Chapter 1: Problem Solutions

Review of Signals and Systems

Signals

■ Problem 1.1

a) $x[n] = -0.5\,\delta[n+1] + \delta[n] + 0.5\,\delta[n-1] + \delta[n-2] - 0.8\,\delta[n-3]$

b) $x[n] = -0.5\,\delta[n+5] + \delta[n+4] + 0.5\,\delta[n+3] + \delta[n+2] - 0.8\,\delta[n+1]$

■ Problem 1.2

a) $I = e^{-1}$

b) $I = e^{-1}$

c) $I = 0$ since the interval of integration does not include the point $t = -1$, where the impulse is centered.

d) $I = 1$

e) $I = \cos^2(0.1\pi)$

f) $I = e$

g) let $\lambda = -\frac{1}{2}t$, then $t = -2\lambda$ and $dt = -2\,d\lambda$. Substitute in the integral to obtain

$$I = \int_{+\infty}^{-\infty} (-2\lambda)^2\,\delta\left(\lambda + \tfrac{1}{2}\right)(-2\,d\lambda) =$$

$$= \int_{-\infty}^{+\infty} 8\lambda^2\,\delta\left(\lambda + \tfrac{1}{2}\right) d\lambda = 2$$

h) let $\lambda = 3t$, then $t = \lambda/3$ and $dt = \left(\frac{1}{3}\right) d\lambda$. Then the integral becomes

$$I = \int_{-\infty}^{+\infty} e^{\lambda/3}\,\delta(\lambda - 1)\left(\tfrac{1}{3}\right) d\lambda = \frac{e^{1/3}}{3}$$

■ Problem 1.3

Amplitude $A = 2$,

Period $T_0 = 0.01\ \text{sec}$, then frequency $F_0 = \frac{1}{T_0} = 100\ \text{Hz} = 0.1\ \text{kHz}$

Phase $\alpha = 0$

Then the signal can be written as $x(t) = 2\cos(200\pi t)$.

■ Problem 1.4

a) Frequency $F_0 = 1 / T_0 = 1 / (3 \times 10^{-3}) = \frac{1}{3} \times 10^3\ \text{Hz}$. Then

$$x(t) = 2.5\cos\left(\frac{2}{3}\, 1000\,\pi t + 15^0\right)$$

b) Digital Frequency $\omega_0 = 2\pi F_0 / F_s = 2\pi / 6 = \pi / 3\ \text{rad.}$ Therefore the sampled sinusoid becomes

$$x[n] = 2.5\cos\left(\frac{\pi}{3} n + 15^0\right)$$

■ Problem 1.5

All sinusoids are distinct. In continuous time there is no ambiguity between frequency and signal.

■ Problem 1.6

First bring all frequencies within the interval $-\pi$ to π. This yields

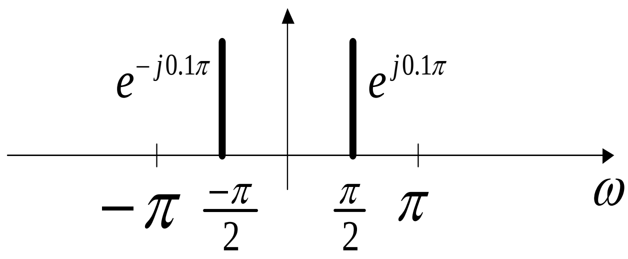
$$x_2[n] = 2\cos(1.5\pi n - 0.1\pi - 2\pi n) = 2\cos(-0.5\pi n - 0.1\pi)$$
$$= 2\cos(0.5\pi n + 0.1\pi)$$

and also

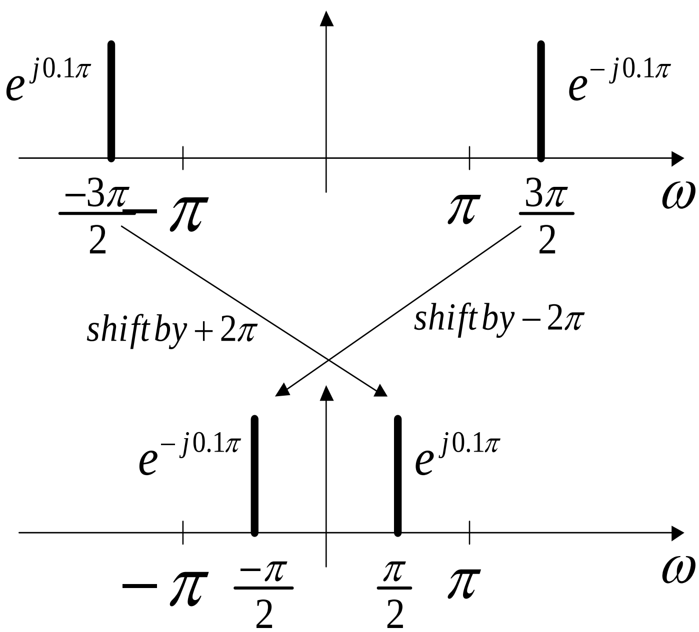
$$x_3[n] = 2\cos(1.5\pi n + 0.1\pi - 2\pi n) = 2\cos(-0.5\pi n + 0.1\pi)$$
$$= 2\cos(0.5\pi n - 0.1\pi)$$

Therefore we can see that $x_1[n] = x_2[n]$ and $x_3[n] = x_4[n]$.

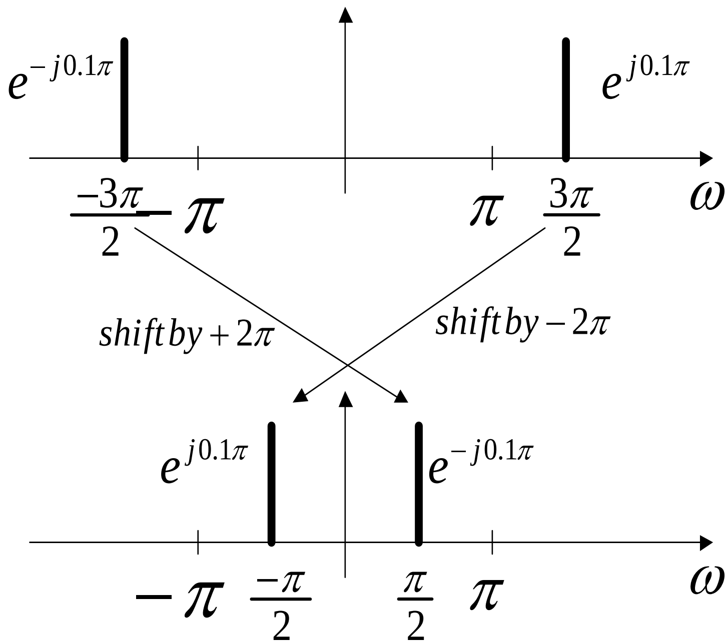
A different way of solving this problem is graphically. The frequency plots for all four signals are shown next. The key point is to understand that in discrete time, all frequencies within the interval $-\pi, \pi$ yield complex exponetials with distinct values:



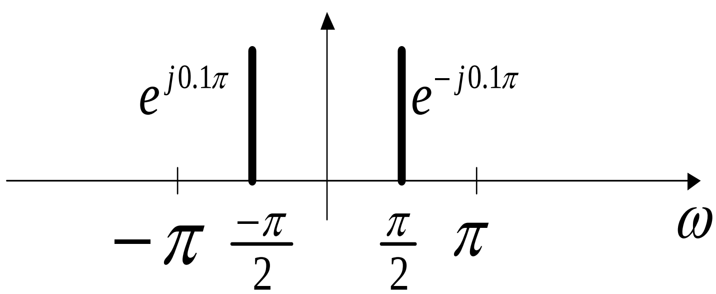
$x_1[n]$ Frequency Representation



$x_2[n]$ Frequency Representation



$x_3[n]$ Frequency Representation



$x_4[n]$ Frequency Representation

Again you see that $x_1[n]$ and $x_2[n]$ have the same representation, and the same for $x_3[n]$ and $x_4[n]$.

■ Problem 1.7

The sinusoid $x[n] = 3\cos(1.9\pi n + 0.2\pi)$ has the same samples, since

$$3\cos(0.1\pi n - 0.2\pi) = 3\cos(-0.1\pi n + 2\pi n + 0.2\pi)$$

■ Problem 1.8

a) since $15^0 = 15 * \pi / 180 = \pi / 12$, we can write

$$x(t) = \frac{3}{2}\left(e^{j\pi/12} e^{j100\pi t} + e^{-j\pi/12} e^{-j100\pi t}\right)$$

b) $x(t) = e^{-j0.1\pi} e^{j10\pi t} + e^{j0.1\pi} e^{-j10\pi t}$

c) $x(t) = -2.5\, j e^{-j0.2\pi} e^{j20\pi t} + 2.5\, j e^{j0.2\pi} e^{-j20\pi t}$

d) $x(t) = 5\, j e^{j1000\pi t} - 5\, j e^{-j1000\pi t}$

e) $x(t) = 1.5\, j e^{-j0.2\pi t} e^{j200\pi t} - 1.5\, j e^{j0.2\pi t} e^{-j200\pi t}$

■ Problem 1.9

a) since $2 + j = 2.2361\, e^{j0.4636}$, we can write $x(t) = 4.4722 \cos(100\pi t + 0.4636)$

b) since $1 + 2\, e^{j0.1\pi} = 2.9672\, e^{j0.2098}$, then $x(t) = 5.9344 \cos(10\pi t + 0.2098)$

c) the signal can be written as $x(t) = \frac{1}{2} \sin(5\pi t + 0.1\pi)\left(\sqrt{2}\, e^{j\frac{\pi}{4}} e^{j10\pi t} + \sqrt{2}\, e^{-j\frac{\pi}{4}} e^{-j10\pi t}\right)$ and therefore

$$x(t) = \frac{1}{\sqrt{2}} \sin(5\pi t + 0.1\pi) \cos\left(10\pi t + \frac{\pi}{4}\right)$$

■ Problem 1.10

a) $\dot{x}(t) = j20\pi \times x(t)$

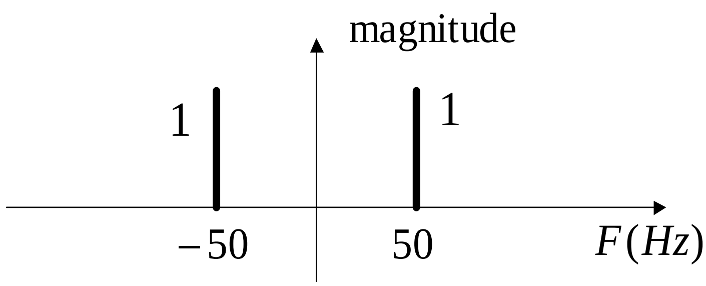
b) $\int x(t)\, dt = \frac{1}{j20\pi} x(t)$

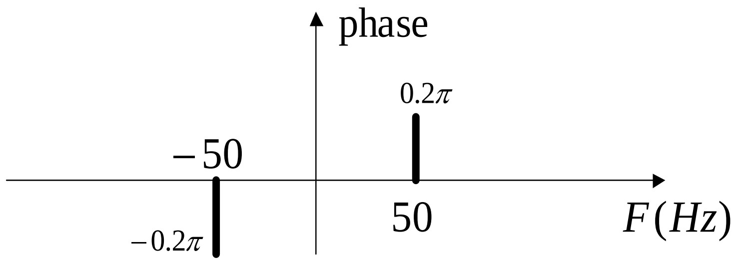
c) $x(t - 0.1) = e^{-j2\pi} x(t) = x(t)$

■ Problem 1.11

In order to solve this problem we have to decompose the signals into complex exponentials.

a) $x(t) = 2\cos(100\pi t + 0.2\pi) = e^{j0.2\pi} e^{j100\pi t} + e^{-j0.2\pi} e^{-j100\pi t}$

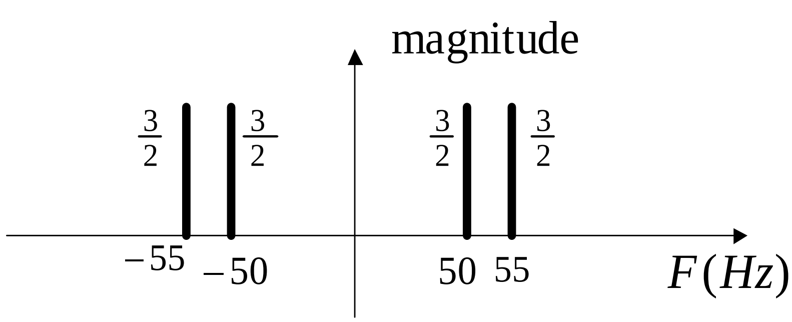


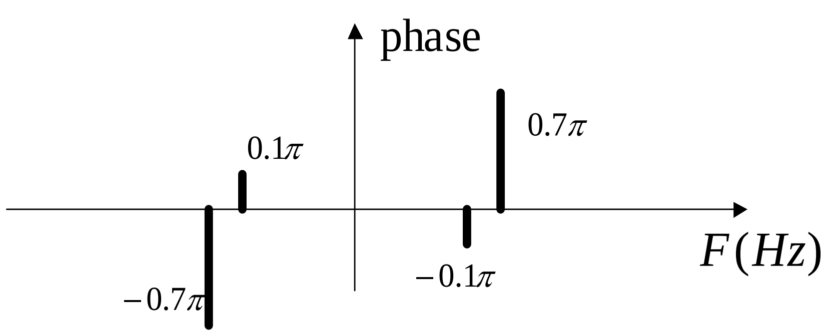


b) $x(t) = \frac{3}{2} e^{-j0.1\pi} e^{j100\pi t} + \frac{3}{2} e^{j0.1\pi} e^{-j100\pi t}$
$- \frac{3}{2j} e^{j0.2\pi} e^{j110\pi t} + \frac{3}{2j} e^{-j0.2\pi} e^{-j110\pi t}$

This becomes

$x(t) = \frac{3}{2} e^{-j0.1\pi} e^{j100\pi t} + \frac{3}{2} e^{j0.1\pi} e^{-j100\pi t}$
$\frac{3}{2} e^{j0.7\pi} e^{j110\pi t} + \frac{3}{2} e^{-j0.7\pi} e^{-j110\pi t}$

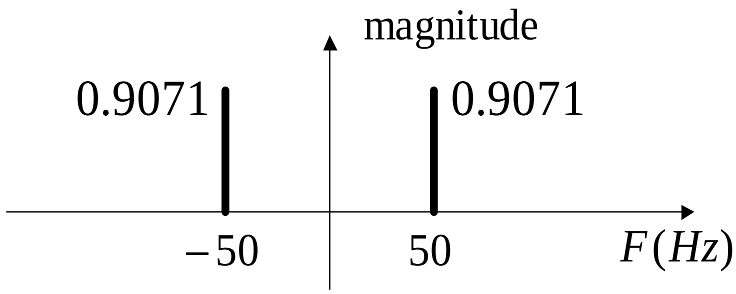


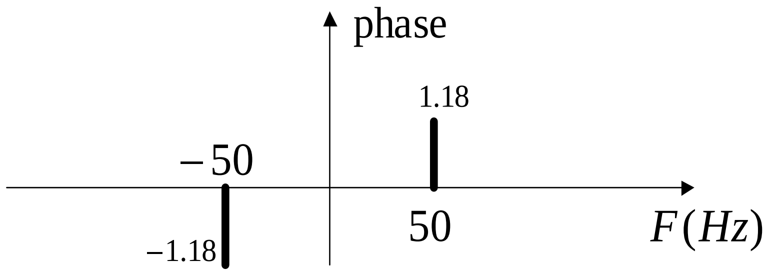


c) $x(t) = e^{-j0.2\pi} e^{j100\pi t} + e^{j0.2\pi} e^{-j100\pi t}$
$- \frac{3}{2j} e^{j0.1\pi} e^{j100\pi t} + \frac{3}{2j} e^{-j0.1\pi} e^{-j100\pi t}$

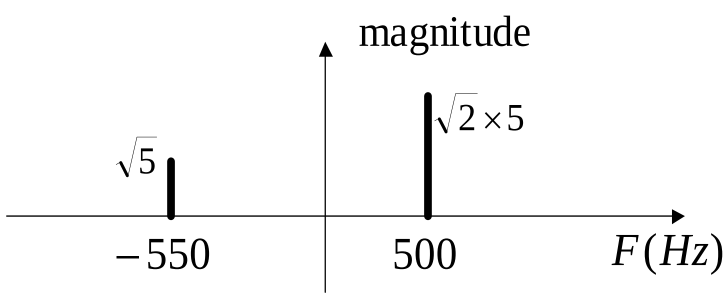
Combining terms we obtain

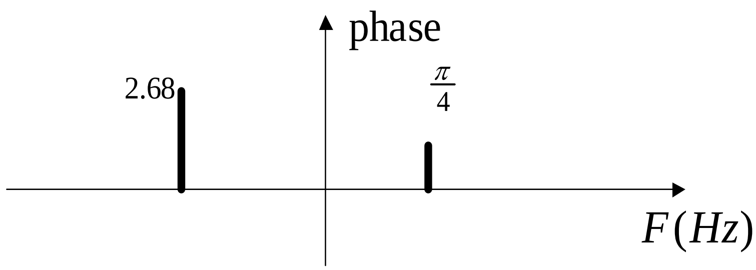
$$x(t) = 0.9071\, e^{j1.1800}\, e^{j100t} + 0.9071\, e^{-j1.1800}\, e^{-j100t}$$





d) $x(t) = \sqrt{2} \times 5\, e^{j\frac{\pi}{4}}\, e^{j1000\pi t} + \sqrt{5}\, e^{j2.67795}\, e^{-j1100\pi t}$

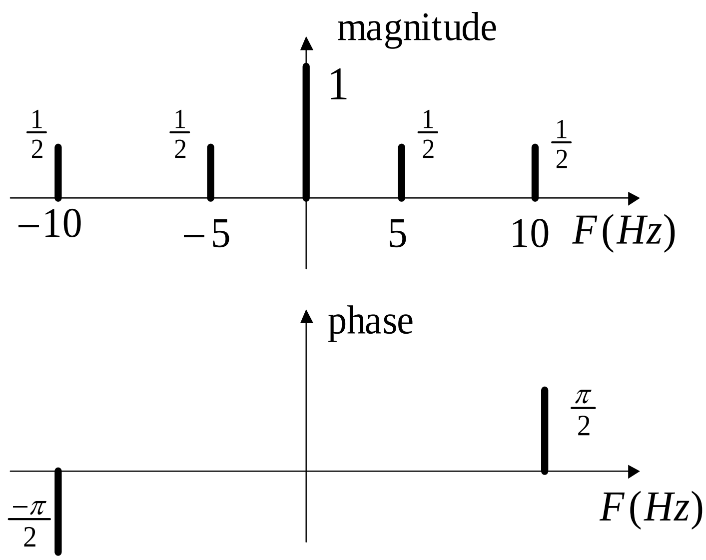




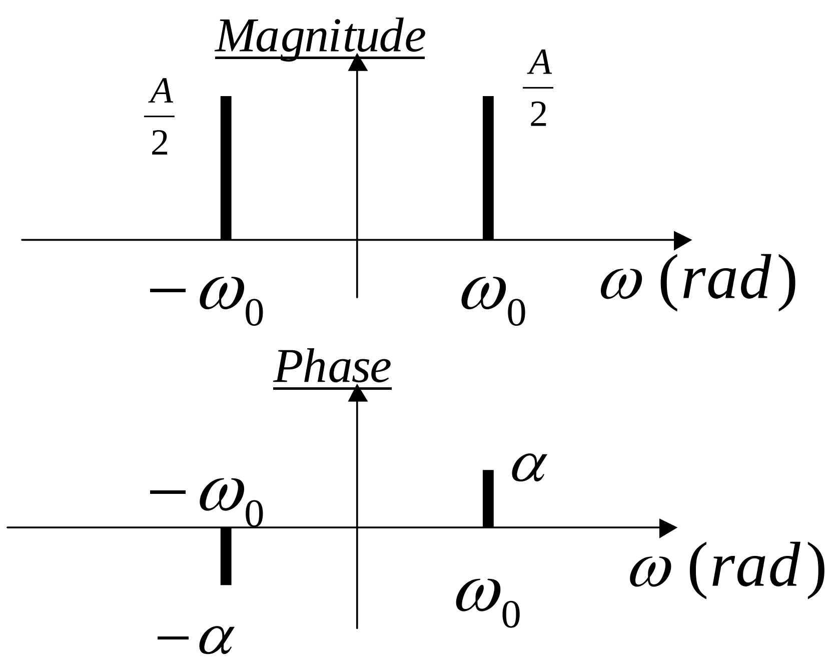
e) $x(t) = e^{j0t} + \frac{1}{2}\, e^{j10\pi t} + \frac{1}{2}\, e^{-j10\pi t} - \frac{1}{2j}\, e^{j20\pi t} + \frac{1}{2j}\, e^{-j20\pi t}$

This becomes

$$x(t) = e^{j0t} + \frac{1}{2}\, e^{j10\pi t} + \frac{1}{2}\, e^{-j10\pi t}\ \frac{1}{2}\, e^{j\frac{\pi}{2}}\, e^{j20\pi t} + \frac{1}{2}\, e^{-j\frac{\pi}{2}}\, e^{-j20\pi t}$$



■ Problem 1.12



Referring to the figure above:

a) $\omega_0 = \pi / 2$

b) $\omega_0 = 2\pi / 3$

c) $\omega_0 = \pi$

d) $\omega_0 = 0.75\,\pi$, since $1000\,\pi / 800 = 1.25\,\pi > \pi$. We want the frequency to be between $-\pi$ and π , we use the fact that this sinusoid has the same samples as $\omega_0 = 2\,\pi - 1.25\,\pi = 0.75\,\pi$.

e) $\omega_0 = 0$, since $1000\,\pi / 500 = 2\,\pi$ and therefore $\omega_0 = 2\,\pi - 2\,\pi = 0$.

■ Problem 1.13

a) Since $\omega_0 = 0.2\,\pi = 2\,\pi F_0 / 2000$ we solve for the frequency as $F_0 = 200$ Hz.

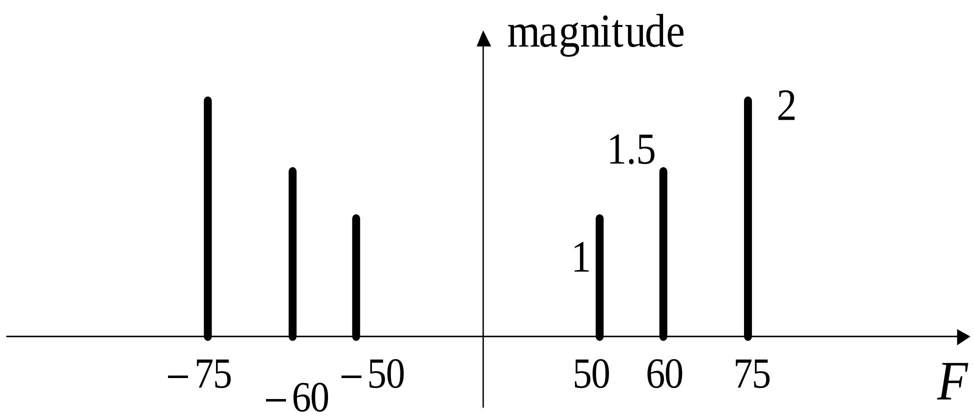
b) Sinusoids with frequencies $F_0 + F_S = 2.2$ kHz, or $F_S - F_0 = 1.8$ kHz have the same sample values.

■ Problem 1.14

a) Since

$$x(t) = e^{j100\pi t} + e^{-j100\pi t} + \frac{3}{2}\,e^{j\frac{\pi}{2}}\,e^{j120\pi t} + \frac{3}{2}\,e^{-j\frac{\pi}{2}}\,e^{-j120\pi t} + 2\,e^{j150\pi t} + 2\,e^{-j150\pi t}$$

we btain the frequency plot shown below.



b) The maximum frequency is $F_{MAX} = 75$ Hz. Therefore the sampling frequency has to be such that $F_S > 150$ Hz.

■ Problem 1.15

The bandwith is 4 kHz and therefore the sampling frequency has to be at least $F_S = 8$ kHz. This means that we need 8000 samples for every second of data, and therefore $16 \times 8000 = 128$ kbytes / sec. For one minute of data we need at least $60 \times 120 = 7200$ kbytes $= 7.2$ Mbytes to store the file.

Systems

■ Problem 1.16

If a property is not specified it is assumed Linear, Time Invariant, Causal, BIBO Stable.

b) Non Linear. In fact if you apply superposition:

$$x_1[n] \to S \to y_1[n] = 2\,x_1[n] - 1$$
$$x_2[n] \to S \to y_2[n] = 2\,x_2[n] - 1$$

and therefore

$$x[n] = x_1[n] + x_2[n] \to S \to y[n] = 2\,(x_1[n] + x_2[n]) - 1$$

which shows that the output is different from $y_1[n] + y_2[n] = 2\,(x_1[n] + x_2[n]) - 2$.

c) Time Varying, due to the time varying coefficient "t". Non BIBO Stable, since $x(t) = u(t)$, a Bounded Input , yields $y(t) = t u(t)$, a Non Bounded Output;

d) Time Varying,due to the coefficient e^{-t};

e) Non Linear. If $x(t) = x_1(t) + x_2(t)$ then the output is $y(t) = |x_1(t) + x_2(t)|$ different from $y_1(t) + y_2(t) = |x_1(t)| + |x_2(t)|$;

f) Non Causal, since the output at any time t depends on a future input at time $t+1$;

h) Non Causal since the output at time n depends on a future intput at time $n+1$;

i) Non Linear (due to the term x^2). Time Varying (due to the time varying coefficient n). Not BIBO Stable, since $y[n] = n u[n] + ...$, ie Non Bounded, when the input $x[n] = u[n]$, a Bounded input.

■ Problem 1.17

a) $y[n] = \sum_{k=-\infty}^{+\infty} u[k]\, 0.5^{n-k}\, u[n-k]$.

If $n \geq 0$:

$$y[n] = \sum_{k=0}^{n} 0.5^{n-k} = \sum_{k=0}^{n} 0.5^{k} = \frac{1-0.5^{n+1}}{1-0.5} = 2 - 0.5^{n}$$

If $n < 0$ then $u[k]\,u[n-k] = 0$ for all k, and therefore $y[n] = 0$.

Therefore $y[n] = (2 - 0.5^n)\, u[n]$.

b) $y[n] = \sum_{k=-\infty}^{+\infty} u[k-2]\, 0.5^{n-k}\, u[n-k]$.

If $n \geq 2$:

$$y[n] = \sum_{k=2}^{n} 0.5^{n-k} = \sum_{k=0}^{n} 0.5^{k} - 0.5^n - 0.5^{n-1} =$$
$$= \frac{1-0.5^{n+1}}{1-0.5} - 0.5^n - 0.5^{n-1} = 2\,(1 - 0.5^{n-2})$$

If $n < 2$ then $u[k-2]\, u[n-k] = 2$ for all k, and therefore $y[n] = 0$.

Therefore $y[n] = (2 - 0.5^{n-2})\, u[n-2]$.

c) $y[n] = \sum_{k=-\infty}^{+\infty} 0.8^k\, u[k]\, 0.5^{n-k}\, u[n-k]$.

If $n \geq 0$:

$$y[n] = \sum_{k=0}^{n} 0.8^k\, 0.5^{n-k} = 0.5^n \sum_{k=0}^{n} 1.6^k = 0.5^n\, \frac{1-1.6^{n+1}}{1-1.6} = 1.25 \times 0.5^n - 2 \times 0.8^n$$

If $n < 0$ then $u[k]\, u[n-k] = 0$ for all k, and therefore $y[n] = 0$.

Therefore $y[n] = (1.25 \times 0.5^n - 2 \times 0.8^n)\, u[n]$.

d) $y[n] = \sum_{k=-\infty}^{+\infty} 1.2^k\, u[-k]\, 0.5^{n-k}\, u[n-k]$

If $n < 0$ then $u[-k]\, u[n-k] = u[n-k]$. This implies

$$y[n] = \sum_{k=-\infty}^{n} 1.2^k\, 0.5^{n-k} = 0.5^n \sum_{k=-\infty}^{n} 2.4^k =$$
$$0.5^n \sum_{k=-n}^{+\infty} 2.4^{-k} =$$
$$0.5^n \left(\sum_{k=0}^{+\infty} 2.4^{-k} - \sum_{k=0}^{-n-1} 2.4^{-k}\right) =$$
$$0.5^n \left(\frac{1}{1-2.4^{-1}} - \frac{1-2.4^{n}}{1-2.4^{-1}}\right) =$$
$$= 1.7143 \times 1.2^n$$

If $n \geq 0$ then $u[-k]\, u[n-k] = u[-k]$. This implies

$$y[n] = \sum_{k=-\infty}^{0} 1.2^k\, 0.5^{n-k} = 0.5^n \sum_{k=-\infty}^{0} 2.4^k =$$
$$= 0.5^n \sum_{k=0}^{+\infty} 2.4^{-k} = 1.7143 \times 0.5^n$$

Finally we can write the answer $y[n] = 1.7143 \times 1.2^n\, u[-n-1] + 1.7143 \times 0.5^n\, u[n]$.

e) $y[n] = \sum_{k=-\infty}^{+\infty} e^{j0.2\pi n-k}\, 0.5^k\, u[k] = \left(\sum_{k=0}^{+\infty} 0.5^k \times e^{-j0.2\pi k}\right) e^{j0.2\pi n}$

Applying the geometric series we obtain

$$y[n] = \frac{1}{1-0.5\,e^{-j0.2\pi}}\,e^{j0.2\pi n} = 1.5059\,e^{j\,(0.2\pi n-0.4585)}$$

f) $y[n] = \sum_{k=-\infty}^{+\infty} e^{j0.2\pi n-k}\,u[n-k]\,0.5^k\,u[k] = \left(\sum_{k=0}^{n} 0.5^k \times e^{-j0.2\pi k}\right) e^{j0.2\pi n}$ if $n \geq 0$. Then, applying the geometric series we obtain

$$y[n] = \left(\frac{1-(0.5\,e^{-j0.2\pi})^{n+1}}{1-0.5\,e^{-j0.2\pi}}\right) e^{j0.2\pi n} =$$
$$= 1.5059\,(e^{j\,(0.2\pi n-0.4585)} - e^{-j0.2\pi}\,0.5^{n+1})$$

and $y[n] = 0$ when $n < 0$. Therefore

$$y[n] = 1.5059\,(e^{j\,(0.2\pi n-0.4585)} - e^{-j0.2\pi}\,0.5^{n+1})\,u[n]$$

g) Since $x[n] = e^{j0.2\pi n} + e^{-j0.2\pi n}$, using the solution to problem e) above we obtain

$$y[n] = 1.5059 \times (e^{j\,(0.2\pi n-0.4585} + e^{-j\,(0.2\pi n-0.4585}) =$$
$$= 3.0118\cos(0.2\pi n - 0.4585)$$

h) Since $x[n] = e^{j0.2\pi n}\,u[n] + e^{-j0.2\pi n}\,u[n]$, using the solution to problem f) above we obtain

$$y[n] = 3.0118\,(\cos(0.2\pi n - 0.4585) - \cos(0.2\pi)\,0.5^{n+1})\,u[n]$$
$$= 3.0118\,(\cos(0.2\pi n - 0.4585) - 1.2183 \times 0.5^n)\,u[n]$$

■ Problem 1.18

a) $h[n] = \delta[n] + \delta[n-1] + \delta[n-2]$

b) $y[n] = u[n] + u[n-1] + u[n-2]$

c) $y[n] = e^{j0.5\pi n} + e^{j0.5\pi n\,(n-1)} + e^{j0.5\pi n\,(n-2)}$. Therefore, since $e^{-j0.5\pi} = -j$, we can write

$$y[n] = (1 - j - 1)\,e^{j0.5\pi n} = e^{(j0.5\pi n - 0.5\pi)}$$

d) $y[n] = \cos(0.5\pi n - 0.5\pi)$

■ Problem 1.19

To determine the impulse response $\delta[n]$ substitute $x[n] = \delta[n]$:

a) $h[n] = 3\,\delta[n] + 2\,\delta[n-1] - \delta[n-2]$

b) $h[n]$ has to satisfy the recursion $h[n] = h[n-1] + \delta[n]$. Therefore:

if $n < 0$ then $h[n] = 0$;

if $n = 0$ then $h[0] = h[-1] + 1 = 1$, since $h[-1] = 0$;

if $n > 0$ then $h[n] = h[n-1]$ that is to say $h[n]$ is a constant, which must be equal to $h[0] = 1$.

Putting everything together you can verify that $h[n] = u[n]$;

c) $h[n] = 2\,\delta[n-2]$

For the next three problems recall that any signal $x[n]$ can be written as $x[n] = \sum_{k=-\infty}^{+\infty} x[k]\,\delta[n-k]$.

d) $y[n] = \sum_{k=0}^{+\infty} 0.5^k\,\delta[n-k] = 0.5^n\,u[n]$

e) $y[n] = \sum_{k=-\infty}^{+\infty} 0.5^{|k|}\,\delta[n-k] = 0.5^{|n|}$

f) $h[n] = \sum_{k=-\infty}^{+\infty} 0.5^{k-2}\,u[k-5]\,\delta[n-k-2] = \sum_{k=-\infty}^{+\infty} 0.5^k\,u[k-3]\,\delta[n-k]$. Therefore the impulse response is $h[n] = 0.5^n\,u[n-3]$.

■ Problem 1.20

We have to check whether or not $\sum_{-\infty}^{+\infty} \left| h[n] \right| < +\infty$. Recall that the geometric series $\sum_{n=0}^{+\infty} a^n < \infty$ if and only if $|a| < 1$.

a) $\sum_{n=0}^{+\infty} 0.5^n < +\infty$, then BIBO Stable;

b) $\sum_{n=0}^{+\infty} 0.5^n + \sum_{n=-\infty}^{-1} 0.5^{-n} = \sum_{n=0}^{+\infty} 0.5^n + \sum_{n=1}^{+\infty} 0.5^{-n} < +\infty$, then BIBO Stable;

c) $\sum_{n=0}^{+\infty} 0.5^n + \sum_{n=-\infty}^{-1} 0.5^n = \sum_{n=0}^{+\infty} 0.5^n + \sum_{n=1}^{+\infty} 0.5^{-n} = +\infty$ then it is NOT BIBO Stable

d) $\sum_{n=-\infty}^{0} 0.5^n = \sum_{n=0}^{+\infty} 2^n = +\infty$ then NOT BIBO Stable

e) $\sum_{n=1}^{+\infty} \frac{1}{n} = +\infty$, then NOT BIBO Stable

f) $\sum_{n=1}^{+\infty} \frac{1}{n^2} < +\infty$, then BIBO Stable

g) $\sum_{n=0}^{+\infty} 1 = +\infty$, then NOT BIBO Stable

h) $\sum_{n=-\infty}^{+\infty} \left| e^{j0.2\pi n} \right| = \sum_{n=-\infty}^{+\infty} 1 = +\infty$, then NOT BIBO Stable

i) $\sum_{n=0}^{+\infty} \left| e^{j0.2\pi n} \right| = \sum_{n=0}^{+\infty} 1 = +\infty$then NOT BIBO Stable

j) $\sum_{n=0}^{+\infty} \left| 0.8^n\,e^{j0.2\pi n} \right| = \sum_{n=0}^{+\infty} 0.8^n < +\infty$ then BIBO Stable

k) $\sum_{n=-\infty}^{+\infty} \left| 0.8^{|n|} \cos(0.2\pi n) \right| \le \sum_{n=0}^{+\infty} 0.8^n + \sum_{n=-\infty}^{-1} 0.8^{-n} = \sum_{n=0}^{+\infty} 0.8^n + \sum_{n=1}^{+\infty} 0.8^n < +\infty$ then BIBO Stable

l) $\sum_{n=-\infty}^{0} 2^n = \sum_{n=0}^{+\infty} 0.5^n < +\infty$ then BIBO Stable

m) $\sum_{n=0}^{+\infty} 2^n = +\infty$ then NOT BIBO Stable

z-Transforms

■ Problem 1.21

a) $X(z) = \sum_{n=0}^{+\infty} 0.5^n z^{-n} = \frac{1}{1-0.5z^{-1}} = \frac{z}{z-0.5}$, ROC $|0.5z^{-1}| < 1$, ie $|z| > 0.5$;

b) $X(z) = \sum_{n=-\infty}^{-1} 0.5^n z^{-n} = \sum_{n=1}^{+\infty} 0.5^{-n} z^n = \frac{1}{1-2z} - 1 = -\frac{z}{z-0.5} =$, ROC $|0.5^{-1} z| < 1$, ie $|z| < 0.5$;

c) $\sum_{n=-\infty}^{0} 0.5^n z^{-n} = \sum_{n=0}^{+\infty} 0.5^{-n} z^n = \frac{1}{1-2z}$, ROC $|z| < 0.5$;

d) $X(z) = \sum_{n=0}^{+\infty} e^{j0.4\pi n} z^{-n} = \frac{1}{1-e^{j0.4\pi} z^{-1}} = \frac{z}{z-e^{j0.4\pi}}$, ROC $|e^{j0.4\pi} z^{-1}| < 1$, ie $|z| > 1$;

e) $X(z) = \sum_{n=-\infty}^{-1} e^{j0.4\pi n} z^{-n} + \sum_{n=0}^{+\infty} e^{j0.4\pi n} z^{-n} = \sum_{n=1}^{+\infty} e^{-j0.4\pi n} z^n + \sum_{n=0}^{+\infty} e^{j0.4\pi n} z^{-n}$. The first series converges when $|z| < 1$ and the second when $|z| > 1$. Therefore there is no common ROC and $X(z)$ for this signal DOES NOT exist!

f) $X(z) = \sum_{n=-\infty}^{-1} e^{j0.4\pi n} z^{-n} = \sum_{n=1}^{+\infty} e^{-j0.4\pi n} z^n = \frac{1}{1-e^{-j0.4\pi} z} - 1 = -\frac{z}{z-e^{j0.4\pi}}$, ROC $|e^{-j0.4\pi} z| < 1$, ie $|z| < 1$;

g) $X(z) = \sum_{n=-\infty}^{-1} 0.5^{-n} z^{-n} + \sum_{n=0}^{+\infty} 0.5^n z^{-n} = \sum_{n=1}^{+\infty} 0.5^n z^n + \sum_{n=0}^{+\infty} 0.5^n z^{-n}$. See the two series:

$\sum_{n=1}^{+\infty} 0.5^n z^n = \frac{1}{1-0.5z} - 1 = -\frac{z}{z-2}$, when $|z| < 2$;

$\sum_{n=0}^{+\infty} 0.5^n z^{-n} = \frac{1}{1-0.5z^{-1}} = \frac{z}{z-0.5}$, when $|z|>0.5$.

Therefore the ROC has to be the intersection of the ROC's, which yields

$$X(z) = \frac{z}{z-0.5} - \frac{z}{z-2}, \text{ ROC: } 0.5 < |z| < 2$$

h) $X(z) = 1 + z^{-1} + z^{-2}$, ROC: $0 < |z|$;

i) $X(z) = z + z^2 + z^3$, ROC $|z| < +\infty$

■ Problem 1.22

Recall

$$Z\{a^n u[n]\} = \frac{z}{z-a} \quad \text{ROC: } |z| > |a|$$

$$Z\{a^n u[-n-1]\} = -\frac{z}{z-a} \quad \text{ROC: } |z| < |a|$$

a) $x[n] = 2 \times 0.5^{n-1} u[n-1]$, then $X(z) = 2 z^{-1} \frac{z}{z-0.5} = \frac{2}{z-0.5}$

b) $x[n] = 0.5^n u[-n-1] + \delta[n]$, then $X(z) = \frac{-z}{z-0.5} + 1 = \frac{-0.5}{z-0.5}$

d) $x[n] = (0.5 \times e^{j0.1\pi})^{''} u[n]$, then $X(z) = \frac{z}{z-0.5\, e^{j0.1\pi}}$

e) $x[n] = (-0.5)^n u[n]$, then $X(z) = \frac{z}{z+0.5}$

f) $x[n] = \frac{1}{2} (0.5\, e^{j0.1\pi})^{''} u[n] + \frac{1}{2} (0.5\, e^{j0.1\pi})^{''} u[n]$, then
$X(z) = \frac{1}{2} \frac{z}{z-0.5\, e^{j0.1\pi}} + \frac{1}{2} \frac{z}{z-0.5\, e^{-j0.1\pi}}$. Combining terms w eobtain

$$X(z) = \frac{z^2 - \cos(0.1\pi)\, z}{z^2 - \cos(0.1\pi) + 0.25}$$

g) $x[n] = 0.5^n u[n] + 0.5^{-n} u[-n-1]$, then $X(z) = \frac{z}{z-0.5} - \frac{z}{z-2} = \frac{-1.5\, z}{z^2 - 2.5\, z + 1}$, ROC: $0.5 < |z| < 2$

h) Using the result in g) above, $X(z) = z^{-1} \frac{-1.5\, z}{z^2 - 2.5\, z + 1} = \frac{-1.5}{z^2 - 2.5\, z + 1}$

■ Problem 1.23

a) $\frac{X(z)}{z} = \frac{2}{z} + \frac{0.5774\, e^{j2.6180}}{z - e^{j2.0944}} + \frac{0.5774\, e^{-j2.6180}}{z - e^{-j2.0944}}$ therefore

$$X(z) = 1 + 0.5774\, e^{j2.6100} \frac{z}{z - e^{j2.0944}} + 0.5774\, e^{-j2.6100} \frac{z}{z - e^{-j2.0944}}$$

Since $|z| > 1$ all terms are causal. Therefore

$$x[n] = \delta[n] + 0.5774\, e^{j(2.0944\, n + 2.6100)} u[n] + 0.5774\, e^{-j(2.0944\, n + 2.6100)} u[n]$$

Combining terms we obtain

$$x[n] = \delta[n] + 1.0548 \cos(2.0944\, n + 2.6100)\, u[n]$$

b) Same $X(z)$ as in the previous problem, but different ROC. All terms are noncausal, and therefore

$$x[n] = \delta[n] - 0.5774\, e^{j(2.0944\, n + 2.6100)} u[-n-1] - 0.5774\, e^{-j(2.0944\, n + 2.6100)} u[-n-1]$$

and, after combining terms

$$x[n] = \delta[n]\ 11.0548 \cos(2.0944\, n + 2.6100)\, u[-n-1]$$

c) $\frac{X(z)}{z} = \frac{0.5}{z} - \frac{2}{z-1} + \frac{1.5}{z-2}$, therefore $X(z) = 0.5 - 2\frac{z}{z-1} + 1.5\frac{z}{z-2}$. Form the ROC all terms are causal. Thereofore

$$x[n] = 0.5\,\delta[n] - 2\,u[n] + 1.5 \times 2^n\, u[n]$$

d) Same $X(z)$ with different ROC. In this case all terms are anticausal,

$$x[n] = 0.5\,\delta[n] + 2\,u[-n-1] - 1.5 \times 2^n\, u[-n-1]$$

e) Same $X(z)$ with different ROC. In this case $\frac{z}{z-1}$ yields a causal term, and $\frac{z}{z-2}$ yields an anti-causal term. Therefore

$$x[n] = 0.5\,\delta[n] - 2\,u[n] - 1.5 \times 2^n\, u[-n-1]$$

f) $\frac{X(z)}{z} = \frac{1}{z^2+1} = \frac{0.5\,j}{z+j} - \frac{0.5\,j}{z-j}$ therefore $X(z) = 0.5\,j\,\frac{z}{z+j} - 0.5\,j\,\frac{z}{z-j}$. From the ROC $|z| > 1$ all terms are causal, therefore

$$x[n] = 0.5\,j e^{j\frac{\pi}{2}n}\, u[n] - 0.5\,j e^{-j\frac{\pi}{2}n}\, u[n]$$

Combining terms:

$$x[n] = \cos\left(\frac{\pi}{2}n + \frac{\pi}{2}\right) u[n]$$

g) Same $X(z)$ but all anticausal:

$$x[n] = -0.5\,j e^{j\frac{\pi}{2}n}\, u[-n-1] + 0.5\,j e^{-j\frac{\pi}{2}n}\, u[-n-1]$$

Combine terms:

$$x[n] = \cos\left(\frac{\pi}{2}n - \frac{\pi}{2}\right) u[-n-1]$$

■ Problem 1.24

a) $H(z) = \frac{1}{1-0.5\,z^{-1}} = \frac{z}{z-0.5}$ Since it is causal, the impulse response is $h[n] = 0.5^n\, u[n]$

b) $H(z) = \frac{1}{1-0.5\,z} = \frac{-2}{z-2}$. Since $\frac{H(z)}{z} = \frac{-2}{z(z-2)} = \frac{1}{z} - \frac{1}{z-2}$ then $H(z) = 1 - \frac{z}{z-2}$. The impulse response is anticausal, form the difference equation, therefore the impulse response is $h[n] = \delta[n] + 2^n\, u[-n-1]$.

c) $H(z) = \frac{1+z^{-1}+z^{-2}}{1-z^{-2}} = \frac{z^2+z+1}{z^2-1}$. Therefore $\frac{H(z)}{z} = \frac{z^2+z+1}{z(z^2-1)} = \frac{-1}{z} + \frac{1.5}{z-1} + \frac{0.5}{z+1}$ which leads to the decomposition $H(z) = -1 + 1.5\,\frac{z}{z-1} + 0.5\,\frac{z}{z+1}$. Since the difference equation is causal, then the impulse response is causal, which leads to $h[n] = -\delta[n] + 1.5\,u[n] + 0.5\,(-1)^n\, u[n]$.

d) $H(z) = \frac{1}{1-z^{-1}-z^{-2}} = \frac{z^2}{z^2-z-1}$, then $\frac{H(z)}{z} = \frac{z}{z^2-z-1} = \frac{0.2764}{z+0.6180} + \frac{0.7236}{z-1.6180}$ and therefore $H(z) = 0.2764\,\frac{z}{z+0.6180} + 0.7236\,\frac{z}{z-1.6180}$. The difference equation is causal and therefore the impulse response is $h[n] = 0.2764\,(-0.6180)^n\, u[n] + 0.7236 \times 1.6180^n\, u[n]$

■ Problem 1.25

a) The system is causal with transfer function $H(z) = \frac{z}{z-0.5}$, with ROC $|z| > 0.5$. The unit circle is in the ROC and therefore the system is BIBO stable. The impulse response is $h[n] = 0.5^n u[n]$.

b) Now $H(z) = -2 \frac{z}{1-2z} = \frac{z}{z-0.5}$, same as before. But since it is clearly anticausal (the output depends from future values only) the ROC is $|z| < 0.5$ and the impulse response $h[n] = -0.5^n u[-n-1]$. The system is NOT BIBO Stable, since the unit circle is not within the ROC.

c) If the input is $x[n] = u[n]$, its z-Tranform is $X(z) = \frac{z}{z-1}$ with ROC $|z| > 1$.

When the system is causal then

$$Y(z) = \frac{z}{z-0.5} \frac{z}{z-1} = -\frac{z}{z-0.5} + 2\frac{z}{z-1}, \text{ ROC } |z| > 1$$

therefore $y[n] = -0.5^n u[n] + 2u[n]$;

When the system is anticausal, the z-Transform is the same, but with different ROC given by the intersection of $|z| > 1$ and $|z| < 0.5$, which is empty. Therefore in this case the output $y[n]$ does not have a z-Transform.

■ Problem 1.26

a) The difference equation can be implemented in a number of ways, so we have not enough information to assess causality or the ROC of the transfer function.

b) The transfer function is $H(z) = \frac{z^{-1}}{1-2.5z^{-1}+z^{-2}} = \frac{z}{z^2-2.5z+1} = \frac{z}{(z-2)(z-0.5)}$. The possible ROC's are the following:

$|z| > 2$, the system is causal , and NOT BIBO stable, since the unit circle is not in the ROC;

$0.5 < |z| < 2$, the system is non causal, but BIBO Stable, since the unit circle is in the ROC;

$|z| < 0.5$, the sytem is anticausal, NOT BIBO Stable since the unit circle is not in the ROC.

c) The step response has transfer function
$Y(z) = \frac{z^2}{(z-2)(z-0.5)(z-1)} = 0.6667\frac{z}{z-0.5} - 2\frac{z}{z-1} + 1.333\frac{z}{z-2}$

When the sytem is causal the ROC is $|z| > 2$, then
$y[n] = 0.66670.5^n u[n] - 2u[n] + 1.333 \times 2^n u[n]$;

When the system is non casual, the ROC is $1 < |z| < 2$, then

$$y[n] = 0.66670.5^n u[n] - 2u[n] - 1.333 \times 2^n u[-n-1];$$

When the system is anticausal, the ROC of the transfer function ($0.5 < |z|$) and the ROC of the input ($|z| > 1$) have empty intersection, and therefore the ouput hs no z-Transform.

Frequency Response

■ Problem 1.27

The transfer function of the system is

$$H(z) = \frac{1}{1-0.5z^{-1}} = \frac{z}{z-0.5}$$

Therefore the system is stable (pole at $z = 0.5$ inside the unit circle) and its frequency response is

$$H(\omega) = \frac{e^{j\omega}}{e^{j\omega}-0.5}$$

a) the input is a complex exponential with frequency $\omega = 0.2\pi$ and therefore the corresponding output is $y[n] = H(0.2\pi)\, e^{j0.2\pi n}$. Substituing for the frequency response $H(\omega)$ we obtain $y[n] = 1.5059\, e^{-j4585}\, e^{j0.2\pi n} = 1.5059\, e^{j(0.2\pi n-0.4585)}$.

b) Since $x[n] = e^{j0.1\pi}\, e^{j0.5\pi n} + e^{-j0.1\pi}\, e^{-j0.5\pi n}$ the response is $y[n] = H(0.5\pi)\, e^{j0.1\pi}\, e^{j0.5\pi n} + H(-0.5\pi)\, e^{-j0.1\pi}\, e^{-j0.5\pi n}$. Substituing for $H(\omega)$ this yields

$$y[n] = 0.8944\, e^{-j0.4636}\, e^{j0.1\pi}\, e^{j0.5\pi n} + 0.8944\, e^{j0.4636}\, e^{-j0.1\pi}\, e^{-j0.5\pi n}$$
$$= 0.4472 \cos(0.5\pi n - 0.1495)$$

c) Since $H(0) = \frac{1}{1-0.5} = 2$ and $H(0.3\pi) = 1.2289\, e^{-j5202}$ then $y[n] = 1 \times 2 + 5 \times 1.2289 \cos(0.3\pi n - 0.5\pi - 0.5202)$ which yields $y[n] = 2 + 6.1443 \cos(0.3\pi n - 2.0910)$;

d) $H(0) = 2$ and $H(\pi) = 0.6667$, then $y[n] = 2 + 0.6667\,(-1)^n$;

e) $H(0.2\pi) = 1.5059\, e^{-j4585}$, therefore
$y[n] = 1.5059 \sin(0.2\pi n - 0.4585) + 1.5059 \cos(0.2\pi n - 0.4585)$;

■ Problem 1.28

The system has transfer function $H(z) = Z\{h[n]\} = Z\{0.8^n\, u[n] + 1.25^n\, u[-n-1]\}$. This yields

$$H(z) = \frac{z}{z-0.8} - \frac{z}{z-1.25}, \quad \text{ROC: } 0.8 < |z| < 1.25$$

The unit circle is within the ROC therefore the system is stable and it has a Frequency Response $H(\omega) = \frac{e^{j\omega}}{e^{j\omega}-0.8} - \frac{e^{j\omega}}{e^{j\omega}-1.25}$.

a) $H(0.2\pi) = 1.0417$, then $y[n] = 1.0417\, e^{j0.2\pi n}$;

b) $H(0.5\pi) = 0.2195$, then $y[n] = 0.4390 \cos(0.5\pi n + 0.1\pi)$;

c) $H(0) = 9$, $H(0.3\pi) = 0.5146$, then $y[n] = 9 + 2.5731 \cos(0.3\pi n - 0.5\pi)$;

d) $H(0) = 9$, $H(\pi) = 0.1111$, therefore $y[n] = 9 + 0.1111\,(-1)^n$;

e) $H(0.2) = 1.0417$, then $y[n] = 1.0417 \sin(0.2\pi n) + 1.0417 \cos(0.2\pi n)$

■ Problem 1.29

The system ha stransfer function $H(z) = \frac{-2z}{1-2z} = \frac{z}{z-0.5}$ and it is anticausal.Therefore the ROC is given by $|z| < 0.5$ and the unit circle is NOT in the ROC. As a consequence the system is UNSTABLE and the steady state frequency response does not exist.

■ Problem 1.30

The transfer function is $H(z) = \frac{z}{1+z^{-1}+z^{-2}}$, the system is causal and the poles are on the unit circle. Therefore ROC: $|z| > 1$, the unit circle is NOT in the ROC and the system is NOT stable. Again the frequency response does not exist!

■ Problem 1.31

a) From the graph $H(0.1\pi) = 6.5\, e^{-j}$. Therefore the output is

$$y[n] = 13 \cos(0.1\pi n - 1)$$

b) From the graph we obtain $H(0) = 8$, $H(0.5\pi) = 1.8\, e^{j\pi}$ and $H(\pi) = 0$. Therefore the output is

$$y[n] = 16 + 1.8 \cos(0.5\pi n + \pi)$$

Time, z-, and Frequency Domains

■ Problem 1.32

a) First we have to determine the digital frequency of the disturbance, as $\omega_0 = 2\pi F_0 / F_s = 0.4\pi$. Therefore the filter must have at least two zeros at $z_{1,2} = e^{\pm j0.4\pi}$, and its transfer function becomes

$$H(z) = b_0 + b_1 z^{-1} + b_2 z^{-2} = \frac{b_0 z^2 + b_1 z + b_2}{z^2}$$
$$= b_0 \frac{(z - e^{j0.4\pi})(z - e^{-j0.4\pi})}{z^2} = b_0 (1 - 0.6180 z^{-1} + z^{-2})$$

The poles are both $z = 0$.

b) With the constraint that $H(0) = 1$, ie that $H(z) = 1$ when $z = 1$ we obtain an ewuation to solve for the coefficient b_0. Therefore

$$H(1) = b_0 (1 - 0.6180 + 1) = 1$$

which yields $b_0 = 0.7236$. Therefore the difference equation of the filter becomes

$$y[n] = 0.7236\, x[n] - 0.4472\, x[n-1] + 0.7236\, x[n]$$

■ Problem 1.33

If we keep the same zeros at $z_{1,2} = e^{\pm j0.4\pi}$, we add two poles close to the zeros as, say, $p_{1,2} = 0.95\, e^{\pm j0.4\pi}$, then the frequency response becomes more selective. In this way the Transfer Function becomes

$$H(z) = b_0 \frac{(z - z_1)(z - z_2)}{(z - p_1)(z - p_2)} = b_0 \frac{z^2 - 0.6180 z + 1}{z^2 - 0.5871 z + 0.9025}$$

In order to have the Frequency Response with $H(0) = 1$, we need to solve for the coefficient b_0 from the equation

$$b_0 \frac{1 - 0.6180 + 1}{1 - 0.5871 + 0.9025} = 1$$

which yields $b_0 = 0.9518$. Finally the difference equation for the filter becomes

$$y[n] = 0.5871\, y[n-1] - 0.9025\, y[n-2] +$$
$$0.9518\, x[n] - 0.5883\, x[n-1] + 0.9518\, x[n-2]$$

■ Problem 1.34

Since $x[n]$ is periodic with period N, by defintion $x[n] = x[n-N]$ for all n, and therefore $x[n] - x[n-N] = 0$ for all n. Therefore in this case the difference equation becomes

$$y[n] = -a_1 y[n-1] - a_2 y[n-2] + 0$$

and if the system is stable $y[n] \to 0$ from any initial condition.

a) When $N = 4$ the filter has Transfer Function

$$H(z) = b_0 \frac{z^4 - 1}{z^2 + a_1 z + a_2}$$

The zeros must be the solutions of the equation $z^4 = 1$, ie $z^4 = e^{jk2\pi}$ with k integer. This yields $z = e^{jk\pi/2}$, for $k = 0, 1, 2, 3$, Therefore the four zeros are $z = 1,\ j,\ -1,\ -j$

b) Choose the poles close to the zeros, inside the unit circle, at $z = \rho,\ j\rho,\ -\rho,\ -j\rho$, with $0 < \rho < 1$, close to one.

■ Problem 1.35

The digital frequencies of the disturbance are $\omega_1 = 2\pi 100/1000 = 0.2\pi$, $\omega_2 = 2\pi 150/1000 = 0.3\pi$, $\omega_3 = 2\pi 200/1000 = 0.4\pi$.

a) zeros at $z_{1,2} = e^{\pm j0.2\pi}$, $z_{3,4} = e^{\pm j0.3\pi}$, $z_{5,6} = e^{\pm j0.4\pi}$, and poles at $p_{1,2} = \rho e^{\pm j0.2\pi}$, $p_{3,4} = \rho e^{\pm j0.3\pi}$, $p_{5,6} = \rho e^{\pm j0.4\pi}$ with ρ again positive, close to one. Say $\rho = 0.9$.

b) From the zeros and the poles the transfer function becomes

$$H(z) = b_0 \frac{z^6 - 3.4116 z^5 + 6.6287 z^4 - 7.9988 z^3 + 6.6287 z^2 - 3.4116 z + 1}{z^6 - 3.0705 z^5 + 5.3692 z^4 - 5.8312 z^3 + 4.3491 z^2 - 2.0145 z + 0.5314}$$

If we want the DC gain to be one, ie $H(\omega) = 1$ when $\omega = 0$, then $b_0 = 0.7664$;

c) The difference equation then becomes

$$\begin{aligned} y[n] = {} & 3.0705\, y[n-1] - 5.3692\, y[n-2] + 5.8312\, y[n-3] - \\ & 4.3491\, y[n-4] + 2.0145\, y[n-5] - 0.5314\, y[n-6] + \\ & + 0.7664\, (x[n] - 3.4116\, x[n-1] + 6.6287\, x[n-2] - \\ & 7.9988\, x[n-3] + 6.6287\, x[n-4] - 3.4116\, x[n-5] + x[n-6]) \end{aligned}$$

■ Problem 1.36

The digital frequency of the signal we want to enhance is $\omega_0 = 2\pi \times 1.5/5 = 0.6\pi$ radians.

a) The poles are $p_{1,2} = \rho e^{\pm j0.6\pi}$ with $0 < \rho < 1$, close to one. We can choose any zeros we like, say $z_{1,2} = \pm 1$ to attenuate the low frequencies ($\omega = 0$) and the high frequencies ($\omega = \pi$).

b) Choose (say) $\rho = 0.9$, and the transfer function becomes

$$H(z) = b_0 \frac{z^2 - 1}{z^2 - 0.5562 z + 0.81}$$

We can choose b_0 to satisfy any normalization we like. For example, if we want the frequency response at frequency ω_0 to have unit magnitude, we impose $|H(0.6\pi)| = 1$ which yields $b_0 = 0.0950$.

c) The difference equation becomes

$$y[n] = 0.5562\, y[n-1] - 0.81\, y[n-2] + 0.0950\, x[n] - 0.0950\, x[n-2]$$

■ Problem 1.37

a) The filter has zeros at $z_{1,2} = \pm j$, poles at $p_{1,2} = \pm j0.9$. Its frequency response is as shown;

b) Notch Filter;

c) $y[n] = -0.81\, y[n-2] + x[n] + x[n-2]$.

Fourier Analysis of Discrete Time Signals

■ Problem 1.38

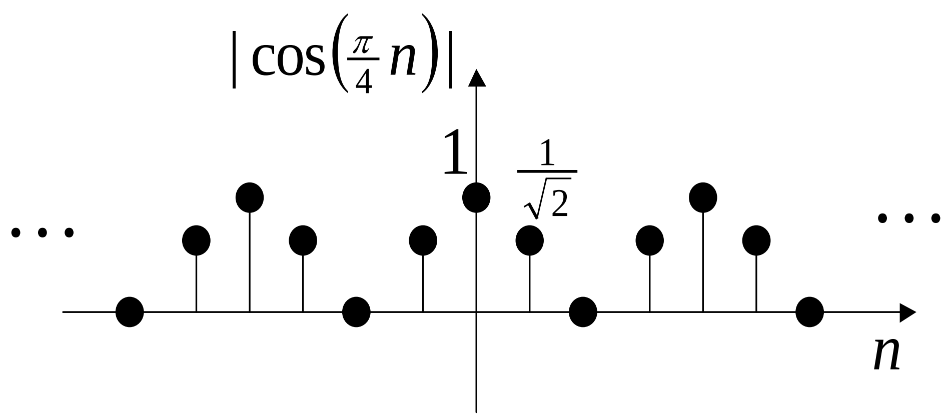
a) Period $N = 3$, therefore $X[k] = \sum_{n=0}^{2} x[n]\, e^{-j \frac{2\pi}{3} kn}$, for $k = 0, 1, 2$. Just substitute for one period as $x[0] = 1,\ x[1] = 2,\ x[2] = 3$ to obtain
$X[0] = 6.000,\ X[1] = -1.5 + j0.8660,\ X[2] = -1.5 - j0.8660$.

b) Period $N = 2$, then $X[k] = \sum_{n=0}^{1} x[n]\, e^{-j \frac{2\pi}{2} kn} = x[0] + (-1)^k x[1]$, for $k = 0, 1$. Therefore $X[0] = 1 - 1 = 0,\ X[1] = 1 - (-1) = 2$.

c) Period $N = 4$, then $X[k] = \sum_{n=0}^{3} x[n]\, e^{-j \frac{2\pi}{4} kn} = \sum_{n=0}^{3} x[n]\, (-j)^{kn}$, for $k = 0, 1, 2, 3$. Substitute the numerical values of the sequence to obtain $X[0] = X[2] = 0,\ X[1] = X[3] = 2$ or, in vector form,

$$X = [0, 2, 0, 2]$$

d) The signal $x[n] = |\cos(\frac{\pi}{4} n)|$ is shown below.



Plot of $x[n] = |\cos(\frac{\pi}{4} n)|$

The period can be seen by inspection as $N = 4$ and therefore the expansion is of the form $x[n] = \frac{1}{4} \sum_{k=0}^{3} X[k]\, e^{-j \frac{\pi}{2} kn}$, with the DFS being

$$X[k] = \sum_{n=0}^{3} x[n]\, j^{kn}, \text{ for } k = 0, 1, 2, 3$$

since $e^{j \frac{\pi}{2}} = j$. Therefore

$$X = [2.4142,\ 1,\ -0.4142,\ 1]$$

■ Problem 1.39

The transfer function of the system is $H(z) = \frac{0.2 z}{z-0.8}$, by inspection. The system is causal and stable therefore we can define the frequency response as

$$H(\omega) = \frac{0.2 e^{j\omega}}{e^{j\omega}-0.8}$$

a) The DFS of the signal is $X = [6,\ -1.5 + j0.866,\ -1.5 - j0.866]$. If we call $y[n]$ the output signal, it is going to be periodic with the same period $N = 3$ and DFS given by $Y[k] = H(k\frac{2\pi}{N})\, X[k]$, for $k = 0, 1, 2$. By substitution we find $Y[0] = \frac{0.2}{1-0.8}(6) = 6$, $Y[1] = \frac{0.2 e^{j\frac{2\pi}{3}}}{e^{j\frac{2\pi}{3}}-0.8}(-1.5 + j0.866) = -0.123 + j0.1846$ and $Y[2] = \frac{0.2 e^{j2\frac{2\pi}{3}}}{e^{j2\frac{2\pi}{3}}-0.8}(-1.5 - j0.866) = -0.123 - j0.1846$. In vector form:

$$Y = [6,\ -0.123 + j0.1846,\ -0.123 - j0.1846]$$

b) Period $N = 2$, therefore $Y[k] = H(k\frac{2\pi}{2})\, X[k]$ for $k = 0, 1$. This yields $Y[k] = \frac{0.2 (-1)^k}{(-1)^k-0.8} X[k]$ for $k = 0, 1$. Substitute te numerical values of $X[k]$ to obtain

$$Y = [0,\ 0.5]$$

c) Period $N = 4$, therefore $Y[k] = H(k\frac{2\pi}{4})\, X[k]$ for $k = 0, 1, 2, 3.$ This yields $Y[k] = \frac{0.2 (-j)^k}{(-j)^k-0.8} X[k]$. Substitute for $X[k]$ to obtain

$$Y = [0,\ 0.2439 + j0.1951,\ 0,\ 0.2439 - j0.1951]$$

■ Problem 1.40

a) The length of the sequence is $N = 4$. Therefore the DFT becomes $X[k] = \sum_{n=0}^{3} x[n]\, e^{-j\frac{2\pi}{4}kn}$ for $k = 0, 1, 2, 3$. Substitute for the sequence to obtain

$$X = [0,\ 0,\ 4,\ 0]$$

b) Length $N = 4$, then again $X[k] = \sum_{n=0}^{3} x[n]\, e^{-j\frac{2\pi}{4}kn}$ for $k = 0, \ldots, 3$ to obtain

$$X = [4,\ 0,\ 0,\ 0]$$

c) Length $N = 3$, then $X[k] = \sum_{n=0}^{2} x[k] = e^{-j\frac{2\pi}{3}kn}$ for $n = 0, 1, 2$. This yields

$$X = [6.0000,\ -1.5000 + j0.8660,\ -1.5000 - j0.8660]$$

d) Length $N = 4$, then again $X[k] = \sum_{n=0}^{3} x[n]\, e^{-j\frac{2\pi}{4}kn}$ for $k = 0, \ldots, 3$ to obtain

$$X = [0, 2, 0, 2]$$

e) Length $N = 4$, then again $X[k] = \sum_{n=0}^{3} x[n]\, e^{-j \frac{2\pi}{4} kn}$ for $k = 0, \ldots, 3$ to obtain

$$X = [1 + j2.4142,\ 1 - j2.4142,\ 1 - j0.4142\, i,\ 1 + j0.4142]$$

f) Length $N = 4$, then again $X[k] = \sum_{n=0}^{3} x[n]\, e^{-j \frac{2\pi}{4} kn}$ for $k = 0, \ldots, 3$ to obtain

$$X = [-4.0961,\ 4.1904 - j2.4938,\ 2.7687,\ 4.1904 + j2.4938]$$

g) Length $N = 8$, then $X[k] = \sum_{n=0}^{7} x[n]\, e^{-j \frac{2\pi}{8} kn}$ for $k = 0, \ldots, 7$ to obtain

$$X = [0,\ 7.0534 + j9.7082,\ 0,\ 0,\ 0,\ 0,\ 0,\ 7.0534 - j9.7082]$$

■ Problem 1.41

Recall the definition $X(\omega) = \text{DTFT}\{x[n]\} = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n}$. Then:

a) $X(\omega) = \sum_{n=-\infty}^{+\infty} 0.8^{|n|}\, e^{-j\omega n} = \sum_{n=-\infty}^{-1} 0.8^{-n}\, e^{-j\omega n} + \sum_{n=0}^{+\infty} 0.8^{n}\, e^{-j\omega n}$. After some simple manipulations:

$$X(\omega) = \sum_{n=0}^{+\infty} 0.8^{n}\, e^{j\omega n} - 1 + \sum_{n=0}^{+\infty} 0.8^{n}\, e^{-j\omega n} = \frac{1}{1 - 0.8\, e^{j\omega}} - 1 + \frac{1}{1 - 0.8\, e^{-j\omega}} = \frac{0.36}{1 - 1.6\cos(\omega) + 0.64}$$

An alternative way would be to use the z-Transforms. Since, in this case

$$x[n] = 1.25^{n}\, u[-n-1] + 0.8^{n}\, u[n]$$

the z-Transform is $X(z) = -\frac{z}{z - 1.25} + \frac{z}{z - 0.8}$, $\text{ROC}: 0.8 < |z| < 1.25$

the unit circle $|z| = 1$ is within the ROC. Therefore just substitute $z = e^{j\omega}$ to obtain:

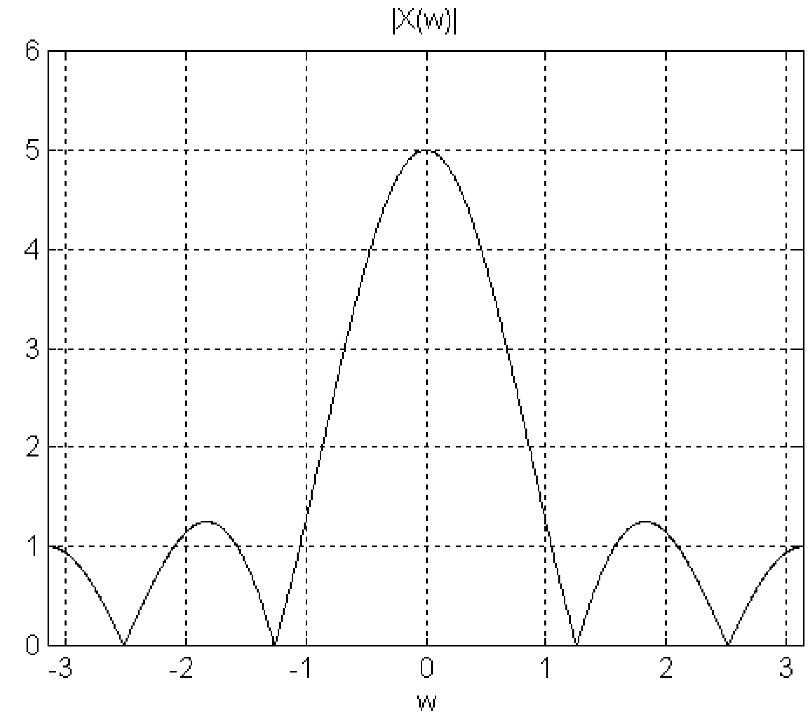
$$X(\omega) = -\frac{e^{j\omega}}{e^{j\omega} - 1.25} + \frac{e^{j\omega}}{e^{j\omega} - 0.8}$$

and the result follows with some algebra.

b) $X(\omega) = \sum_{n=0}^{+\infty} 0.5^{n}\, e^{-j\omega n} = \frac{1}{1 - 0.5\, e^{-j\omega}} = \frac{e^{j\omega}}{e^{j\omega} - 0.5}$

c) $x[n] = 1$ when $0 \le n \le 4$ and $x[n] = 0$ otherwise. Therefore

$$X(\omega) = \sum_{n=0}^{4} e^{-j\omega n} = \frac{1 - e^{-j5\omega}}{1 - e^{-j\omega}} = e^{-j1.5\omega}\, \frac{\sin(2.5\,\omega)}{\sin(\omega)}$$

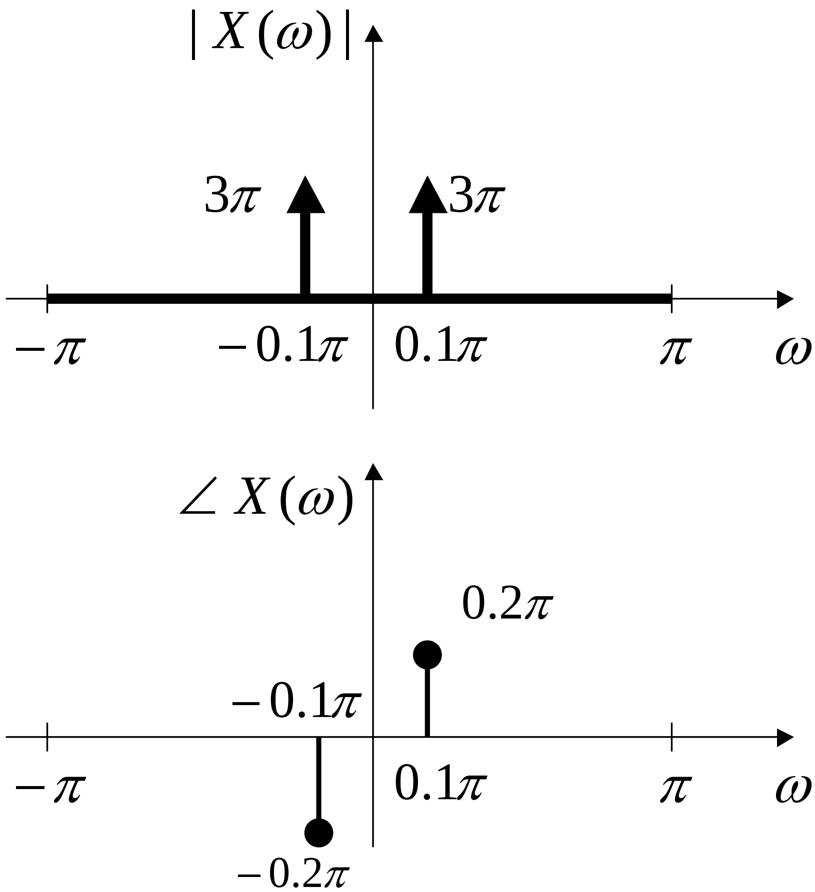


magnitude $|X(\omega)|$

d) $x[n] = 1.5\, e^{j0.2\pi}\, e^{j0.1\pi n} + 1.5\, e^{-j0.2\pi}\, e^{-j0.1\pi n}$.

Recall that $\mathrm{DTFT}\{e^{j\omega_0 n}\} = 2\pi\delta(\omega - \omega_0)$ and therefore we obtain

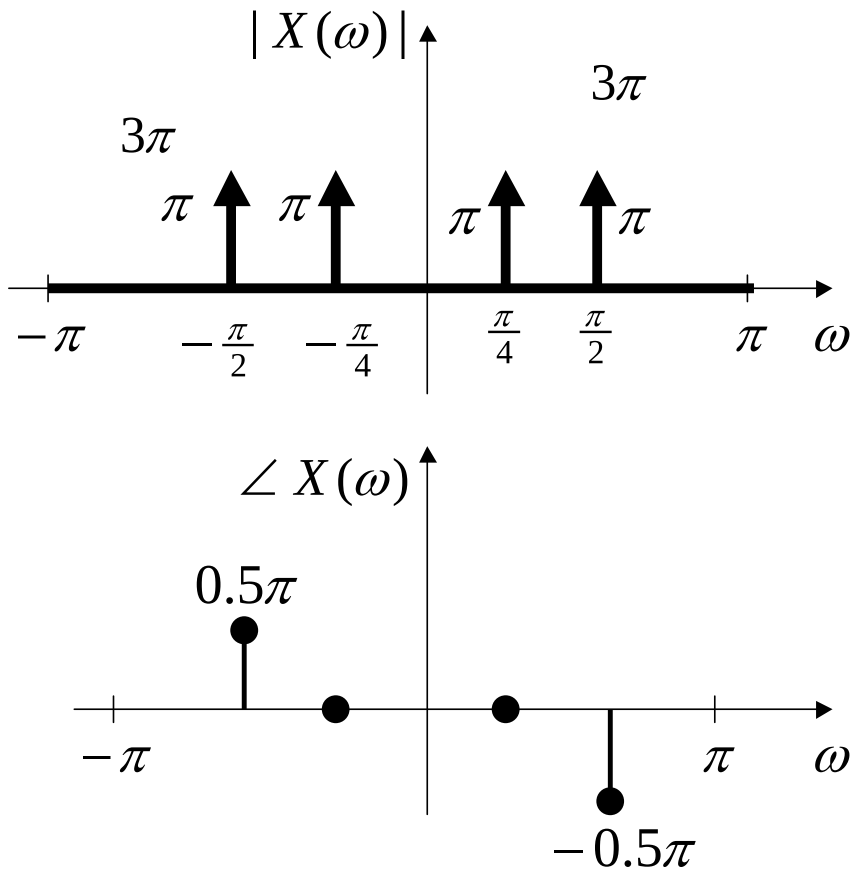
$$X(\omega) = 3\pi e^{j0.2\pi}\,\delta(\omega - 0.1\pi) + 3\pi e^{-j0.2\pi}\,\delta(\omega + 0.1\pi)$$



X (ω) magnitude and phase

e) $x[n] = -0.5\, je^{j0.5\pi n} + 0.5\, je^{-j0.5\pi n} + 0.5\, e^{j0.25\pi n} + 0.5\, e^{-j0.25\pi n}$. Therefore the DTFT becomes

$$X(\omega) = \pi e^{-j0.5\pi}\,\delta(\omega - 0.5\pi) + \pi e^{j0.5\pi}\,\delta(\omega + 0.5\pi) + \\ + \pi\delta(\omega - 0.25\pi) + \pi\delta(\omega + 0.25\pi)$$



X(ω) magnitude and phase

f) $x[n] = 2\,(-0.5\,j e^{j0.5\pi n} + 0.5\,j e^{-j0.5\pi n})^2 =$

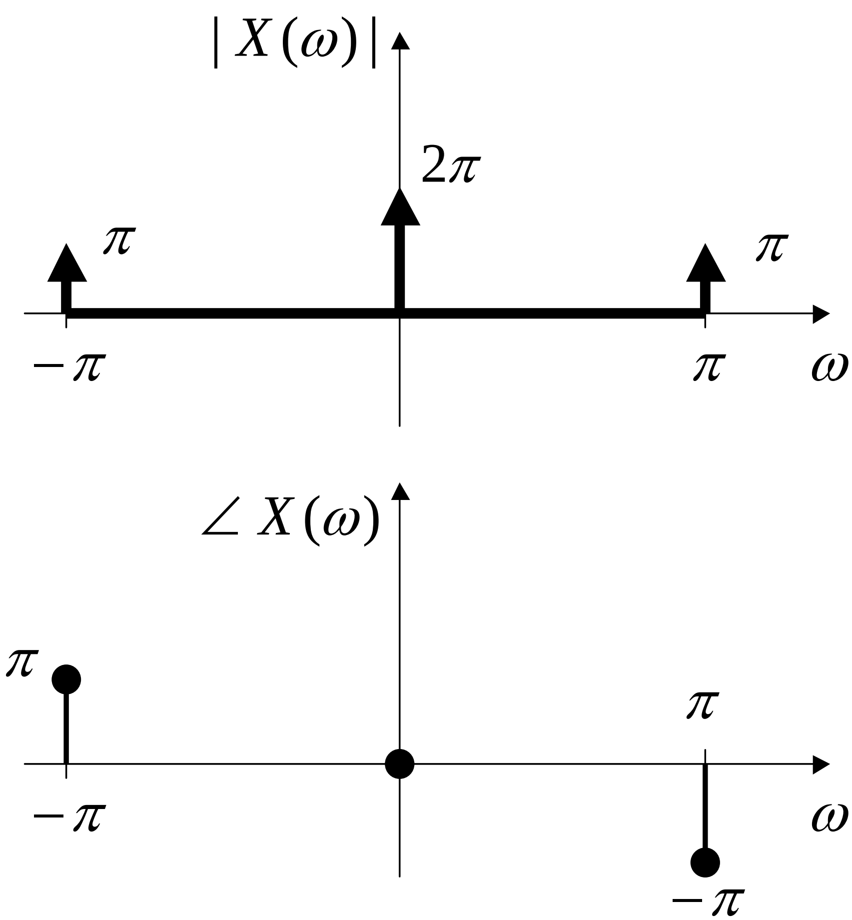
$= 2\,(-0.25\,e^{j\pi n} - 0.25\,e^{-j\pi n} + 2 \times 0.25) =$

$= 1 - e^{j\pi n}$

Therefore $X(\omega) = 2\pi\delta(\omega) - 2\pi\delta(\omega - \pi) =$

$= 2\pi\delta(\omega) + \pi e^{-j\pi}\delta(\omega - \pi) + \pi e^{j\pi}\delta(\omega + \pi)$

Notice that we split the term $2\pi\delta(\omega - \pi) = \pi e^{-j\pi}\delta(\omega - \pi) + \pi e^{j\pi}\delta(\omega + \pi)$ to preserve the simmetry of the DTFT.

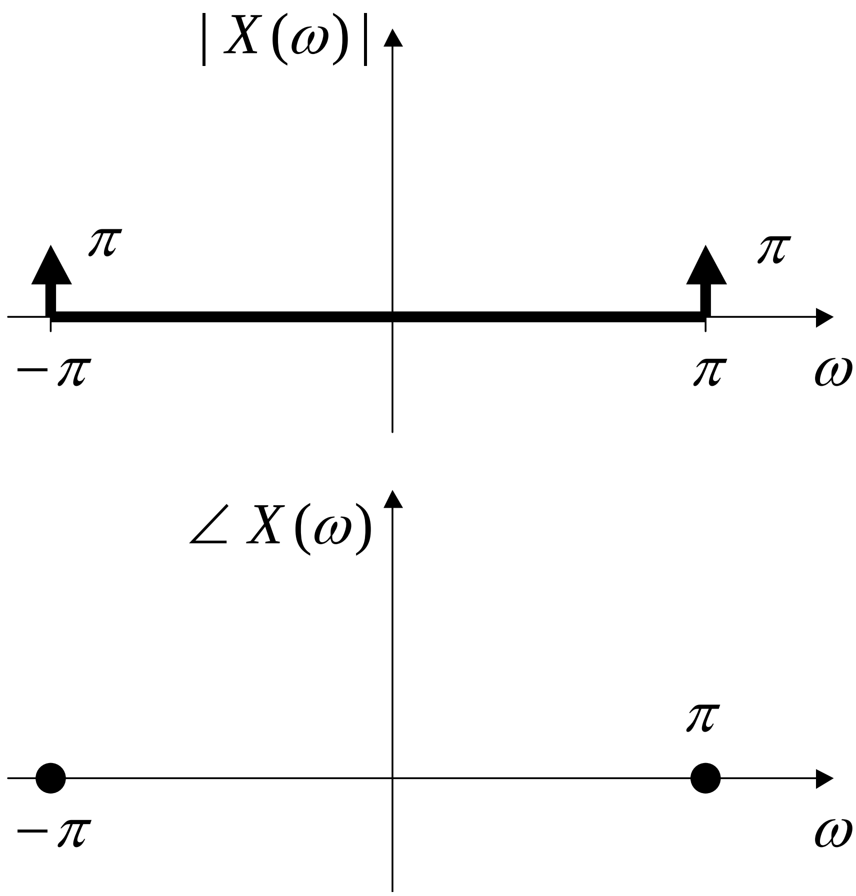


$X(\omega)$ magnitude and phase

g) $x[n] = (-1)^n = e^{-j\pi n} = 0.5\, e^{j\pi n} + 0.5\, e^{-j\pi n}$ therefore

$$X(\omega) = \pi\delta(\omega - \pi) + \pi\delta(\omega + \pi)$$

again we split it to preserve the simmetry of the DTFT.



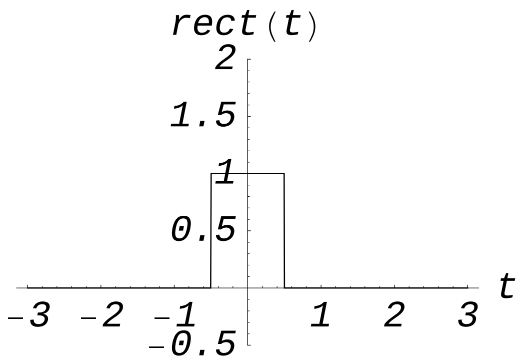
X (ω) magnitude and phase

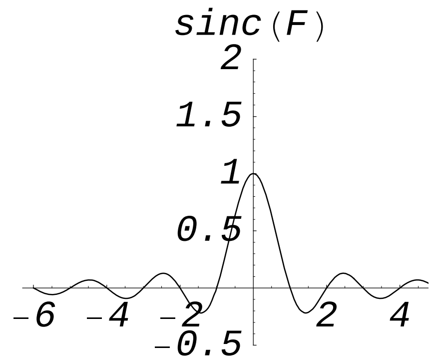
Fourier Transform

■ Problem 1.42

You know that

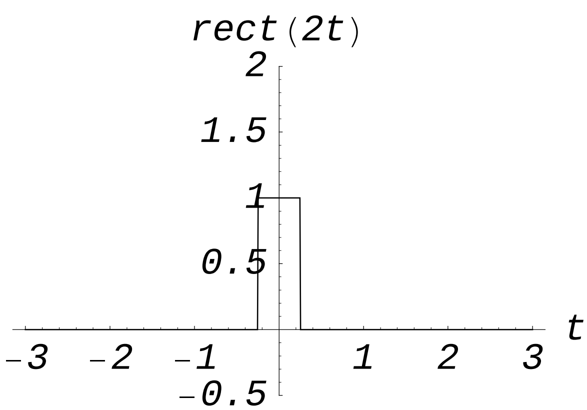
$$FT\{rect(t)\} = sinc(F)$$

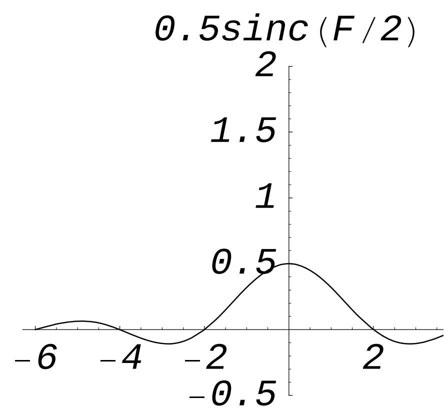




Apply the properties of the Fourier Transform:

a) $\text{FT}\{\text{rect}(2t)\} = \frac{1}{2}\,\text{sinc}\left(\frac{F}{2}\right)$



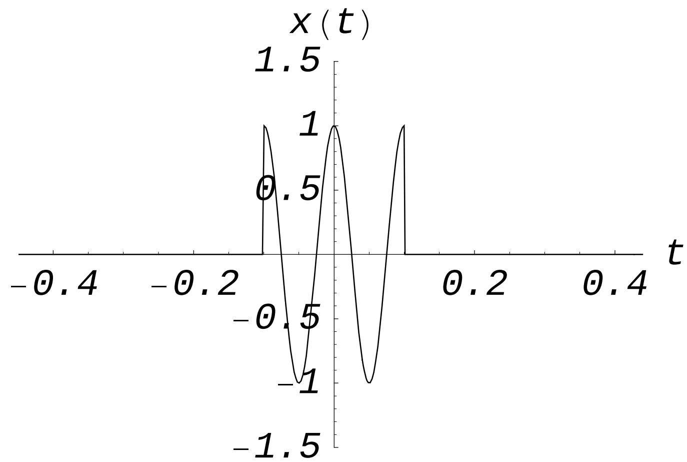


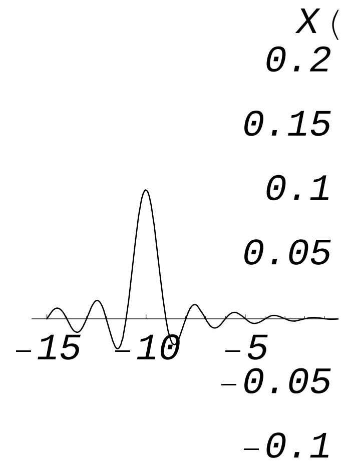
b) $\text{FT}\{\text{rect}(2t-5)\} = \text{FT}\{\text{rect}(2(t-2.5))\} = e^{-j2\pi F\,2.5}\,\text{FT}\{\text{rect}(2t)\}$ which yields

$$\text{FT}\{\text{rect}(2t-5)\} = e^{-j5\pi F}\,\frac{1}{2}\,\text{sinc}\left(\frac{F}{2}\right)$$

c) $\text{FT}\{\text{rect}(5t)\cos(20\pi t)\} = \text{FT}\{\text{rect}(5t)\,\frac{1}{2}\,(e^{j2\pi 10t} + e^{-j2\pi 10t})\}$. Since $\text{FT}\{\text{rect}(5t)\} = \frac{1}{5}\,\text{sinc}\left(\frac{F}{5}\right)$ then

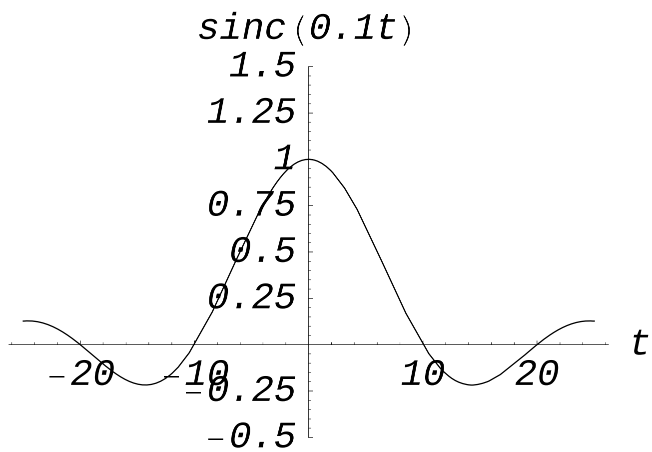
$$\text{FT}\{\text{rect}(5t)\cos(20\pi t)\} = \frac{1}{10}\,\text{sinc}(F-10) + \frac{1}{10}\,\text{sinc}(F+10)$$





d) $\mathrm{FT}\{\mathrm{sinc}(0.1\,t)\}$. By the duality property, $\mathrm{FT}\{\mathrm{sinc}(t)\} = \mathrm{rect}(-F)$. Since the "rect" function is symmetric, we can write $\mathrm{FT}\{\mathrm{sinc}(t)\} = \mathrm{rect}(F)$. Therefore

$$\mathrm{FT}\{\mathrm{sinc}(0.1\,t)\} = 10\,\mathrm{rect}(10\,F)$$



e) $\mathrm{FT}\{\mathrm{sinc}(0.1\,t)\cos(20\,\pi t + 0.1\,\pi)\} =$. Therefore
$e^{j0.1\pi}\,\mathrm{FT}\{e^{j2\pi 10t}\,\mathrm{sinc}(0.1\,\pi t)\} + e^{-j0.1\pi}\,\mathrm{FT}\{e^{-j2\pi 10t}\,\mathrm{sinc}(0.1\,\pi t)\}$

$$X(F) = e^{j0.1\pi}\,0.1\,\mathrm{rect}(10\,(F-10)) + e^{-j0.1\pi}\,0.1\,\mathrm{rect}(10\,(F+10))$$

f) Since $\mathrm{FT}\{\mathrm{rect}(at)\} = \frac{1}{|a|}\,\mathrm{sinc}\left(\frac{F}{a}\right)$, then $\mathrm{IFT}\{\mathrm{sinc}\left(\frac{F}{a}\right)\} = |a|\,\mathrm{rect}(at)$. Therefore let $a = 1/10$ and then

$$\mathrm{IFT}\{\mathrm{sinc}(10\,F)\} = 0.1\,\mathrm{rect}(0.1\,t)$$

g) $\mathrm{IFT}\{\mathrm{rect}(10\,F)\} = 0.1\,\mathrm{sinc}(0.1\,t)$ for the same reasons as in f);

h) Recall that $\text{FT}\left\{\sum_{n=-\infty}^{+\infty}\delta(t-nT_0)\right\}=F_0\sum_{k=-\infty}^{+\infty}\delta(F-kF_0)$, where $F_0=1/T_0$. Also recall that

$$\sum_{n=-\infty}^{+\infty}x(nT_0)\,\delta(t-nT_0)=x(t)\sum_{n=-\infty}^{+\infty}\delta(t-nT_0).$$

With this in mind we can write
$\sum_{n=-\infty}^{+\infty}\text{sinc}(0.2\,n)\,\delta(t-0.1\,n)=\text{sinc}(2\,t)\sum_{n=-\infty}^{+\infty}\delta(t-0.1\,n)$, and therefore

$$\text{FT}\left\{\sum_{n=-\infty}^{+\infty}\text{sinc}(0.2\,n)\,\delta(t-0.1\,n)\right\}=\text{FT}\{\text{sinc}(2\,t)\}*\text{FT}\left\{\sum_{n=-\infty}^{+\infty}\delta(t-0.1\,n)\right\}$$

$$=0.5\,\text{rect}(0.5\,F)*10\sum_{k-\infty}^{+\infty}\delta(F-10\,k)=5\sum_{k-\infty}^{+\infty}\text{rect}(0.5\,(F-10\,k))$$

An alternative way to arrive at the same inswer is by using the "rep" and "comb" operators. Recall that

$$\text{FT}\{\text{comb}_{T_0}\,x(t)\}=F_0\,\text{rep}_{F_0}\,X(F)$$

In our case $T_0=0.1$, $F_0=1/T_0=10$, $x(t)=\text{sinc}(2\,t)$,
$X(F)=\text{FT}\{\text{sinc}(2\,t)\}=0.5\,\text{rect}(0.5\,F)$. Therefore

$$\text{FT}\{\text{comb}_{0.1}\,\text{sinc}(2\,t)\}=10\,\text{rep}_{10}\,0.5\,\text{rect}(0.5\,F)$$

i) Again apply duality to obtain

$$\text{FT}\{\text{rep}_{T_0}\,x(t)\}=F_0\,\text{comb}_{F_0}\,X(F)$$

Therefore $\text{FT}\left\{\sum_{n=-\infty}^{+\infty}\text{rect}(t-2\,n)\right\}=\text{FT}\{\text{comb}_2\,\text{rect}(t)\}=0.5\,\text{rep}_{0.5}\,\text{sinc}(F)$, and therefore

$$\text{FT}\left\{\sum_{n=-\infty}^{+\infty}\text{rect}(t-2\,n)\right\}=0.5\sum_{k=-\infty}^{+\infty}\text{sinc}(0.5\,k)\,\delta(F-0.5\,k)$$

j) By the same arguments as in the previous problem, we can write

$$\text{FT}\left\{\sum_{n=-\infty}^{+\infty}\text{rect}(t-n)\right\}=\text{FT}\{\text{comb}_1\,\text{rect}(t)\}=\text{rep}_1\,\text{sinc}(F)$$

and therefore

$$\text{FT}\left\{\sum_{n=-\infty}^{+\infty}\text{rect}(t-n)\right\}=\sum_{k=-\infty}^{+\infty}\text{sinc}(k)\,\delta(F-k)$$

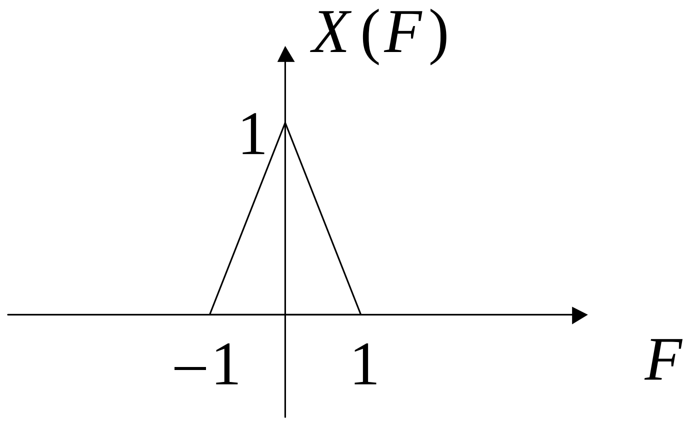
But now recall that $\text{sinc}(k)=0$ for all $k\neq 0$. Also it is easy to see that $\sum_{n=-\infty}^{+\infty}\text{rect}(t-n)=1$ for all t. Therefore

$$\text{FT}\{1\}=\delta(F)$$

since all other terms $\text{sinc}(k)\,\delta(F-k)$, for $k\neq 0$, in the summation, are zero.

■ Problem 1.43

Let a signal $x(t)$ have Fourier Transform as shown.

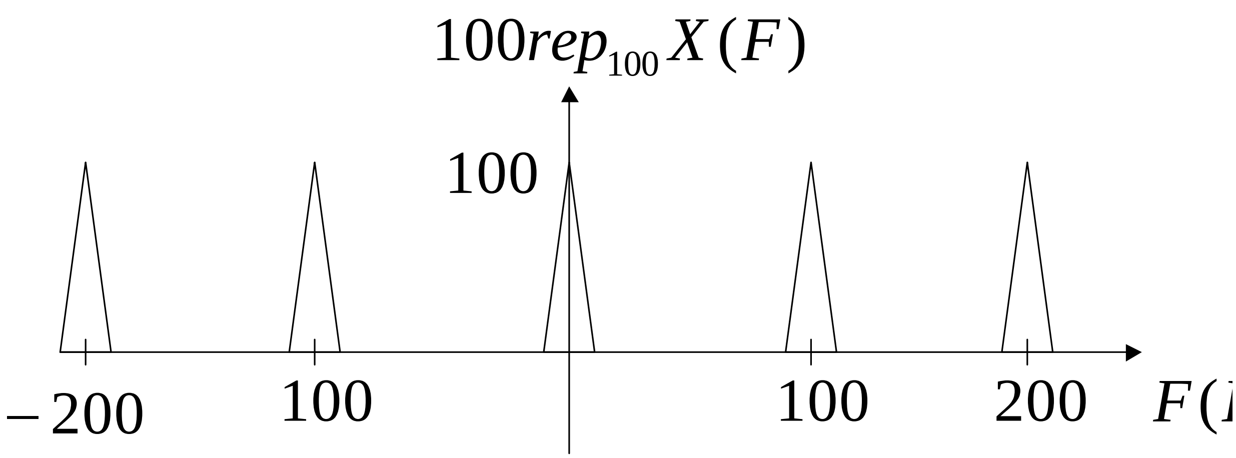


Sketch the following:

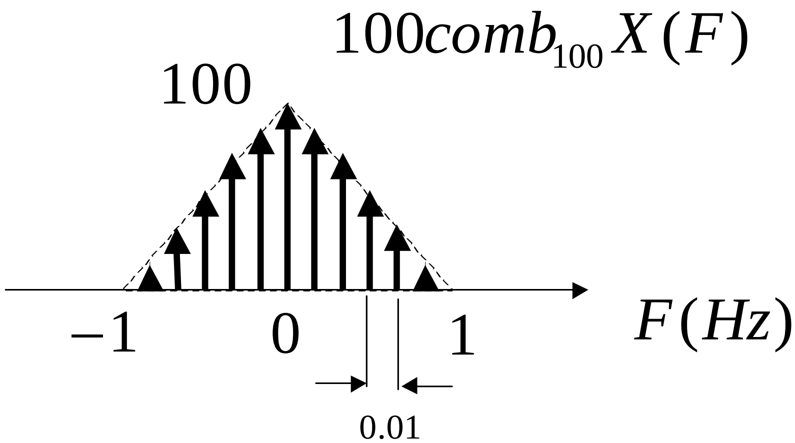
a) $\sum_{n=-\infty}^{+\infty} x(0.01\,n)\,\delta(t-0.01\,n) = x(t)\sum_{n=-\infty}^{+\infty}\delta(t-0.01\,n) = \text{comb}_{0.01}\,x(t)$. Therefore its Fourier Transform is given by

$$\text{FT}\left\{\sum_{n=-\infty}^{+\infty} x(0.01\,n)\,\delta(t-0.01\,n)\right\} = 100\ \text{rep}_{100}\,X(F)$$

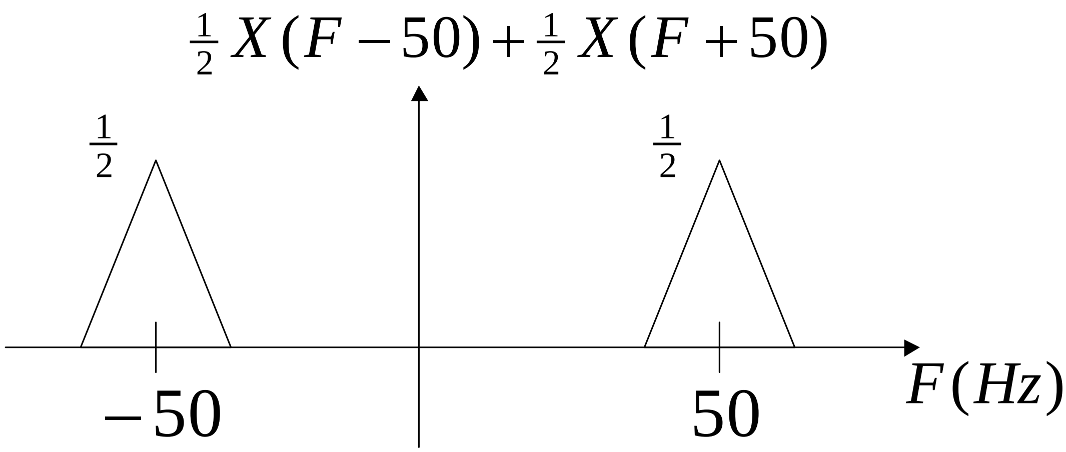
shown below



b) $\sum_{n=-\infty}^{+\infty} x(t-0.01\,n) = \text{rep}_{0.01}\,x(t)$. Therefore its Fourier Transform is $100\ \text{comb}_{100}\,X(F)$ shown below



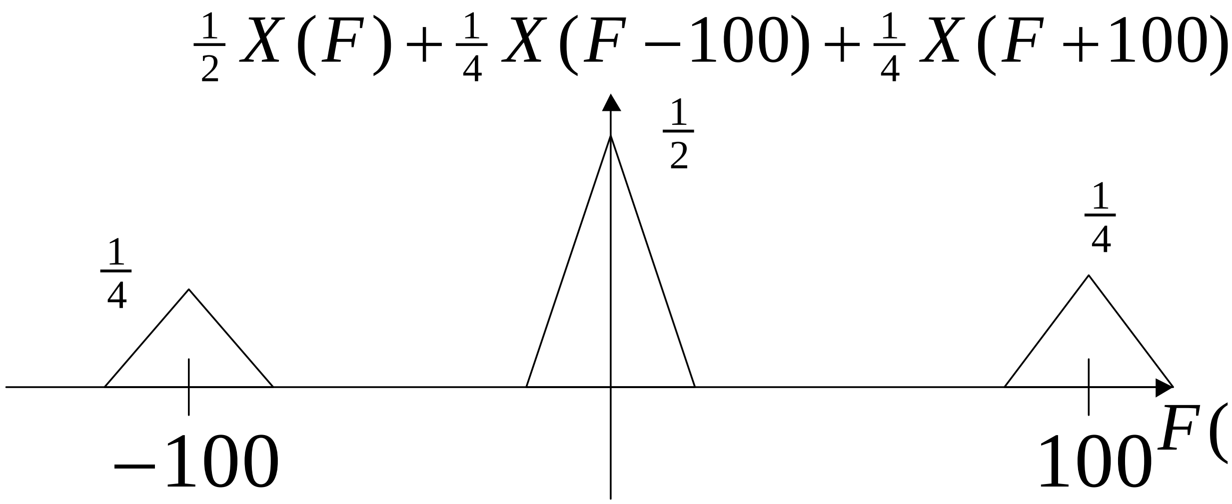
c) FT $\{x(t)\cos(100\pi t)\} = \frac{1}{2} X(F - 50 + \frac{1}{2} X(F + 50)$



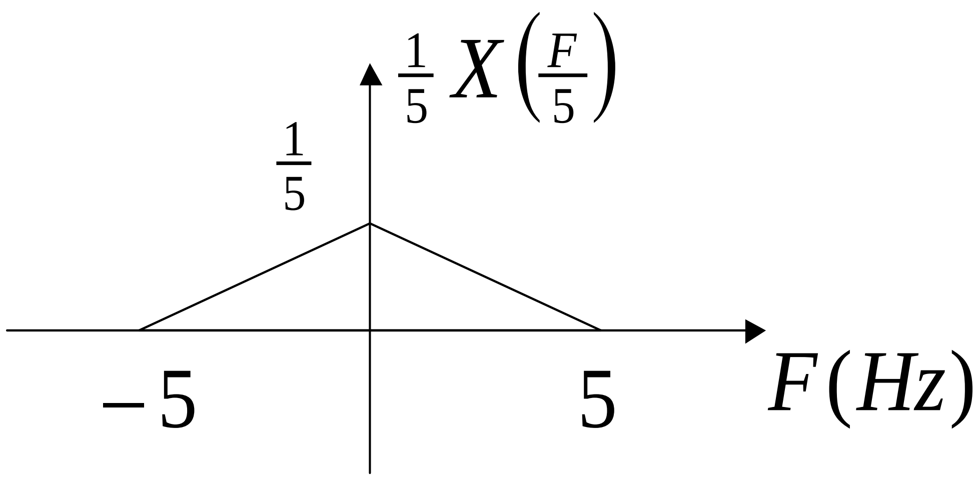
d) Since $\cos^2(100\pi t) = \left(\frac{1}{2} e^{j100\pi t} + \frac{1}{2} e^{-j100\pi t}\right)^2 = \frac{1}{4}\left(e^{j200\pi t} + 2 + e^{-j200\pi t}\right)$ then

$$FT\{x(t)\cos^2(100\pi t)\} = \frac{1}{2} X(F) + \frac{1}{4} X(F - 100) + \frac{1}{4} X(F + 100)$$

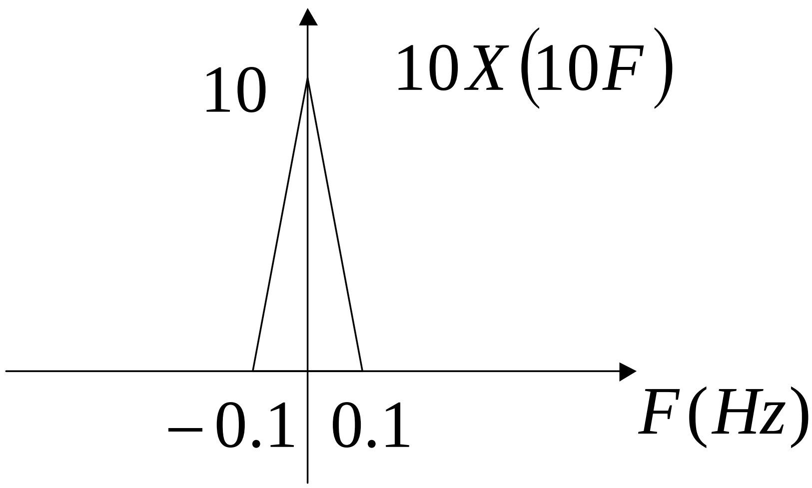
shown below.



e) From the propertises, $FT\{x(5t)\} = \frac{1}{5} X\left(\frac{F}{5}\right)$ shown below:



f) From the properties again, $FT\{x(0.1t)\} = 10 X(10F)$, as shown:


10
10X(10F)
−0.1
0.1
F(Hz)