

# Netted Matrices

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## Abstract:

We prove that powers of 4-netted matrices (the entries satisfy a four-term recurrence  $\delta a_{i,j} = \alpha a_{i-1,j} + \beta a_{i-1,j-1} + \gamma a_{i,j-1}$ ) preserve this property of nettedness, that is the entries of the  $e$ -th power satisfy  $\delta_e a_{i,j}^{(e)} = \alpha_e a_{i-1,j}^{(e)} + \beta_e a_{i-1,j-1}^{(e)} + \gamma_e a_{i,j-1}^{(e)}$ , where the coefficients are all instances of the same sequence,  $x_{e+1} = (\beta + \delta)x_e - (\beta\delta + \alpha\gamma)x_{e-1}$ . Also, we find a matrix  $Q_n(a, b)$  and a vector  $v$ , such that  $Q_n(a, b)^e \cdot v$  acts as a shifting on the general second-order recurrence sequence with parameters  $a, b$ . It generalizes the known property  $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^e \cdot (1, 0)^t = (F_{e-1}, F_e)^t$ . We use it to prove a few interesting identities. In the last section we prove some results about congruences satisfied by the matrix  $Q_n(a, b)$ .

**Keywords.** *Binomial Coefficients; Matrices; Sequences; Congruences.*

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# 1 Introduction

In [4], Peele and Stănică studied  $n \times n$  matrices with the  $(i, j)$  entry the binomial coefficient  $\binom{i-1}{j-1}$  (matrix  $L_n$ ), respectively  $\binom{i-1}{n-j}$  (matrix  $R_n$ ) and derived many interesting results on powers of these matrices.  $L_n$  was easily subdued, but curiously enough, closed forms for entries of powers of  $R_n$ , say  $R_n^e$ , were not found. However, recurrences among various entries of  $R_n^e$  were proved and precise results on congruences modulo any prime  $p$  were found. To accomplish that, the authors of [4] proved that the entries  $a_{i,j}^{(e)}$  of the  $e$ -th power of  $R_n$ , satisfy

$$F_{e-1}a_{i,j}^{(e)} = F_e a_{i-1,j}^{(e)} + F_{e+1}a_{i-1,j-1}^{(e)} - F_e a_{i,j-1}^{(e)},$$

where  $F_e$  is the Fibonacci sequence,  $F_{e+1} = F_e + F_{e-1}$ ,  $F_0 = 0$ ,  $F_1 = 1$ . As we shall see in our first result, this is not a singular phenomenon. The goal of this paper is two-fold: we prove the results of [4] for a class of matrices, containing  $R_n$ , where the entries satisfy any four-term recurrence (we call these *3-netted* matrices), and we find a possible generalization of the  $Q$ -matrix, namely a matrix  $Q_n(a, b)$ , with the property that any power multiplied by a fixed vector gives an  $n$ -tuple of consecutive terms of the general Pell or Fibonacci sequence. It generalizes the known property  $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^e \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} F_{e-1} \\ F_e \end{pmatrix}$ . We also find the generating function for the entries of powers of these matrices. As applications we find some interesting identities for general Fibonacci (or Pell) numbers. In the fifth section we provide a few results on the order of these matrices modulo a prime.

## 2 Matrices with Entries Satisfying a General Four-Term Recurrence

Define a tableau with elements  $a_{i,j}$ ,  $i \geq 0$ ,  $j \geq 0$ , which satisfy (for  $i \geq 1, j \geq 1$ )

$$\delta a_{i,j} = \alpha a_{i-1,j} + \beta a_{i-1,j-1} + \gamma a_{i,j-1}, \quad (1)$$

with the boundary conditions

$$\beta a_{i,0} + \gamma a_{i+1,0} = 0, \forall 1 \leq i \leq n-1 \quad (2)$$

$$\delta a_{i+1,n+1} - \alpha a_{i,n+1} = 0, \forall 1 \leq i \leq n-1. \quad (3)$$

We remark that if the 0-th and  $(n+1)$ -th column are made up of zeros, then the conditions (2) and (3) are fulfilled.

In our main result of this section we prove that (1) is preserved for higher powers of the  $n \times n$  matrix  $(a_{i,j})_{i=1\dots n, j=1\dots n}$ . Precisely, we prove

**Theorem 1.** *The entries of the  $e$ -th power of the matrix  $R = (a_{i,j})_{i=1\dots n, j=1\dots n}$  satisfy the recurrence*

$$\delta_e a_{i,j}^{(e)} = \alpha_e a_{i-1,j}^{(e)} + \beta_e a_{i-1,j-1}^{(e)} + \gamma_e a_{i,j-1}^{(e)}, \quad i, j \leq n,$$

where the sequences  $\alpha_e, \beta_e, \gamma_e, \delta_e$  are all instances of the sequence  $x_e$  satisfying

$$x_{e+1} = (\beta + \delta)x_e - (\beta\delta + \alpha\gamma)x_{e-1},$$

with initial conditions  $(\delta_1 = \delta; \delta_2 = \delta^2 - \alpha\gamma); (\alpha_1 = \alpha; \alpha_2 = \alpha(\delta + \beta)); (\beta_1 = \beta; \beta_2 = \beta^2 - \alpha\gamma)$  and  $(\gamma_1 = \gamma; \gamma_2 = \gamma(\beta + \delta))$ .

*Proof.* We prove by induction on  $e$  that there exists a relation among the entries of any  $2 \times 2$  cells, namely

$$\delta_e a_{i,j}^{(e)} = \alpha_e a_{i-1,j}^{(e)} + \beta_e a_{i-1,j-1}^{(e)} + \gamma_e a_{i,j-1}^{(e)}.$$

The above relation is certainly true for  $e = 1$ . We evaluate, for  $i \geq 2$ ,

$$\begin{aligned}
\alpha \delta_{e-1} a_{i-1,j}^{(e)} &= \sum_{s=1}^n \alpha \delta_{e-1} a_{i-1,s} a_{s,j}^{(e-1)} \\
&= \sum_{s=1}^n \alpha a_{i-1,s} \left( \alpha_{e-1} a_{s-1,j}^{(e-1)} + \beta_{e-1} a_{s-1,j-1}^{(e-1)} + \gamma_{e-1} a_{s,j-1}^{(e-1)} \right) \\
&= \sum_{s=1}^n (\delta a_{i,s} - \beta a_{i-1,s-1} - \gamma a_{i,s-1}) \left( \alpha_{e-1} a_{s-1,j}^{(e-1)} + \beta_{e-1} a_{s-1,j-1}^{(e-1)} \right) \\
&\quad + \sum_{s=1}^n \alpha \gamma_{e-1} a_{i-1,s} a_{s,j-1}^{(e-1)} = \sum_{s=1}^n \delta a_{i,s} \left( \alpha_{e-1} a_{s-1,j}^{(e-1)} + \beta_{e-1} a_{s-1,j-1}^{(e-1)} \right) \\
&\quad - \gamma \alpha_{e-1} a_{i,j}^{(e)} - \beta \beta_{e-1} a_{i-1,j-1}^{(e)} - \gamma \beta_{e-1} a_{i,j-1}^{(e)} - \beta \alpha_{e-1} a_{i-1,j}^{(e)} \\
&\quad + \alpha \gamma_{e-1} a_{i-1,j-1}^{(e)} - \gamma \alpha_{e-1} \left( a_{i,0} a_{0,j}^{(e-1)} - a_{i,n} a_{n,j}^{(e-1)} \right) \\
&\quad - \beta \beta_{e-1} \left( a_{i-1,0} a_{0,j-1}^{(e-1)} - a_{i-1,n} a_{n,j-1}^{(e-1)} \right) \\
&\quad - \gamma \beta_{e-1} \left( a_{i,0} a_{0,j-1}^{(e-1)} - a_{i,n} a_{n,j-1}^{(e-1)} \right) \\
&\quad - \beta \alpha_{e-1} \left( a_{i-1,0} a_{0,j}^{(e-1)} - a_{i-1,n} a_{n,j}^{(e-1)} \right).
\end{aligned}$$

Using the boundary conditions (2) and (3), we obtain, for  $i \geq 2$ ,

$$\begin{aligned}
\alpha\delta_{e-1}a_{i-1,j}^{(e)} &= (\alpha\gamma_{e-1} - \beta\beta_{e-1})a_{i-1,j-1}^{(e)} - \gamma\alpha_{e-1}a_{i,j}^{(e)} - \gamma\beta_{e-1}a_{i,j-1}^{(e)} \\
&\quad - \beta\alpha_{e-1}a_{i-1,j}^{(e)} + \sum_{s=1}^n \delta a_{i,s} \left( \delta_{e-1}a_{s,j}^{(e-1)} - \gamma_{e-1}a_{s,j-1}^{(e-1)} \right) \\
&\quad + \left( \alpha_{e-1}a_{n,j}^{(e-1)} + \beta_{e-1}a_{n,j-1}^{(e-1)} \right) (\beta a_{i-1,n} + \gamma a_{i,n}) \\
&\quad - \left( \alpha_{e-1}a_{0,j}^{(e-1)} + \beta_{e-1}a_{0,j-1}^{(e-1)} \right) (\beta a_{i-1,0} + \gamma a_{i,0}) \\
&= (\alpha\gamma_{e-1} - \beta\beta_{e-1})a_{i-1,j-1}^{(e)} - \gamma\alpha_{e-1}a_{i,j}^{(e)} - \gamma\beta_{e-1}a_{i,j-1}^{(e)} \\
&\quad - \beta\alpha_{e-1}a_{i-1,j}^{(e)} + \delta\delta_{e-1}a_{i,j}^{(e)} - \delta\gamma_{e-1}a_{i,j-1}^{(e)} \\
&\quad + \left( \alpha_{e-1}a_{n,j}^{(e-1)} + \beta_{e-1}a_{n,j-1}^{(e-1)} \right) (\delta a_{i,n+1} - \alpha a_{i-1,n+1}) \\
&= (\alpha\gamma_{e-1} - \beta\beta_{e-1})a_{i-1,j-1}^{(e)} + (\delta\delta_{e-1} - \gamma\alpha_{e-1})a_{i,j}^{(e)} \\
&\quad - (\gamma\beta_{e-1} + \delta\gamma_{e-1})a_{i,j-1}^{(e)} - \beta\alpha_{e-1}a_{i-1,j}^{(e)}.
\end{aligned}$$

Thus,

$$\begin{aligned}
(\delta\delta_{e-1} - \gamma\alpha_{e-1})a_{i,j}^{(e)} &= (\alpha\delta_{e-1} + \beta\alpha_{e-1})a_{i-1,j}^{(e)} + (\beta\beta_{e-1} - \alpha\gamma_{e-1})a_{i-1,j-1}^{(e)} \\
&\quad + (\gamma\beta_{e-1} + \delta\gamma_{e-1})a_{i,j-1}^{(e)}.
\end{aligned}$$

Therefore, we obtain the system of sequences

$$\delta_e = \delta\delta_{e-1} - \gamma\alpha_{e-1} \quad (4)$$

$$\alpha_e = \alpha\delta_{e-1} + \beta\alpha_{e-1} \quad (5)$$

$$\beta_e = \beta\beta_{e-1} - \alpha\gamma_{e-1} \quad (6)$$

$$\gamma_e = \gamma\beta_{e-1} + \delta\gamma_{e-1}. \quad (7)$$

From (4) we get  $\alpha_{e-1} = (\delta/\gamma)\delta_{e-1} - (1/\gamma)\delta_e$ , which replaced in (5) gives the recurrence

$$\delta_{e+1} = (\beta + \delta)\delta_e - (\beta\delta + \gamma\alpha)\delta_{e-1}.$$

Similarly,  $\alpha_{e+1} = (\beta + \delta)\alpha_e - (\beta\delta + \gamma\alpha)\alpha_{e-1}$ ;  $\beta_{e+1} = (\beta + \delta)\beta_e - (\beta\delta + \gamma\alpha)\beta_{e-1}$ ;  $\gamma_{e+1} = (\beta + \delta)\gamma_e - (\beta\delta + \gamma\alpha)\gamma_{e-1}$ . The initial conditions are  $(\delta_1 = \delta; \delta_2 = \delta^2 - \alpha\gamma)$ ,  $(\alpha_1 = \alpha; \alpha_2 = \alpha(\delta + \beta))$ ,  $(\beta_1 = \beta; \beta_2 = \beta^2 - \alpha\gamma)$ ,  $(\gamma_1 = \gamma; \gamma_2 = \gamma(\beta + \delta))$ .  $\square$

**Example 2.** As examples of tableaux satisfying our conditions, we have  $a_{i,j}^1 = \binom{i-1}{j-1}$  ( $\delta = 1, \alpha = 1, \beta = 1, \gamma = 0$ ),  $a_{i,j}^2 = \binom{i-1}{n-j}$  ( $\delta = 0, \alpha = 1, \beta = 1, \gamma = -1$ ),  $a_{i,j}^3 = \binom{n-i}{n-j}$  ( $\delta = 1, \alpha = 0, \beta = -1, \gamma = 1$ ). Other examples are given by the alternating matrices  $(-1)^{i+j}a_{i,j}^k$  (or  $(-1)^{i-1}a_{i,j}^k$  or  $(-1)^{j-1}a_{i,j}^k$ , etc.),  $k = 1, 2, 3$ . In the next section we present more examples.

### 3 Higher Order Fibonacci Matrices

In this section we uncover a very interesting side of our previous results. A matrix of the form  $M = \begin{pmatrix} 0 & 1 \\ 1 & m \end{pmatrix}$  is called a *Fibonacci* or *Q-matrix*. It is known that if the sequence  $U_{e+1} = mU_e + U_{e-1}$ ,  $U_0 = 0$ ,  $U_1 = 1$ , then  $M^e = \begin{pmatrix} U_{e-1} & U_e \\ U_e & U_{e+1} \end{pmatrix}$  and  $M^e \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} U_{e-1} \\ U_e \end{pmatrix}$ . Next, we find a matrix  $Q_n(a, b)$  such that  $Q_n(a, b)^e \cdot v$  is a vector of  $n$  consecutive terms of the sequence  $U_n$ , for any power  $e$  and any vector  $v$  of alternating consecutive terms in the sequence  $U_{n+1} = aU_n + bU_{n-1}$ . Let  $I_n$  be the identity matrix of dimension  $n$  and  $M^t$  be the *transpose* of a matrix  $M$ .

Let  $a_{i,j} = a_{i,j}^{(1)} = a^{i+j-n-1}b^{n-j} \binom{i-1}{n-j}$  and  $Q_n(a, b) = (a_{i,j})_{i,j}$ . We use our previous results to show

**Theorem 3.** Let  $w = ((-1)^n U_{n-1}, (-1)^{n-1} U_{n-2}, \dots, -U_0)^t$ . Then

$$Q_n(a, b)^{e+1} \cdot w = (U_{(n-1)e}, U_{(n-1)e+1}, \dots, U_{(n-1)(e+1)})^t \text{ and} \quad (8)$$

$$U_{e-1}a_{i,j}^{(e)} + U_e a_{i,j-1}^{(e)} = U_e a_{i-1,j}^{(e)} + U_{e+1} a_{i-1,j-1}^{(e)}, \quad (9)$$

where  $a_{i,j}^{(e)}$  are the entries of  $Q_n(a,b)^e$  and  $U_e$  is the sequence satisfying  $U_{e+1} = aU_e + bU_{e-1}$ ,  $U_0 = 0, U_1 = 1$ . Moreover,  $Q_n(a,b)$  is unique with the property  $a_{1,j} = 0, j < n, a_{i,n} = a^{i-1}$  and  $a_{i,j} = a a_{i-1,j} + b a_{i-1,j+1}$ .

*Proof.* First, the  $i$ -th entry of  $Q_n(a,b)^{e+1} \cdot w$  is

$$\begin{aligned} \sum_{j=1}^n (-1)^{n+1-j} a_{i,j}^{(e+1)} U_{n-j} &= \sum_{j=1}^n (-1)^{n+1-j} \sum_{k=1}^n a_{i,k} a_{k,j}^{(e)} U_{n-j} \\ &= \sum_{k=1}^n a_{i,k} \sum_{j=1}^n (-1)^{n+1-j} a_{k,j}^{(e)} U_{n-j} \\ &= \sum_{j=1}^n a_{i,k} U_{(n-1)(e-1)+k-1}. \end{aligned}$$

The initial condition and the step of induction (on  $e$ ) will both follow if we can prove that the matrix  $Q_n$  acts as an index-translation on our sequence  $U_l$ , namely

$$\sum_{k=1}^n a_{i,k} U_{t+k} = U_{t+n+i-1}, \quad t \geq -1.$$

Let  $W_i = \sum_{k=1}^n a_{i,k} U_{t+k}$  ( $t$  is assumed fixed). First,  $W_1 = \sum_{k=1}^n a_{1,k} U_{t+k} = a_{1,n} U_{t+n} = U_{t+n}$ . Then,  $W_2 = \sum_{k=1}^n a_{2,k} U_{t+k} = a_{2,n-1} U_{t+n-1} + a_{2,n} U_{t+n} = b U_{t+n-1} + a U_{t+n} = U_{t+n+1}$ .

Now, for  $1 \leq i \leq n-1$ ,

$$\begin{aligned} W_{i+1} &= \sum_{k=1}^n a_{i+1,k} U_{t+k} = \sum_{k=1}^n (a a_{i,k} + b a_{i,k+1}) U_{t+k} \\ &= a W_i + \sum_{k=1}^n a_{i,k+1} b U_{t+k} = a W_i + \sum_{k=1}^{n-1} a_{i,k+1} (U_{t+k+2} - a U_{t+k+1}) \\ &\stackrel{u=k+1}{=} a W_i + \sum_{u=2}^n a_{i,u} U_{t+u+1} - a \sum_{u=2}^n a_{i,u} U_{t+u} \\ &= b V_i - b a_{i,1} U_{t+2} + a a_{i,1} U_{t+1} \stackrel{a_{i,1}=0 \text{ if } i \leq n-1}{=} V_i, \end{aligned}$$

where

$$\begin{aligned}
V_i &= \sum_{u=1}^n a_{i,u} U_{t+u+1} = \sum_{u=1}^n (a a_{i-1,u} + b a_{i-1,u+1}) U_{t+u+1} \\
&\stackrel{a_{i+1,n+1}=0}{=} a V_{i-1} + b \sum_{u=1}^{n-1} a_{i-1,u+1} U_{t+u+1} \stackrel{u+1=s}{=} a V_{i-1} + b \sum_{u=2}^n a_{i-1,s} U_{t+s} \\
&= a V_{i-1} + b W_{i-1} - b a_{i-1,1} U_{t+1} \stackrel{a_{i-1,1}=0 \text{ if } i \leq n-1}{=} a V_{i-1} + b W_{i-1}.
\end{aligned}$$

Using  $V_i = W_{i+1}$  in the previous recurrence, we get  $W_{i+1} = a W_i + b W_{i-1}$ . Therefore,  $W_i = U_{t+n+i-1}$ , since  $W_1 = U_{t+n}$ ,  $W_2 = U_{t+n+1}$ .

Using Theorem 1, with  $\delta = 0$ ,  $\alpha = b$ ,  $\beta = a$ ,  $\gamma = -1$ , we get the recurrence between the entries of the higher power of  $Q_n$ , namely  $U_{e-1} a_{i,j}^{(e)} + U_e a_{i,j-1}^{(e)} = U_e a_{i-1,j}^{(e)} + U_{e+1} a_{i-1,j-1}^{(e)}$ , with the appropriate initial conditions. The fact that  $Q_n$  is the unique matrix with the given properties follows easily observing that such a matrix could be defined inductively as follows: let  $Q_1 = 1$ . Assume  $Q_{n-1} = (a_{i,j})_{i,j=1,2,\dots,n-1}$  and construct  $Q_n$  by bordering  $Q_{n-1}$  with the first column and the last row (left and bottom). The first column is  $(0, 0, \dots, 0, 1)^t$  and the last row is given by:  $a_{n,n} = a^{n-1}$  and  $a_{n,j} = a a_{n-1,j} + b a_{n-1,j+1}$ .  $\square$

**Definition 4.** We call such a matrix  $Q_n(a, b)$  a generalized Fibonacci or  $Q$ -matrix of dimension  $n$  and parameters  $a, b$ .

**Example 5.** We give here the first few powers of  $Q_3(a, b)$ ,

$$\begin{aligned}
Q_3(a, b) &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & b & a \\ b^2 & 2ab & a^2 \end{pmatrix}, \quad Q_3(a, b)^2 = \begin{pmatrix} b^2 & 2ab & a^2 \\ ab^2 & b(2a^2 + b) & a(a^2 + b) \\ a^2 b^2 & 2ab(a^2 + b) & (a^2 + b)^2 \end{pmatrix}, \\
Q_3(a, b)^3 &= \begin{pmatrix} a^2 b^2 & 2ab(a^2 + b) & (a^2 + b)^2 \\ ab^2(a^2 + b) & b(2a^4 + 4a^2 b + b^2) & a^5 + 3a^3 b + 2ab^2 \\ b^2(a^2 + b)^2 & 2ab(a^4 + 3a^2 b + 2b^2) & (a^3 + 2ab)^2 \end{pmatrix},
\end{aligned}$$

## 4 Some Generating Functions and an Inverse

Although we cannot find simple closed forms for *all* entries of  $Q_n(a, b)^e$ , we prove

**Theorem 6.** *The generating function for  $a_{i,j}^{(e)}$  is*

$$B_n^{(e)}(x, y) = \frac{(U_{e-1} + U_e y)(bU_{e-1} + yU_e)^{n-1}}{U_{e-1} + U_e y - x(U_e + U_{e+1}y)}$$

*Proof.* Multiplying the recurrence (9) by  $x^{i-1}y^{j-1}$  and summing for  $i, j \geq 2$ , we get

$$\begin{aligned} & U_{e-1} \sum_{i,j \geq 2} a_{i,j}^{(e)} x^{i-1} y^{j-1} + U_e y \sum_{i,j \geq 2} a_{i,j-1}^{(e)} x^{i-1} y^{j-2} = \\ & U_e x \sum_{i,j \geq 2} a_{i-1,j}^{(e)} x^{i-2} y^{j-1} + U_{e+1} x y \sum_{i,j \geq 2} a_{i-1,j-1}^{(e)} x^{i-2} y^{j-2} \end{aligned}$$

Thus,

$$\begin{aligned} & U_{e-1} \left( B_n^{(e)}(x, y) - \sum_{i \geq 1} a_{i,1}^{(e)} x^{i-1} - \sum_{j \geq 1} a_{1,j}^{(e)} y^{j-1} + a_{1,1}^{(e)} \right) \\ & \quad + U_e y \left( B_n^{(e)}(x, y) - \sum_{j \geq 1} a_{1,j}^{(e)} y^{j-1} \right) = \\ & U_e x \left( B_n^{(e)}(x, y) - \sum_{i \geq 1} a_{i,1}^{(e)} x^{i-1} \right) + U_{e+1} x y B_n^{(e)}(x, y). \end{aligned}$$

Solving for  $B_n^{(e)}(x, y)$ , we get

$$\begin{aligned} & B_n^{(e)}(x, y) (U_{e-1} + U_e y - x(U_e + U_{e+1}y)) = \\ & (U_{e-1} - U_e x) \sum_{i \geq 1} a_{i,1}^{(e)} x^{i-1} + (U_{e-1} + U_e y) \sum_{j \geq 1} a_{1,j}^{(e)} y^{j-1} - U_{e-1} a_{1,1}^{(e)}. \end{aligned} \tag{10}$$

We need to find  $a_{i,1}^{(e)}$  and  $a_{1,j}^{(e)}$ . As in [4], we prove

$$\begin{aligned} & a_{1,j}^{(e)} = b^{n-j} U_{e-1}^{n-j} U_e^{j-1} \binom{n-1}{j-1} \\ & a_{i,1}^{(e)} = b^{n-1} U_{e-1}^{n-i} U_e^{i-1}. \end{aligned} \tag{11}$$

There is no difficulty to show the relations for  $e = 1, 2$ . Assume  $e \geq 3$ . First we deal with the elements in the first row,

$$\begin{aligned}
a_{1,j}^{(e+1)} &= \sum_{s=1}^n a_{1,s}^{(e)} a_{s,j} = \sum_{s=1}^n b^{n-s} U_{e-1}^{n-s} U_e^{s-1} a^{s+j-n-1} b^{n-j} \binom{n-1}{s-1} \binom{s-1}{n-j} \\
&= \sum_{s=1}^n U_{e-1}^{n-s} U_e^{s-1} a^{s+j-n-1} b^{2n-j-s} \binom{n-1}{j-1} \binom{j-1}{n-s} \\
&= b^{n-j} a^{j-1} U_e^{n-1} \binom{n-1}{j-1} \sum_{s=1}^n \left( \frac{bU_{e-1}}{aU_e} \right)^{n-s} \binom{j-1}{n-s} \\
&= b^{n-j} a^{j-1} U_e^{n-1} \binom{n-1}{j-1} \left( 1 + \frac{bU_{e-1}}{aU_e} \right)^{j-1} = b^{n-j} U_e^{n-j} U_{e+1}^{j-1} \binom{n-1}{j-1}.
\end{aligned}$$

Now we find the elements in the first column,

$$\begin{aligned}
a_{i,1}^{(e+1)} &= \sum_{s=1}^n a_{i,s} a_{s,1}^{(e)} = \sum_{s=1}^n a^{i+s-n-1} b^{n-s} \binom{i-1}{n-s} b^{n-1} U_{e-1}^{n-s} U_e^{s-1} \\
&= a^{i-1} b^{n-1} U_e^{n-1} \sum_{s=1}^n \left( \frac{bU_{e-1}}{aU_e} \right)^{n-s} \binom{i-1}{n-s} \\
&= a^{i-1} b^{n-1} U_e^{n-1} \left( 1 + \frac{bU_{e-1}}{aU_e} \right)^{i-1} = b^{n-1} U_e^{n-i} U_{e+1}^{i-1}.
\end{aligned}$$

Using (11), we get

$$\begin{aligned}
\sum_{j \geq 1} a_{1,j}^{(e)} y^{j-1} &= \sum_{j \geq 1} U_{e-1}^{n-j} U_e^{j-1} b^{n-j} y^{j-1} \binom{n-1}{j-1} \\
&= b^{n-1} \sum_{s \geq 0} U_{e-1}^{(n-1)-s} \left( \frac{yU_e}{b} \right)^s \binom{n-1}{s} \\
&= (bU_{e-1} + yU_e)^{n-1}
\end{aligned}$$

Using (10),  $(U_{e-1} - U_e x) \sum_{i \geq 1} U_{e-1}^{n-i} U_e^{i-1} b^{n-1} x^{i-1} = b^{n-1} U_{e-1}^n$  and  $U_{e-1} a_{1,1}^{(e)} = b^{n-1} U_{e-1}^n$ ,

we deduce the result.  $\square$

The inverse of  $Q_n(a, b)$  is not difficult to find. We have

**Theorem 7.** *The inverse of  $Q_n(a, b)$  is*

$$Q_n(a, b)^{-1} = \left( (-1)^{n+i+j+1} a^{n+1-i-j} b^{i-n} \binom{n-i}{j-1} \right)_{i,j}.$$

*Proof.* Straightforward. □

In general, finding *simple* closed forms for the entries of powers of  $Q_n(a, b)$  seems to be a very difficult matter. We can derive (after some work) simple formulas for the entries of the second row and column of  $Q_n(a, b)^e$ .

**Proposition 8.** *We have  $a_{2,j}^{(e)} = b^{n-j} U_{e-1}^{n-j-1} U_e^j \binom{n-2}{j-1} + b^{n-j} U_{e-1}^{n-j} U_e^{j-2} U_{e+1} \binom{n-2}{j-2}$ , and  $a_{i,2}^{(e)} = (n-i) b^{n-2} U_{e-1}^{n-i-1} U_e + (i-1) b^{n-2} U_{e-1}^{n-i} U_e^{i-2} U_{e+1}$ .*

**Remark 9.** *Since  $b^{n-j} a_{j,n}^{(e-1)} = a_{j,1}^{(e)}$  and  $a_{n,j}^{(e-1)} = a_{1,j}^{(e)}$ , we get closed forms for the last row and column of  $Q_n(a, b)^e$ , as well.*

By taking some particular cases of our previous results we get some very interesting binomial sums. For instance,

**Corollary 10.** *We have*

1.  $\sum_{j=1}^n (-1)^{n+1-j} a^{i+j-n-1} b^{n-j} \binom{i-1}{n-j} U_{n-j} = U_{i-1}.$
2.  $\sum_{j=1}^n \sum_{k=1}^n (-1)^{n+1-j} a^{i+j+2k-2n-2} b^{2n-j-k} \binom{i-1}{n-k} \binom{k-1}{n-j} U_{n-j} = U_{n+i-2}.$
3.  $\sum_{j=1}^n U_{l-1}^{n-j} U_l^{j-1} U_{(n-1)p+j-1} b^{n-j} \binom{n-1}{j-1} = U_{(n-1)(l+p)},$  for any  $l, p.$
4.  $\sum_{j=1}^n U_{(n-1)p+j-1} U_{l-1}^{n-j-1} U_l^{j-2} b^{n-j} \left[ U_l^2 \binom{n-1}{j-1} + (-1)^l \binom{n-2}{j-2} \right] = U_{(n-1)(l+p)+1},$  for any  $l, p.$

*Proof.* Using Theorem 3, with  $e = 1, 2$ , we obtain the first two identities. Now, with the help of Theorem 3 and the trivial identity  $Q_n(a, b)^{l+p} = Q_n(a, b)^l Q_n(a, b)^p$ , we get

$$\begin{aligned} \left( Q_n(a, b)^l Q_n(a, b)^p \right) \cdot v &= Q_n(a, b)^l \cdot (U_{(n-1)p}, U_{(n-1)p+1}, \dots, U_{(n-1)(p+1)})^t \\ &= (U_{(n-1)(l+p)}, U_{(n-1)(l+p)+1}, \dots, U_{(n-1)(l+p+1)}) \cdot \end{aligned}$$

Since  $a_{1,j}^{(l)} = U_{l-1}^{n-j} U_l^{j-1} b^{n-j} \binom{n-1}{j-1}$ , we obtain the third identity.

Using  $a_{2,j}^{(l)} = b^{n-j} U_{l-1}^{n-j-1} U_l^j \binom{n-2}{j-1} + b^{n-j} U_{l-1}^{n-j} U_l^{j-2} U_{l+1} \binom{n-2}{j-2}$  and Cassini's identity (see [2], p. 292) (usually given for the Fibonacci numbers, but certainly true for the sequence  $U_l$ , as well, as the reader can check easily),  $U_{l-1} U_{l+1} - U_l^2 = (-1)^l$ , we get the fourth identity. □

**Corollary 11.** *In general, we have*

$$\sum_{j=1}^n U_{(n-1)p+j-1} a_{i,j}^{(l)} = U_{(n-1)(l+p)+i-1}$$

for any  $i, l, p$ .

## 5 Order of $Q_n(a, b)$ modulo $p$

Let  $a, b \in \mathbf{Z}$  and  $p \in \mathbf{Z}$  prime. Using the recurrence among the entries of  $Q_n(a, b)$ , and reasoning as in [4], we prove the following

**Theorem 12.** *If  $e$  is the least positive integer (entry point) such that  $U_e \equiv 0 \pmod{p}$ , then*

$$Q_{2k}(a, b)^e \equiv (-1)^{(k+1)e} U_{e-1} I_{2k} \pmod{p}$$

$$Q_{2k+1}(a, b)^e \equiv (-1)^{ke} I_{2k+1} \pmod{p}.$$

Moreover,  $Q_n(a, b)^{4e} \equiv I_n \pmod{p}$ . Furthermore, considering the parity of  $e$ , we have

$$Q_n(a, b)^{2e} \equiv I_n \pmod{p} \text{ if } e \text{ even}$$

$$Q_n(a, b)^{2e} \equiv r^{n-1} I_n \pmod{p} \text{ if } e \equiv 3 \pmod{4}$$

$$Q_n(a, b)^{2e} \equiv (-r)^{n-1} I_n \pmod{p} \text{ if } e \equiv 1 \pmod{4},$$

where  $r \equiv \frac{U_{(e+1)/2}}{U_{(e-1)/2}} \pmod{p}$ , so  $r^2 \equiv -1 \pmod{p}$ .

*Proof.* Using (9), if  $U_e \equiv 0 \pmod{p}$ , then

$$U_{e-1} a_{i,j}^{(e)} \equiv U_{e+1} a_{i-1,j-1}^{(e)}.$$

Since  $p$  divides neither  $U_{e-1}$  nor  $U_{e+1}$  (otherwise it would divide  $U_1 = 1$ ), we get

$$a_{i,j}^{(e)} \equiv 0 \pmod{p}, \text{ if } i \neq j,$$

$$a_{i,i}^{(e)} \equiv a_{i-1,i-1}^{(e)} \equiv \cdots \equiv a_{1,1}^{(e)} \equiv U_{e-1}^{n-1} \pmod{p}.$$

Therefore

$$Q_n(a, b)^e \equiv U_{e-1}^{n-1} I_n \pmod{p}.$$

Using Cassini's identity  $U_{l-1}U_{l+1} - U_l^2 = (-1)^l$ , for  $l = e$ , we get, if  $n = 2k$ ,

$$U_{e-1}^{n-1} = U_{e-1}^{2k-1} \equiv (U_{e-1}^2)^k U_{e-1}^{-1} \equiv (-1)^{ke} U_{e-1}^{-1} \equiv (-1)^{(k+1)e} U_{e-1} \pmod{p},$$

since  $U_{e-1}^2 \equiv U_{e+1}^2 \equiv (-1)^e \pmod{p}$ . If  $n = 2k + 1$ , then

$$U_{e-1}^{n-1} = U_{e-1}^{2k} \equiv (U_{e-1}^2)^k \equiv (-1)^{ke} \pmod{p}.$$

The previous two congruences replaced in  $Q_n(a, b)^e \equiv U_{e-1}^{n-1} I_n \pmod{p}$ , prove the first claim.

Lemma 3.4 of [3] implies

$$U_{e-1} \equiv (-1)^{\frac{e-2}{2}} \text{ if } e \text{ even}$$

$$U_{e-1} \equiv r(-1)^{\frac{e-3}{2}}, r^2 \equiv -1 \pmod{p}, \text{ if } e \text{ odd}.$$

The residue  $r$  in the previous relation is just  $r \equiv \frac{U_{(e+1)/2}}{U_{(e-1)/2}} \pmod{p}$ . Thus, if  $e$  even, then  $U_{e-1}^2 \equiv 1 \pmod{p}$ , so  $Q_n(a, b)^{2e} \equiv I_n \pmod{p}$ , for any  $n$ . The remaining cases are similar.  $\square$

Similarly, we can prove

**Theorem 13.** 1. If  $p \mid U_{p-1}$ , then  $Q_n(a, b)^{p-1} \equiv I_n \pmod{p}$ .

2. If  $p \mid U_{p+1}$ , then  $Q_{2k+1}(a, b)^{p+1} \equiv I_{2k+1} \pmod{p}$  and  $Q_{2k}(a, b)^{p+1} \equiv -I_{2k} \pmod{p}$ .

A consequence of Theorem 1 of [1] is

**Lemma 14.** For a prime  $p$  which divides  $f(x) = x^2 - ax - 1$  for some integer  $x$ , the sequence  $\{U_e\}_e$ , satisfying the recurrence  $U_e = aU_{e-1} + U_{e-2}$ , has a period  $p - 1 \pmod{p}$ , provided  $p$  is not a divisor of  $D = a^2 + 4$ .

Our final result is

**Theorem 15.** Let  $p$  be a prime divisor of  $x^2 - ax - 1$ , for some integer  $x$  and  $\gcd(p, a^2 + 4) =$

1. Then,  $Q_n(a, 1)^{p-1} \equiv I_n \pmod{p}$ .

*Proof.* Straightforward, using Lemma 14 and Theorem 3 or Theorem 12.  $\square$

## 6 Further Comments

We observed that netted matrices defined using three-term or four-term recurrences with constant coefficients (we call these 3 or 4-netted matrices) preserve a *four-term* recurrence among the entries of their powers. We ask the question:

*What is the order of the recurrence for higher powers of a 5-netted, etc., matrices?*

We might attempt to prove that a  $k$ -netted matrix will preserve a  $k$ -term recurrence. However, that is not true, and it can be seen from our work, since a two-term recurrence is not preserved (see our first theorem). Our guess is that a  $k^2$ -netted matrix preserves a  $k^2$ -term recurrence among the entries of its higher powers. Our guess is based on work already done and on many computer hours running examples. However, it is too early to promote it to a conjecture.

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