} and $\hat{H}_N$ are the frequency responses of h_{N+1} and $\hat{h}_N$.

b) If there were two different solutions $\hat{h}_N$ and $\hat{g}_N$ of the same order N having the sign alternation property, the difference $\hat{H}_N (\omega) - \hat{G}_N (\omega)$ would have N + 1 roots against the assumption of being polynomials (in cos (ω)) of order N.

■ Problem 4.13

A Hilbert Transform is a filter with frequency response

$$H_d (\omega) = -j\,\mathtt{sign} (\omega)$$

with sign $(\omega) = \pm 1$ for $\omega \neq 0$ being the signum function.

a) Plot the magnitude and phase of the filter;

b) Determine the impulse response $h_d [n]$;

c) Determine a causal approximation h[0], ..., h[N] using a rectangular window. Plot the magnitude of the frequency response for various values of N, say N = 40, 60, 120. Does it converge everywhere? In this case what would you call the transition region?

Solution

a) Since $| H_d (\omega) | = | -j\,\mathtt{sign} (\omega) | = 1$ for all ω, and

$$\mathtt{Phase} (H (\omega)) = \begin{cases} -\pi / 2 & \text{if } \omega > 0 \\ +\pi / 2 & \text{if } \omega < 0 \end{cases}$$

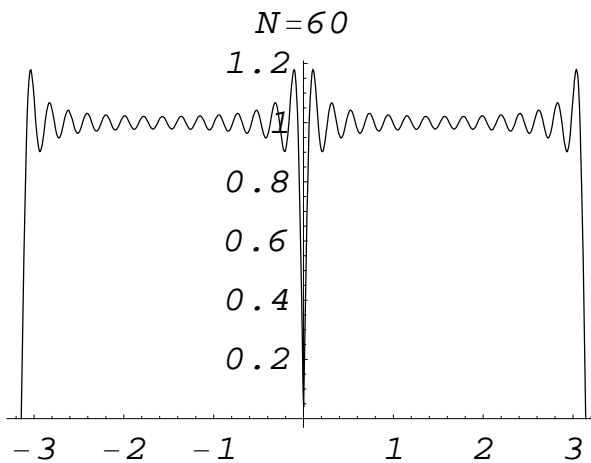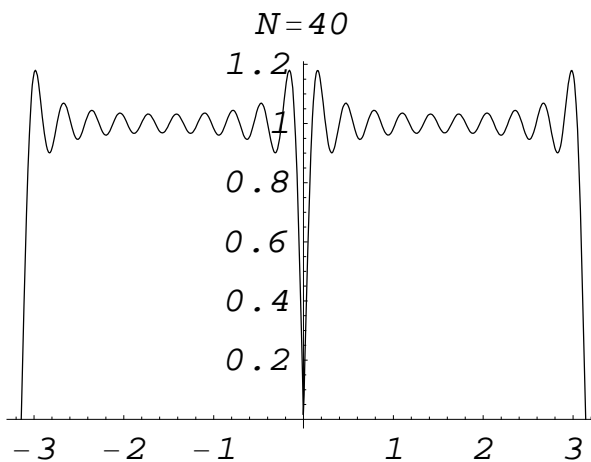we obtain the plot below.
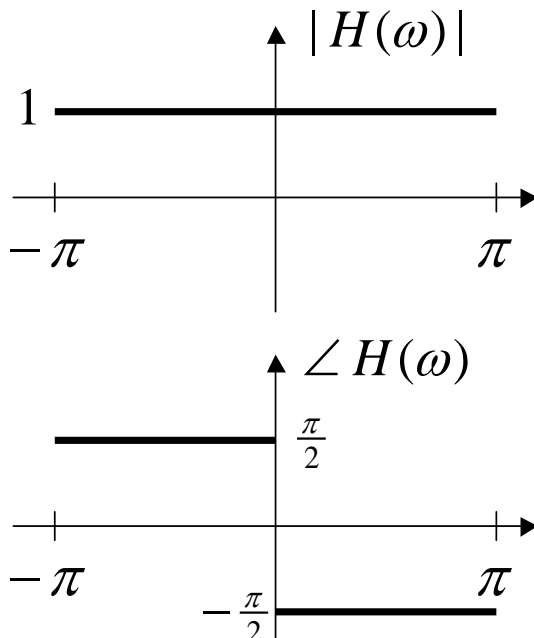
b) The impulse response becomes

$$h_d[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} -j\,\text{sign}(\omega)\, e^{j\omega n}\, d\omega = \frac{2\,\text{Sin}[\frac{n\pi}{2}]^2}{n\pi} \text{ if } n \neq 0$$
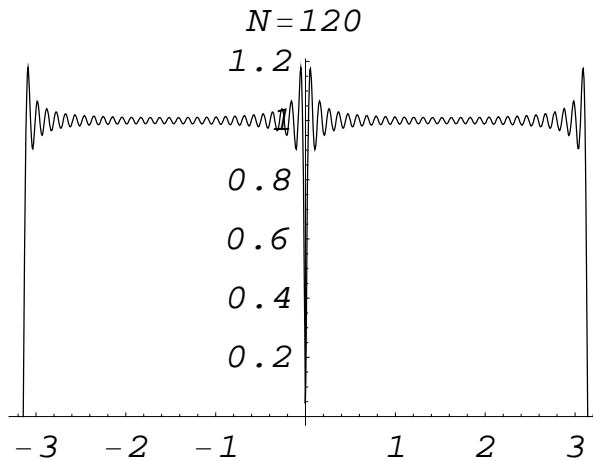
and $h_d[0] = 0$.

c) The plots of

$$H_L(\omega) = \sum_{n=-L}^{+L} h_d[n]\, e^{-j\omega n}$$

are shown below (magnitude only) for $L = 20,\ 40,\ 60$ corresponding to the orders $N = 40,\ 60,\ 120$ respectively. The transition region is defined around the frequency $\omega = 0$.

$$N = 120$$

Problems on IIR Filters

■ Problem 4.14

Using the Bilinear Transformation, determine the order N and the cut off frequency Ω_c of the analog prototype filter for the following discrete time design:

a) passband 8 kHz;

b) stopband 9 kHz;

c) passband ripple 0.5 dB;

d) stopband attenuation 40 dB;

e) sampling frequency F_s = 44 kHz.

Solution

First we define the problem in the digital frequency domain:

$$\omega_p = \frac{2 \pi F_p}{F_s} = 1.1424 \, \text{rad}$$
$$\omega_S = \frac{2 \pi F_S}{F_s} = 1.2852 \, \text{rad}$$

Then we determine the specifications of the Analog Prototype:

$$\Omega_p = F_s \, 2 \, \text{Tan} \, (\tfrac{\omega_p}{2}) = 56554.2 \, \text{rad} \, / \, \text{sec}$$
$$\Omega_S = F_s \, 2 \, \text{Tan} \, (\tfrac{\omega_S}{2}) = 65876.0 \, \text{rad} \, / \, \text{sec}$$

Now from the Passband Ripple we determine ϵ as

$$\delta_p = 10^{-0.5/20} = 0.944$$

$$\epsilon = \sqrt{\left(\tfrac{1}{\delta_p}\right)^2 - 1} = 0.349$$

Then we determine the order of the filter

$$N \geq \frac{\log\left(\frac{1-\delta_s^2}{\epsilon^2 \, \delta_s^2}\right)}{2 \log\left(\frac{\Omega_s}{\Omega_p}\right)} = 37.0762$$

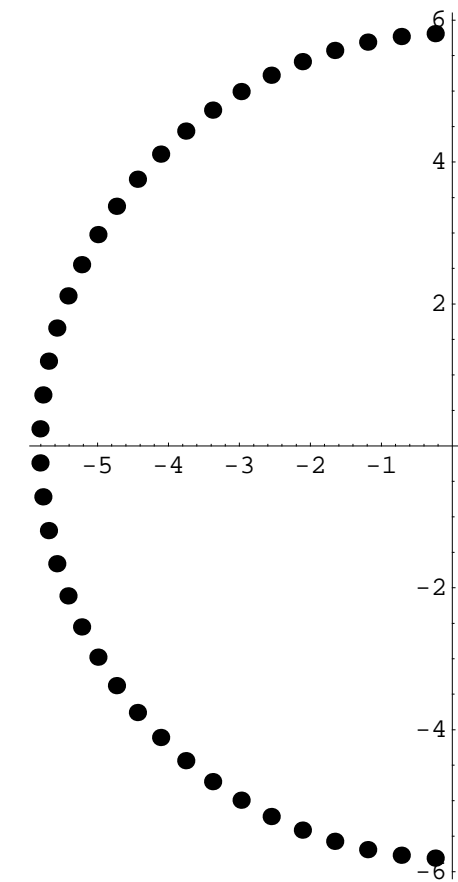which yields $N = 38$. Finally we determine the cut off frequency of the filter as

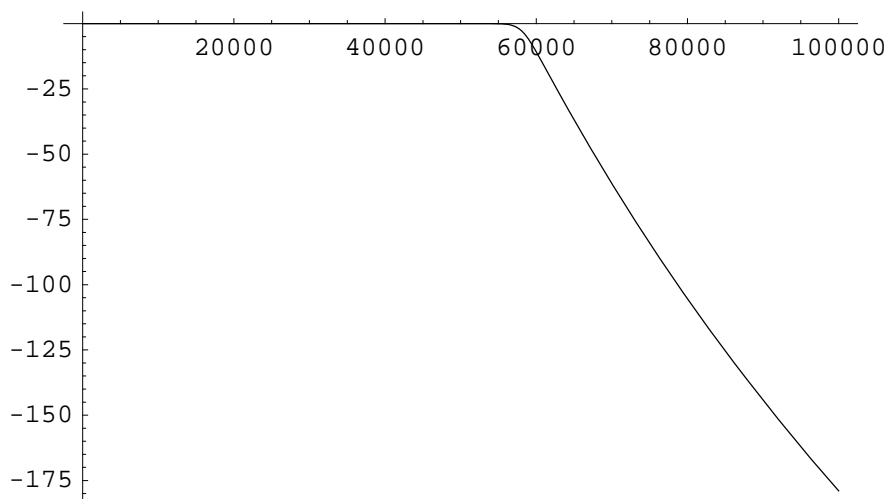$$\Omega_c = \Omega_p \, / \, \epsilon^{1/N} = 58141.4 \, \text{rad} \, / \, \text{sec}$$

and therefore its frequency response becomes

$$H \, (\Omega) = \frac{1}{1 + \epsilon^2 \, \left(\frac{\Omega}{\Omega_p}\right)^{2N}} = \frac{1}{1 + \left(\frac{\Omega}{(\Omega_p / \epsilon^{1/N})}\right)^{2N}}$$

The following plot shows the poles of the filter in the s-plane:

The frequency response (magnitude only) is shown next:

■ **Problem 4.15**

A 4-th order Butterworth filter has cut off frequency $\Omega_c = 200\,\pi$ `rad / sec`.

a) Determine the zeros and poles of the transfer function;

b) What would be its passband and stopband frequencies if we want 1dB ripple in the passband and 40dB attenuation in the stopband?

c) If we apply a Bilinear Transformation with sampling frequency `F`$_s$ = 1 `kHz`, determine the zeros and poles in the z-plane.

Solution

a) Recall that an `N` – `th` order Butterworth Filter has poles on a circle with radius
$\Omega_c = 200\,\pi$ `rad / sec` spaced by an angle of $360 / 2\,N = 360 / 8 = 45$ `degrees.` The poles are shown in the figure below
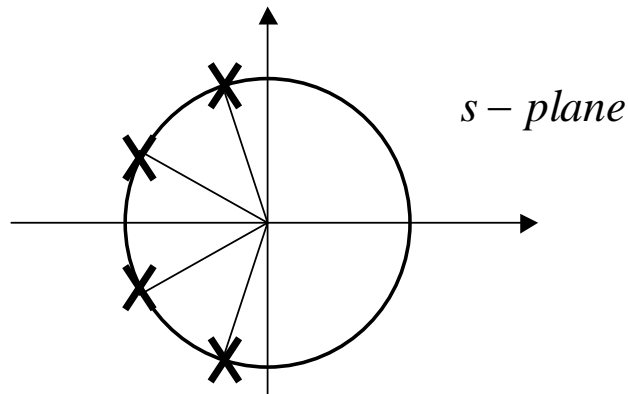
and they are given by $p_{1,4} = 200\,\pi e^{\pm j 5\pi/8}$ and $p_{2,3} = 200\,\pi e^{\pm j 7\pi/8}$. All zeros are at `s` = ∞.

b) From the frequency response $H\;(\Omega)\;=\;\sqrt{\dfrac{1}{1+\left(\frac{\Omega}{200\,\pi}\right)^8}}$ we solve for Ω_p and Ω_S , as

$$H\;(\Omega_p)\;=\;10^{-1/20} = 530.673\;\text{rad / sec}$$
$$H\;(\Omega_S)\;=\;10^{-40/20} = 1986.89\;\text{rad / sec}$$

c) Applying the formula for the Bilinear Transformation each pole is mapped as

$$z\;=\;-\;\frac{s+\frac{2}{T_s}}{s-\frac{2}{T_s}}$$

This yields poles in the z-plane at $0.673045 \pm j0.433479$ and $0.53675 \pm j0.143193$ and four zeros at $z = -1$.

■ Problem 4.16

Repeat Problem 4.14 using a Chebyshev filter. Which one would you choose if complexity is an issue?

Solution

The specifications of the prototype filter are the same, since they depend on the original specifications of the filter and on the bilinear transformation. Recall them here for convenience:

$$\omega_p = \frac{2\pi F_p}{F_s} = 1.1424 \text{ rad}$$
$$\omega_S = \frac{2\pi F_S}{F_s} = 1.2852 \text{ rad}$$

$$\Omega_p = F_s \, 2 \operatorname{Tan}\left(\frac{\omega_p}{2}\right) = 56554.2 \text{ rad / sec}$$
$$\Omega_S = F_s \, 2 \operatorname{Tan}\left(\frac{\omega_S}{2}\right) = 65876.0 \text{ rad / sec}$$

$$\delta_p = 10^{-1/20} = 0.944$$
$$\delta_S = 10^{-40/20} = 0.01$$

$$\epsilon = \sqrt{\left(\frac{1}{\delta_p}\right)^2 - 1} = 0.349$$

Then use the formulas for the Chebyshev Filter, from section 4.4, to obtain $N = 12$ (the complexity of the filter) and the frequency response as shown below.

The filter order is determined from the formula

$$N \geq \frac{\log\left(\frac{\sqrt{1-\delta_S^2\,(\epsilon^2+1)} + \sqrt{1-\delta_S^2}}{\epsilon\,\delta_S}\right)}{\log\left(\frac{\Omega_S}{\Omega_p} + \sqrt{\left(\frac{\Omega_S}{\Omega_p}\right)^2 - 1}\right)}$$

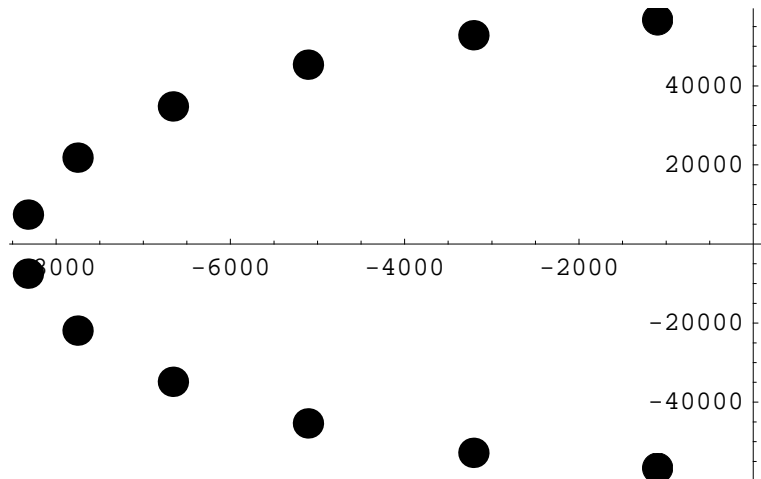which yields $N = 12$. Then the poles of the filter in the s-plane are computed as

$$\beta = \left(\frac{\sqrt{\epsilon^2 + 1} + 1}{\epsilon} \right)^{1/N} ;$$

$$r_1 = \frac{\Omega_p \, (\beta^2 + 1)}{2 \, \beta} ;$$

$$r_2 = \frac{\Omega_p \, (\beta^2 - 1)}{2 \, \beta} ;$$

$$p = r_2 \cos \left(\frac{\pi \, (2 \, k + 1)}{2 \, N} + \frac{\pi}{2} \right) + j \, r_1 \sin \left(\frac{\pi \, (2 \, k + 1)}{2 \, N} + \frac{\pi}{2} \right) , \quad k = 0, \ \dots, \ N - 1$$

The poles in the s-plane are shown below. Notice the two different scales for the Real and Imaginary axis

The plot of the frequency response is shown below. If you compare it with the Butterworth filter in Problem 4.14, notice that you obtain the same attenuation with a lower complexity ($N = 12$ for Chebyshev and $N = 38$ for Butterworth).
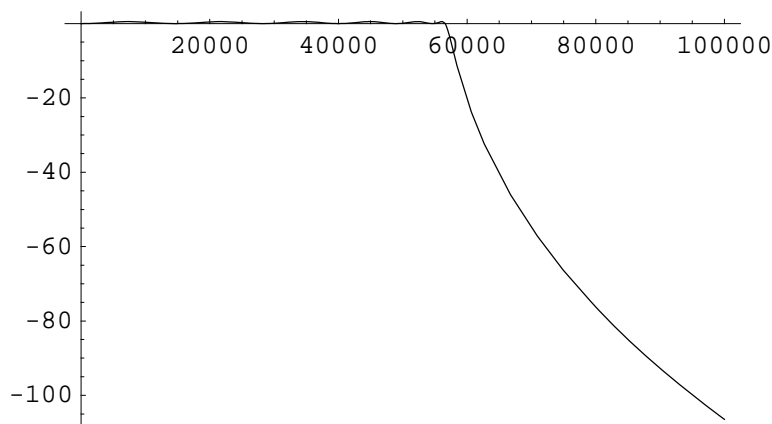
■ Problem 4.17

We want to implement the analog filter with transfer function

$$H(s) = \frac{2s+1}{s^2+s+1}$$

by a discrete time approximation, using the Bilinear Transformation. Let $F_s = 10\,Hz$ be the sampling frequency.

a) Determine zeros and poles of both the analog filter $H(s)$ and the discrete time implementation $H(z)$;

b) Determine the Linear Difference Equation of the discrete time implementation;

c) Plot the frequency responses of the digital filter $H(\omega)$ and the analog filter $H(\Omega)$. Verify that you can obtain one from the other by the appropriate frequency transformation.

Solution

a) The zeros of the analog system are $s = -1/2$, and $s = \infty$ (yes this is a zero too!). The poles are the solution of $s^2 + s + 1 = 0$ which yields $s = e^{\pm j2\pi/3}$

Applying the mapping s→z

$$z = -\frac{s+\frac{2}{T_s}}{s-\frac{2}{T_s}}$$

we can verify that the zeros are mapped as

$$s = -1/2 \quad \rightarrow \quad z = 0.9512$$
$$s = \infty \quad \rightarrow \quad z = -1$$

and the poles

$$s = e^{\pm j2\pi/3} \quad \rightarrow \quad z = 0.947743 \pm j\,0.0822827$$

b) Since we know the zeros and the poles we can write the transfer function to be

$$H(z) = K\,\frac{(z+1)\,(z-0.9512)}{(z-0.947743-j\,0.0822827)\,(z-0.947743+j\,0.0822827)}$$

with the constant K to be determined. Combining terms we obtain

$$H(z) = K\,\frac{z^2+0.0488\,z-0.9512}{z^2-1.8955\,z+0.9050}$$

We can determine K by matching the value at one frequency component, say at $s = 0 \rightarrow z = 1$:

$$H(s)\,|_{s=0} = \frac{2s+1}{s^2+s+1} = 1 \quad \rightarrow \quad H(z)\,|_{z=1} = K\,\frac{z^2+0.0488\,z-0.9512}{z^2-1.8955\,z+0.9050}\,|_{z=1} = 10.2737\,K$$

Equating the two we obtain $K = 1/10.2737 = 0.0973361$. Finally the transfer function

$$H(z) = 0.0973361 \, \frac{z^2 + 0.0488\, z - 0.9512}{z^2 - 1.8955\, z + 0.9050}$$

and the difference equation

$$y[n] - 1.8955\, y[n-1] + 0.9050\, y[n-2] =$$
$$0.09763361\, (x[n] + 0.0488\, x[n-1] - 0.9512\, x[n-2])$$

■ Problem 4.18

An integrator has transfer function

$$H(s) = \frac{1}{s}$$

a) Determine a discrete time implementation using Euler's approximation, with sampling frequency $F_s = 10$ Hz. Sketch the frequency response, and verify that it approximates the integrator for low frequencies;

b) Same as a), using the Bilinear Transformation.

Solution

a) By the Euler approximation

$$s = \frac{1 - z^{-1}}{T_s}$$

we obtain

$$H(z) = 0.1 \, \frac{z}{z-1}$$

The frequency response is

$$H(\omega) = 0.1 \, \frac{e^{j\omega}}{e^{j\omega} - 1} = 0.1 \, \frac{e^{j\omega}}{2\, j\, e^{-j\omega/2} \sin(\omega/2)}$$

For ω small, we can approximate $e^{j\omega} = 1$ and $\sin(\omega/2) = \omega/2$, and therefore we see that, for small ω the discrete time system behaves like an integrator with frequency response

$$H(\omega) = \frac{1}{j\omega\, F_s} = \frac{1}{j2\pi F} \Big|_{F = \omega F_s / 2\pi}$$

The following plot compares the frequency response $|H(\omega)|$ with the ideal integrator. We can see that in this case the Euler approximation gives a good approximation.
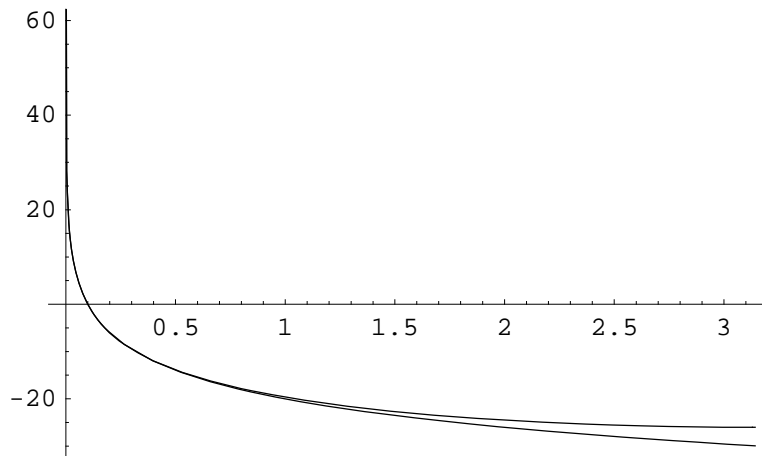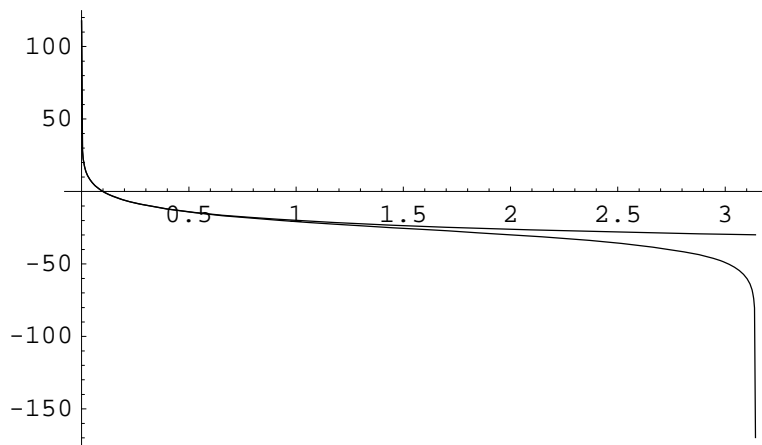
b) With the Bilinear Transformation we have

$$H(z) = \frac{T_s}{2} \frac{1+z^{-1}}{1-z^{-1}}$$

and the frequency response is

$$H(\omega) = \frac{0.1}{2} \frac{\cos\left(\frac{\omega}{2}\right)}{j\sin\left(\frac{\omega}{2}\right)}$$

The following plot compares the frequency response $|H(\omega)|$ with the ideal integrator. Even here we can see that in this case the Euler approximation gives a good approximation. Notice that at $z = -1$ (ie $\omega = \pi$), the frequency response is zero, and in the dB plot it goes to $-\infty$.

■ **Problem 4.19**

You want to design an analog Band Pass Filter which passes the frequencies in the interval

$$5 \text{ kHz} \le F \le 6 \text{ kHz}$$

with 1dB ripple in the passband. Let the the filter be Butterworth with order $N = 4$.

a) Determine the frequency transformation Low Pass to Band Pass you would use;

b) Determine the frequency response of the corresponding Low Pass Filter, together with its zeros and poles;

c) Determine the zeros and poles of the Band Pass Filter and its transfer function.

Solution

From the specifications we determine the lower and upper frequencies as

$$\Omega_L = 2\,\pi\,5000$$
$$\Omega_U = 2\,\pi\,6000$$

Then we choose a prototype Butterworth filter of order $N = 4$ and with arbitrary   cut off frequency, say

$$\Omega_C = 1$$

which has poles at

$$p_k = \Omega_C\,e^{\frac{i\,\pi}{2}}\,e^{\frac{i\,(2\,k+1)\,\pi}{2\times4}}, \quad k = 0,\,1,\,2,\,3$$

In order to apply the proper transformation (Low Pass to Band Pass)

$$q\,(s) = \frac{\Omega_C\,(s^2 + \Omega_L\,\Omega_U)}{s\,(\Omega_U - \Omega_L)}$$

we compute the poles of the bandpass filter from the equations

$$q\,(s) = p_k, \quad k = 0,\,1,\,2,\,3$$

Each equation is quadratic and it yields two solutions. As a total we have $2\times4 = 8$ poles for the bandpass filter which are given by

```
poles = {-1303.33 + 37418.3 i, -1101.14 - 31613.4 i,
   -3004.15 + 35515.3 i, -2800.76 - 33110.8 i, -3004.15 - 35515.3 i,
   -2800.76 + 33110.8 i, -1303.33 - 37418.3 i, -1101.14 + 31613.4 i}
```

The filter has also four zeros at $s = 0$, due to the fact that $q\,(0) = \infty$. From the zeros and the poles we determine the transfer function. The magnitude of the frequency response is shown below
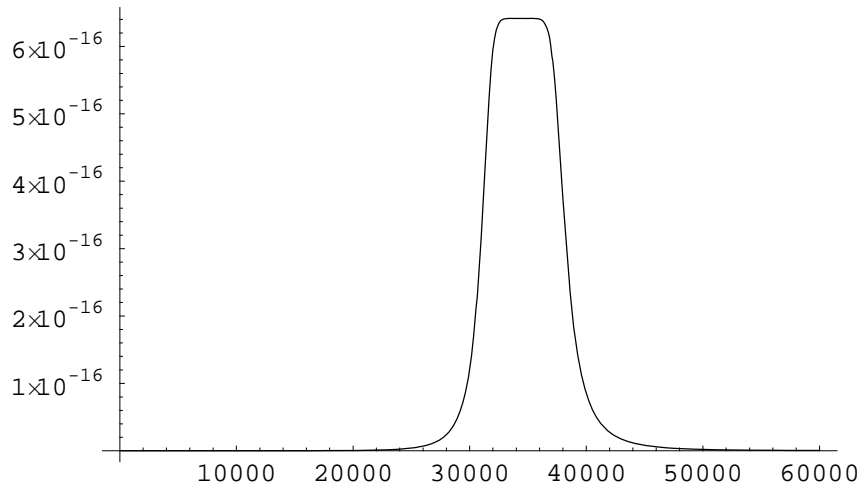
Problems Involving Material from Previous Chapters

■ **Problem 4.20**

Recall that, when we apply a digital filter to a continuous time signal, we need two analog filters: Anti-Aliasing and Reconstruction. Due to hardware constraints and cost, these two filters cannot have a very large order, and they represent a constraint in our design. Suppose we want to design a digital filter for a signal having bandwidth of 8 kHz, and we have to use a 5 pole Butterworth filter (commercially available for low cost) for both antialiasing and reconstruction filters. Also we want 1dB passband ripple and 50dB attenuation in the stopband.

a) What do you think would be the minimum sampling frequency we can have, with these two analog filters. (Hint: recall what is the passband and what is the stopband of the analog antialising and reconstruction filters in terms of signal bandwidth and sampling frequency);

b) The 5 pole Butterworth filter you buy, often is based on switched capacitor technology. This allows to select the cut off frequency Ω_c fairly easily, by adjusting the frequency of an oscillator. Given the sampling frequency you determined in question Q1, determine a suitable value for the cut off frequency Ω_c of the filter.

Solution

a) From the specifications of the problem, we have to pass all frequencies up to $8\ \mathtt{kHz}$. This yields

$$\Omega_{\mathtt{p}} = 2\ \pi\ \mathtt{8000}\ \text{rad/sec}$$

We want 1dB ripple in the passband. This gives a value of ϵ as

$$\epsilon = \mathtt{0.509}$$

Therefore the frequency response becomes

$$\mid\,\mathtt{H}\ (\Omega)\ \mid\ =\ \sqrt{\frac{1}{1+\epsilon^2\ \left(\frac{\Omega}{\Omega_{\mathtt{p}}}\right)^{10}}}$$

with ϵ and $\Omega_{\mathtt{p}}$ as given. Now we have to find the stopband frequency, from the requirement of 50dB attenuation. This leads to the equation

$$\mid\,\mathtt{H}\ (\Omega)\ \mid\ =\ \sqrt{\frac{1}{1\ +\ \epsilon^2\ \left(\frac{\Omega}{\Omega_{\mathtt{p}}}\right)^{10}}}\ =\ \mathtt{10}^{-\mathtt{50/20}}$$

which we solve for Ω. This yields the stopband frequency

$$\Omega_{\mathtt{S}}\ =\mathtt{181950.}\ \text{rad/sec}$$

or, in Hertz,

$$\mathtt{F}_{\mathtt{S}}\ =\ \frac{\Omega_{\mathtt{S}}}{2\ \pi}\ =\ \mathtt{28958.2\ Hz}$$

Now the sampling frequency $\mathtt{F}_{\mathtt{s}}$ has to be such that

$$\mathtt{F}_{\mathtt{S}}\ \le\ \mathtt{F}_{\mathtt{s}}\ -\ \mathtt{8000\ Hz}$$

which yields a sampling frequency

$$\mathtt{F}_{\mathtt{s}}\ \ge\ \mathtt{F}_{\mathtt{S}}\ +\ \mathtt{8000}\ =\ \mathtt{28958.2}\ +\ \mathtt{8000}\ =\ \mathtt{36.9582\ kHz}$$

b) The cut-off frequency $\Omega_{\mathtt{C}}$ of the filter is obtained by solving

$$\mid\,\mathtt{H}\ (\Omega)\ \mid\ =\ \sqrt{\frac{1}{1\ +\ \epsilon^2\ \left(\frac{\Omega}{\Omega_{\mathtt{p}}}\right)^{10}}}\ =\ \frac{1}{\sqrt{2}}$$

which yields $\Omega_{\mathtt{C}} = \mathtt{57537.7\ rad\ /\ sec.}$ The figure below illustrates the problem.
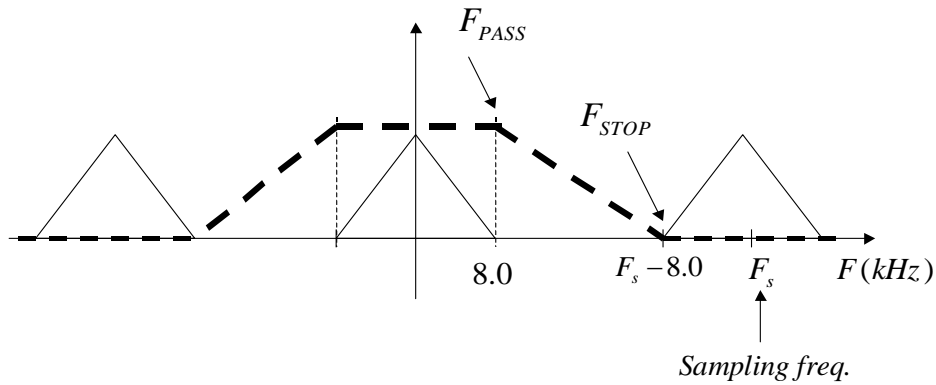
■ **Problem 4.21**

In your CD the data is sampled at 44.1kHz (CD quality), and we want to have a good sound quality up to 21kHz. If you had a to use an analog Butterworth filter as a reconstruction filter, what would be the order of your filter? Do you think you can reasonably build a filter with that complexity? (I do not think so either!) The chapter on Multi Rate DSP is going to show you how the CD technology solves this problem.

Solution

Since we want the filter to pass the signal and reject all frequencies above $F_s / 2$, we can see that passband and stopband frequencies become

$$\Omega_p = 2 \pi 21000 \text{ rad} / \text{sec}$$
$$\Omega_S = 2 \pi 44100 / 2 = 2 \pi 22050 \text{ rad} / \text{sec}$$

Assuming a 1dB passband ripple and 40dB attenuation in the stopband, this would yield a frequency response of the form

$$| H (\Omega) | = \sqrt{\frac{1}{1 + \epsilon^2 \left(\frac{\Omega}{\Omega_p}\right)^{2N}}}$$

with ϵ=0.509 and Ω_P as given. For an attenuation of 40dB we obtain $N > 108$ after solving

$$| H (\Omega_S) | = \sqrt{\frac{1}{1 + \epsilon^2 \left(\frac{\Omega_S}{\Omega_P}\right)^{2N}}} \leq 0.01$$

for the order N.