The Lehmer matrix and its recursive analogue

Emrah Kilic[1], Pantelimon Stănică[2]

[1]TOBB Economics and Technology University, Mathematics Department
06560 Sogutozu, Ankara, Turkey; ekilic@etu.edu.tr
[2]Naval Postgraduate School, Department of Applied Mathematics
833 Dyer Rd., Monterey, CA 93943, USA; pstanica@nps.edu

Abstract

This paper considers the Lehmer matrix and its recursive analogue. The determinant of Lehmer matrix is derived explicitly by both its LU and Cholesky factorizations. We further define a generalized Lehmer matrix with (i,j) entries $g_{ij} = \dfrac{\min\{u_{i+1}, u_{j+1}\}}{\max\{u_{i+1}, u_{j+1}\}}$ where u_n is the nth term of a binary sequence $\{u_n\}$. We derive both the LU and Cholesky factorizations of this analogous matrix and we precisely compute the determinant.

1 Introduction

D.H. Lehmer (see [2]) constructed an $n \times n$ symmetric matrix $A = (a_{ij})_{i,j}$ whose (i,j) entry is

$$a_{ij} = \frac{\min\{i,j\}}{\max\{i,j\}} = \begin{cases} i/j & j \geq i, \\ j/i & i > j. \end{cases}$$

Define the second order recurrence $\{U_n(p,q)\}$ as follows:

$$U_n(p,q) = pU_{n-1}(p,q) - qU_{n-2}(p,q),$$

where $U_0(p,q) = 0$ and $U_1(p,q) = 1$ for $n > 1$.

As an interesting example, we mention that the set of natural numbers can be obtained from the sequence $\{U_n(p,q)\}$ by taking $p = 2,\ q = 1$. Throughout this paper, we consider the case $q = -1$ and we denote $u_n = U_n(p,-1)$.

We now define an $n \times n$ generalized Lehmer matrix, namely $\mathcal{F}_n = (g_{ij})_{1 \le i,j \le n}$ defined below:

$$g_{ij} = \frac{\min\{u_{i+1}, u_{j+1}\}}{\max\{u_{i+1}, u_{j+1}\}} = \begin{cases} \frac{u_{i+1}}{u_{j+1}} & \text{if } j \ge i, \\ \\ \frac{u_{j+1}}{u_{i+1}} & \text{if } i > j. \end{cases}$$

where u_n is the nth term of the sequence $\{u_n\}$. In this paper, we obtain the general LU factorization and other explicit formulas for both the Lehmer matrix and its recursive analogue.

The Lehmer matrix is part of a family of matrices known as test matrices, which are used to evaluate the accuracy of matrix inversion programs since the exact inverses are known (see [1, 2]). It is hoped that our generalized Lehmer matrix will add to the literature of special matrices with known inverse.

2 The Lehmer Matrix

We start by obtaining the LU factorization of the Lehmer matrix A. Using the inverses of L and U, we obtain the explicit form for the inverse of A, whose inverse is well-known, thus obtaining another proof of this result.

We define the $n \times n$ invertible lower triangular matrix $L = (\ell_{ij})$ where $\ell_{ij} = j/i$ for $i \ge j$ and 0 otherwise. Next, we define the $n \times n$ invertible upper triangular matrix $U = (u_{ij})$ with $u_{ij} = \frac{2i-1}{ij}$ for

$i \leq j$ and 0 otherwise. For example, when $n = 5$, we get

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 1 & 0 & 0 \\ \frac{1}{4} & \frac{2}{4} & \frac{3}{4} & 1 & 0 \\ \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ 0 & \frac{3}{4} & \frac{3}{6} & \frac{3}{8} & \frac{3}{10} \\ 0 & 0 & \frac{5}{9} & \frac{5}{12} & \frac{5}{15} \\ 0 & 0 & 0 & \frac{7}{16} & \frac{7}{20} \\ 0 & 0 & 0 & 0 & \frac{9}{25} \end{bmatrix}.$$

The following result holds.

Theorem 1. *For $n > 0$, the LU factorization of Lehmer matrix is given by*

$$A = LU$$

where L and U were defined previously.

Proof. We split the proof into three cases.
Case 1: $i = j$. By $\sum_{k=1}^{t} (2k-1) = t^2$, then

$$a_{ii} = \sum_{k=1}^{n} \ell_{ik} u_{ki} = \sum_{k=1}^{i} \ell_{ik} u_{ki} = \sum_{k=1}^{i} \frac{k}{i} \frac{(2k-1)}{k.i} = \sum_{k=1}^{i} \frac{2k-1}{i^2} = 1.$$

Case 2: $i > j$. Thus

$$\begin{aligned} a_{ij} &= \sum_{k=1}^{n} \ell_{ik} u_{kj} = \sum_{k=1}^{j} \ell_{ik} u_{kj} = \sum_{k=1}^{j} \frac{k}{i} \frac{(2k-1)}{kj} \\ &= \sum_{k=1}^{j} \frac{2k-1}{ij} = \frac{1}{ij} \sum_{k=1}^{j} 2k-1 = \frac{j}{i}. \end{aligned}$$

Case 3: $j > i$. Then

$$\begin{aligned} a_{ij} &= \sum_{k=1}^{n} \ell_{ik} u_{kj} = \sum_{k=1}^{i} \ell_{ik} u_{kj} = \sum_{k=1}^{i} \frac{k}{i} \frac{(2k-1)}{kj} \\ &= \sum_{k=1}^{i} \frac{2k-1}{ij} = \frac{1}{ij} \sum_{k=1}^{i} 2k-1 = \frac{i}{j}, \end{aligned}$$

which completes the proof. $\square$

We display an example below:

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{2} & 1 & \frac{2}{3} & \frac{2}{4} & \frac{2}{5} \\ \frac{1}{3} & \frac{2}{3} & 1 & \frac{3}{4} & \frac{3}{5} \\ \frac{1}{4} & \frac{2}{4} & \frac{3}{4} & 1 & \frac{4}{5} \\ \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 1 & 0 & 0 \\ \frac{1}{4} & \frac{2}{4} & \frac{3}{4} & 1 & 0 \\ \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ 0 & \frac{3}{4} & \frac{3}{6} & \frac{3}{8} & \frac{3}{10} \\ 0 & 0 & \frac{5}{9} & \frac{5}{12} & \frac{5}{15} \\ 0 & 0 & 0 & \frac{7}{16} & \frac{7}{20} \\ 0 & 0 & 0 & 0 & \frac{9}{25} \end{bmatrix}.$$

As a consequence of Theorem 1, we obtain an explicit value of the determinant of the Lehmer matrix in the following corollary.

Corollary 1. *For $n > 0$,*

$$\det A = \frac{(2n)!}{2^n \, (n!)^3}$$

Proof. The proof follows from the LU factorization of matrix A by considering $\det A = \det U = \prod_{i=1}^{n} \frac{2i-1}{i^2}$. $\square$

The nth Catalan number is given in terms of binomial coefficients by

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!n!}.$$

Thus we may note that

$$\det A = \frac{(n+1)}{2^n n!} C_n.$$

We continue our analysis by determining the $L_1 L_1^T$ (named after Cholesky) factorization of the Lehmer matrix, where L_1 is a lower triangular matrix. The Cholesky factorization was obtained for a different kind of matrix defined using binary sequences by the second author in [3].

Theorem 2. *The Cholesky factorization of the Lehmer matrix is given by*

$$A = L_1 L_1^T$$

where $L_1 = (f_{ij})$ *is a lower triangular matrix with* $f_{ij} = \frac{\sqrt{2j-1}}{i}$ *for all* $i \geq j$.

Proof. If $i > j$, then

$$\begin{aligned} a_{ij} &= \sum_{r=1}^{n} f_{ir} f_{jr} = \sum_{r=1}^{j} f_{ir} f_{jr} = \sum_{r=1}^{j} \frac{\sqrt{2r-1}}{i} \frac{\sqrt{2r-1}}{j} \\ &= \frac{1}{ij} \sum_{r=1}^{j} (2r-1) = \frac{j}{i}. \end{aligned}$$

If $i = j$, then

$$\begin{aligned} a_{ii} &= \sum_{r=1}^{n} f_{ir}^2 = \sum_{r=1}^{i} f_{ir}^2 = \sum_{r=1}^{i} \left(\frac{\sqrt{2r-1}}{i} \right)^2 \\ &= \frac{1}{i^2} \sum_{r=1}^{i} (2r-1) = \frac{i^2}{i^2} = 1. \end{aligned}$$

Finally, if $i < j$, then

$$a_{ij} = \sum_{r=1}^{n} f_{ir} f_{jr} = \sum_{r=1}^{i} f_{ir} f_{jr} = \frac{1}{ij} \sum_{r=1}^{i} (2r-1) = \frac{i}{j},$$

which proves the theorem. □

As an example, for $n = 5$ and $p = 1$ (the Fibonacci sequence case), we have

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{2} & 1 & \frac{2}{3} & \frac{2}{4} & \frac{2}{5} \\ \frac{1}{3} & \frac{2}{3} & 1 & \frac{3}{4} & \frac{3}{5} \\ \frac{1}{4} & \frac{2}{4} & \frac{3}{4} & 1 & \frac{4}{5} \\ \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{\sqrt{3}}{3} & \frac{\sqrt{5}}{3} & 0 & 0 \\ \frac{1}{4} & \frac{\sqrt{3}}{4} & \frac{\sqrt{5}}{4} & \frac{\sqrt{7}}{4} & 0 \\ \frac{1}{5} & \frac{\sqrt{3}}{5} & \frac{\sqrt{5}}{5} & \frac{\sqrt{7}}{5} & \frac{\sqrt{9}}{5} \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ 0 & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{5} \\ 0 & 0 & \frac{\sqrt{5}}{3} & \frac{\sqrt{5}}{4} & \frac{\sqrt{5}}{5} \\ 0 & 0 & 0 & \frac{\sqrt{7}}{4} & \frac{\sqrt{7}}{5} \\ 0 & 0 & 0 & 0 & \frac{\sqrt{9}}{5} \end{bmatrix}.$$

By Theorem 2, we find that, since $A = L_1 L_1^T$, we have that $\det(A) = \prod_{i=1}^{n} f_{ii}^2 = \prod_{t=1}^{n} \frac{2i-1}{i^2} = \frac{(2n)!}{2^n (n!)^3}$, that is, Corollary 1.

3 The Inverse of the Lehmer Matrix

Now we find an explicit formula for the inverse of the Lehmer matrix. For this purpose, we use its LU factorization as $A^{-1} = U^{-1}L^{-1}$. We first derive the inverses of the matrices L and U.

Lemma 1. *Let $L^{-1} = (t_{ij})$ denote the inverse of L. Then*

$$t_{ij} = \begin{cases} 1 & \text{if } i = j, \\ -\frac{j}{i} & \text{if } i = j+1, \\ 0 & \text{otherwise,} \end{cases}$$

Proof. The proof can be easily checked from the product $L^{-1}L$. □

Lemma 2. *Let $U^{-1} = (w_{ij})$ denote the inverse of U. Then*

$$w_{ij} = \begin{cases} \frac{i^2}{2i-1} & \text{if } i = j \\ -\frac{i(i+1)}{2i+1} & \text{if } i+1 = j, \\ 0 & \text{otherwise,} \end{cases}$$

Proof. The proof follows from the product $U^{-1}U$. □

The inverse of the Lehmer matrix is found in the following theorem.

Theorem 3. *For $n > 0$, let $A^{-1} = (b_{ij})$, then*

$$b_{ij} = \begin{cases} \frac{4i^3}{4i^2-1} & \text{if } i = j < n \\ \frac{n^2}{2n-1} & \text{if } i = j = n, \\ -\frac{i(i+1)}{2i+1} & \text{if } |i-j| = 1, \\ 0 & \text{otherwise,} \end{cases}$$

Proof. Since $A^{-1} = U^{-1}L^{-1}$, using the previous two lemmas, we obtain for $1 \le i \le n-1$,

$$\begin{aligned} b_{ii} &= \sum_{k=1}^{n} w_{ik}t_{ki} = w_{ii} + w_{i,i+1}t_{i+1,i} \\ &= \frac{i^2}{2i-1} + \frac{i\,(i+1)}{2i+1}\frac{i}{(i+1)} = \frac{i^2}{2i-1} + \frac{i^2}{2i+1} = \frac{4i^3}{4i^2-1}. \end{aligned}$$

When $i = j = n$, it is easy to see that $b_{nn} = w_{nn} = \frac{n^2}{2n-1}$. If $i = j+1$, then

$$\begin{aligned} b_{i+1,i} &= \sum_{k=1}^{n} w_{i+1,k}t_{ki} = w_{i+1,i+1}t_{i+1,i} \\ &= \frac{(i+1)^2}{2i+1}\left(\frac{-i}{i+1}\right) = -\frac{i\,(i+1)}{2i+1}. \end{aligned}$$

The last case $j = i+1$ can be similarly done, and the proof is complete. □

Therefore we recover the known fact that the inverse of the Lehmer matrix is a symmetric tridiagonal matrix.

We give the following example as a consequence of the above theorem: for $n = 4$,

$$\begin{aligned} A^{-1} &= \begin{bmatrix} \frac{4}{3} & -\frac{2}{3} & 0 & 0 \\ -\frac{2}{3} & \frac{32}{15} & -\frac{6}{5} & 0 \\ 0 & -\frac{6}{5} & \frac{108}{35} & -\frac{12}{7} \\ 0 & 0 & -\frac{12}{7} & \frac{16}{7} \end{bmatrix} \\ &= \begin{bmatrix} 1 & -\frac{2}{3} & 0 & 0 \\ 0 & \frac{4}{3} & -\frac{6}{5} & 0 \\ 0 & 0 & \frac{9}{5} & -\frac{12}{7} \\ 0 & 0 & 0 & \frac{16}{7} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 0 & -\frac{2}{3} & 1 & 0 \\ 0 & 0 & -\frac{3}{4} & 1 \end{bmatrix} \end{aligned}$$

We also give a relation between the terms of inverse of the Lehmer matrix and triangular numbers. Recall that the nth triangular number T_n is defined as the sum of the first n natural numbers, that is, $T_n = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$. We can re-write $A^{-1} = (b_{ij})$ as $b_{ij} = -\frac{2T_i}{2i+1}$ for $|i - j| = 1$, and $b_{ii} = \frac{4i^3}{4i^2-1}$.

4 Recursive Analogue of the Lehmer Matrix

In this section we investigate the same questions for our generalized recursive analogue of the Lehmer matrix $\mathcal{F}_n$ defined in the first section, namely, $\mathcal{F}_n = (g_{ij})$:

$$g_{ij} = \frac{\min\{u_{i+1}, u_{j+1}\}}{\max\{u_{i+1}, u_{j+1}\}} = \begin{cases} \frac{u_{i+1}}{u_{j+1}} & \text{if } j \geq i, \\ \frac{u_{j+1}}{u_{i+1}} & \text{if } i > j. \end{cases}$$

where u_n is the nth term of the sequence $\{u_n\}$.

For example, when $n = 5$ and $p = 1$, the matrix $\mathcal{F}_5$ takes the following form:

$$\mathcal{F}_5 = \begin{bmatrix} 1 & \frac{u_2}{u_3} & \frac{u_2}{u_4} & \frac{u_2}{u_5} & \frac{u_2}{u_6} \\ \frac{u_2}{u_3} & 1 & \frac{u_3}{u_4} & \frac{u_3}{u_5} & \frac{u_3}{u_6} \\ \frac{u_2}{u_4} & \frac{u_3}{u_4} & 1 & \frac{u_4}{u_5} & \frac{u_4}{u_6} \\ \frac{u_2}{u_5} & \frac{u_3}{u_5} & \frac{u_4}{u_5} & 1 & \frac{u_5}{u_6} \\ \frac{u_2}{u_6} & \frac{u_3}{u_6} & \frac{u_4}{u_6} & \frac{u_5}{u_6} & 1 \end{bmatrix}.$$

In order to give the LU factorization of the matrix $\mathcal{F}_n$, we define two triangular matrices.

Define the $n \times n$ unit lower triangular matrix $L_2 = (c_{ij})$ with $c_{ij} = \frac{u_{j+1}}{u_{i+1}}$ for all $i \geq j$ and $u_{ij} = 0$ for all $i < j$.

For example, when $n = 5$, the matrix takes the form:

$$L_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{u_2}{u_3} & 1 & 0 & 0 & 0 \\ \frac{u_2}{u_4} & \frac{u_3}{u_4} & 1 & 0 & 0 \\ \frac{u_2}{u_5} & \frac{u_3}{u_5} & \frac{u_4}{u_5} & 1 & 0 \\ \frac{u_2}{u_6} & \frac{u_3}{u_6} & \frac{u_4}{u_6} & \frac{u_5}{u_6} & 1 \end{bmatrix}.$$

Before defining an upper triangular matrix for the LU factorization of the matrix $\mathcal{F}_n$, we need to introduce a new sequence $\{t_n\}$ by the following relation:

$$t_n = (p-1)\, u_n + u_{n-1}, \text{ that is, } t_n = u_{n+1} - u_n, \ n > 1,$$

where u_n is defined as before.

Define the $n \times n$ upper triangular matrix $U_2 = (d_{ij})$ with $d_{1j} = \frac{u_2}{u_{j+1}}$ for $1 \leq j \leq n$, $d_{ij} = \frac{(u_i + u_{i+1}) t_i}{u_{i+1} u_{j+1}}$ for $1 < i \leq j \leq n$.

From the definition of the sequence $\{t_n\}$, we rewrite the matrix U_2 with $d_{1j} = \frac{u_2}{u_{j+1}}$ for $1 \leq j \leq n$, $d_{ij} = \frac{u_{i+1}^2 - u_i^2}{u_{i+1} u_{j+1}}$ for $1 < i \leq j \leq n$.

For example, when $n = 4$, the matrix takes the form:

$$U_2 = \begin{bmatrix} 1 & \frac{u_2}{u_3} & \frac{u_2}{u_4} & \frac{u_2}{u_5} & \frac{u_2}{u_6} \\ 0 & \frac{u_3^2 - u_2^2}{u_3^2} & \frac{u_3^2 - u_2^2}{u_3 u_4} & \frac{u_3^2 - u_2^2}{u_3 u_5} & \frac{u_3^2 - u_2^2}{u_3 u_6} \\ 0 & 0 & \frac{u_4^2 - u_3^2}{u_4^2} & \frac{u_4^2 - u_3^2}{u_4 u_5} & \frac{u_4^2 - u_3^2}{u_4 u_6} \\ 0 & 0 & 0 & \frac{u_5^2 - u_4^2}{u_5^2} & \frac{u_5^2 - u_4^2}{u_5 u_6} \\ 0 & 0 & 0 & 0 & \frac{u_6^2 - u_5^2}{u_6^2} \end{bmatrix}.$$

Theorem 4. *For $n > 0$, the factorization of matrix $\mathcal{F}_n = (g_{ij})$ is given by*

$$\mathcal{F}_n = L_2 U_2,$$

where U_2 and L_2 were defined previously.

Proof. Let $L_2U_2 = (h_{ij})$ .We consider two cases, $i > j$ and $i \leq j$. For the first case, we write

$$\begin{aligned}
h_{ij} &= \sum_{m=1}^{n} c_{im} d_{mj} = \sum_{m=1}^{j} c_{im} d_{mj} \\
&= c_{i1} d_{1j} + \sum_{m=2}^{j} \left(\frac{u_{m+1}}{u_{i+1}} \frac{\left(u_{m+1}^2 - u_m^2\right)}{u_{m+1} u_{j+1}} \right) \\
&= \frac{u_2^2}{u_{i+1} u_{j+1}} + \frac{1}{u_{i+1} u_{j+1}} \sum_{m=2}^{j} \left(u_{m+1}^2 - u_m^2\right) \\
&= \frac{u_2^2}{u_{i+1} u_{j+1}} + \frac{1}{u_{i+1} u_{j+1}} \left(u_{j+1}^2 - u_2^2\right) = \frac{u_{j+1}}{u_{i+1}} = g_{ij}.
\end{aligned}$$

If $i \leq j$, then similarly

$$\begin{aligned}
h_{ij} &= \sum_{m=1}^{n} c_{im} d_{mj} = \sum_{m=1}^{i} c_{im} d_{mj} \\
&= c_{i1} d_{1j} + \sum_{m=2}^{i} \left(\frac{u_{m+1}}{u_{i+1}} \frac{\left(u_{m+1}^2 - u_m^2\right)}{u_{m+1} u_{j+1}} \right) \\
&= \frac{u_2^2}{u_{i+1} u_{j+1}} + \frac{1}{u_{i+1} u_{j+1}} \sum_{m=2}^{i} \left(u_{m+1}^2 - u_m^2\right) \\
&= \frac{u_{i+1}}{u_{j+1}} = g_{ij},
\end{aligned}$$

and the claim is shown. □

Now we can find the value of $\det(\mathcal{F}_n)$ by considering its LU factorization.

Corollary 2. *For $n > 0$,*

$$\det(\mathcal{F}_n) = \prod_{i=2}^{n} \left(\frac{u_{i+1}^2 - u_i^2}{u_{i+1}^2} \right).$$

As a special cases of the matrix $\mathcal{F}_n$, we take the matrix $\mathcal{F}_n^0$ obtained using the Fibonacci sequence, that is, $F_{n+1} = F_n + F_{n-1}, F_0 = 0, F_1 = 1$. The determinant of this matrix becomes

$$\det\left(\mathcal{F}_n^0\right) = \frac{F_{n-1}!F_{n+2}!}{2\left(F_{n+1}!\right)^2},$$

where $F_n!$ is the Fibonomial factorial, that is, $F_n! = F_1 F_2 \cdots F_n$.

Next we give the Cholesky factorization of the generalized Lehmer matrix $\mathcal{F}_n$. For this purpose we define a lower triangular matrix $L_3 = (m_{ij})$ with $m_{i,1} = \frac{u_2}{u_{i+1}}$ for $1 \leq i \leq n$, $m_{ij} = \frac{1}{u_{i+1}}\sqrt{u_{j+1}^2 - u_j^2}$ for $1 < j \leq i \leq n$ and 0 otherwise.

When $n = 4$, the matrix L_3 takes the form:

$$L_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{u_2}{u_3} & \frac{1}{u_3}\sqrt{u_3^2 - u_2^2} & 0 & 0 \\ \frac{u_2}{u_4} & \frac{1}{u_4}\sqrt{u_3^2 - u_2^2} & \frac{1}{u_4}\sqrt{u_4^2 - u_3^2} & 0 \\ \frac{u_2}{u_5} & \frac{1}{u_5}\sqrt{u_3^2 - u_2^2} & \frac{1}{u_5}\sqrt{u_4^2 - u_3^2} & \frac{1}{u_5}\sqrt{u_5^2 - u_4^2} \end{bmatrix}.$$

The proof of the next theorem is analogous to the proof of Theorem 4, so it will be omitted.

Theorem 5. *The Cholesky factorization of the recursive analogue of the Lehmer matrix is given by*

$$\mathcal{F}_n = L_3 L_3^T$$

where L_3 is the lower triangular matrix defined previously.

5 The Inverse of the Generalized Lehmer Matrix

Here we give the inverse of the recursive analogue of the Lehmer matrix $\mathcal{F}_n^{-1}$ by considering its LU factorization. Before this, we give the inverses of the matrices L_2 and U_2 in the following lemmas, stated without proofs, as they are immediate.

Lemma 3. *Let $U_2^{-1} = (\hat{w}_{ij})$ denote the inverse of U_2. Then*

$$\hat{w}_{ij} = \begin{cases} 1 & \text{if } i = j = 1 \\ -\frac{u_{i+1}^2}{u_i^2 - u_{i+1}^2} & \text{if } 1 < i = j, \\ \frac{u_{i+1}u_{i+2}}{u_{i+1}^2 - u_{i+2}^2} & \text{if } i+1 = j, \\ 0 & \text{otherwise,} \end{cases}$$

Lemma 4. *Let $L_2^{-1} = \left(\hat{t}_{ij}\right)$ denote the inverse of L. Then*

$$\hat{t}_{ij} = \begin{cases} 1 & \text{if } i = j, \\ -\frac{u_i}{u_{i+1}} & \text{if } i = j+1, \\ 0 & \text{otherwise,} \end{cases}$$

Thus the inverse of the matrix $\mathcal{F}_n$ is found in the following theorem.

Theorem 6. *For $n > 0$, let $\mathcal{F}_n^{-1} = (q_{ij})$, then $q_{11} = \frac{u_3^2}{u_3^2 - u_2^2}$, $q_{nn} = \frac{u_{n+1}^2}{u_{n+1}^2 - u_n^2}$, $q_{i,i+1} = q_{i+1,i} = \frac{u_{i+1}u_{i+2}}{u_{i+1}^2 - u_{i+2}^2}$ for $1 \leq i \leq n-1$, $q_{ii} = \frac{u_{i+1}^2\left(u_{i+2}^2 - u_i^2\right)}{\left(u_{i+1}^2 - u_i^2\right)\left(u_{i+2}^2 - u_{i+1}^2\right)}$ for $2 \leq i \leq n-1$ and 0 otherwise.*

Proof. Since $\mathcal{F}_n^{-1} = U_2^{-1}L_2^{-1}$, the proof follows from the previous two lemmas and from matrix multiplication. □

For example, for $n = 4$,

$$\mathcal{F}_5^{-1} = \begin{bmatrix} \frac{u_3^2}{u_3^2 - u_2^2} & \frac{u_2u_3}{u_2^2 - u_3^2} & 0 & 0 \\ \frac{u_2u_3}{u_2^2 - u_3^2} & \left(\frac{u_3^2}{u_3^2 - u_2^2}\right)\left(\frac{u_4^2 - u_2^2}{u_4^2 - u_3^2}\right) & \frac{u_3u_4}{u_3^2 - u_4^2} & 0 \\ 0 & \frac{u_3u_4}{u_3^2 - u_4^2} & \left(\frac{u_4^2}{u_4^2 - u_3^2}\right)\left(\frac{u_5^2 - u_3^2}{u_5^2 - u_4^2}\right) & \frac{u_4u_5}{u_4^2 - u_5^2} \\ 0 & 0 & \frac{u_4u_5}{u_4^2 - u_5^2} & \frac{u_5^2}{u_5^2 - u_4^2} \end{bmatrix}.$$

6 Further comment

With a bit more care, one can certainly remove the constraint $q = -1$ on the sequence U_n, and prove similar results like in the present paper for the corresponding generalized Lehmer matrix.

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