$F_1F_2F_3F_4F_5F_6F_8F_{10}F_{12} = 11!$

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Abstract

In this paper, we show that the equality appearing in the title gives the largest solution of the diophantine equation

$$F_{n_1} \dots F_{n_k} = m_1! \dots m_t!,$$

where $0 < n_1 < \dots < n_k$ and $1 \le m_1 \le m_2 \le \dots \le m_t$ are integers.

1 Introduction

Recall that the Fibonacci sequence denoted by $(F_n)_{n\ge 0}$ is the sequence of integers given by $F_0 = 0,\ F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all $n \ge 0$.

There are many papers in the literature which address diophantine equations involving Fibonacci numbers. A long standing problem asking whether 0, 1, 8 and 144 are the only perfect powers in the Fibonacci sequence was recently confirmed by Bugeaud, Mignotte and Siksek [2]. An extension of such a result to diophantine equations involving perfect powers in products of Fibonacci numbers whose indices form an arithmetic progression was obtained in [7]. For example, the only instance in which a product of consecutive terms in the Fibonacci sequence is a perfect power is the trivial case $F_1F_2 = 1$.

There are also a few papers in the literature which address diophantine equations involving members of the Fibonacci sequence and factorials. For example, in [6] it is shown that the largest solution of the diophantine equation $F_n = m_1! \dots m_t!$ in positive integers n and $2 \le m_1 \le m_2 \le \dots \le m_t$ is $F_{12} = 2!^2\, 3!^2 = 3!\, 4!$, while in [4] it is shown that the largest solution of the diophantine equation $F_n = m_1! \pm m_2!$ is $F_{12} = 3! + 4!$.

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2 Main Result

In this note, we extend the main result from [6] and we prove the following result.

Theorem 1. *The largest solution of the diophantine equation*

$$F_{n_1}F_{n_2}\dots F_{n_k} = m_1!\dots m_t! \tag{1}$$

with positive integers $3 \leq n_1 < \cdots < n_k$ *and* $2 \leq m_1 \leq m_2 \leq \cdots \leq m_t$ *is*

$$F_3F_4F_5F_6F_8F_{10}F_{12} = 11! \tag{2}$$

In the above theorem, we did not allow the indices n_j to be 1 or 2 because $F_1 = F_2 = 1$, and we imposed the restriction $m_i \geq 2$ for the same reason because $0! = 1! = 1$. Note that the numbers m_i are not necessarily distinct for $i = 1, \dots, t$, while the numbers n_j are distinct for $j = 1, \dots, k$. We imposed the restriction that the indices n_j are distinct, for if not, then the above equation (1) will have infinitely many solutions (for example, raising the equality (2) to any power will produce another solution).

By the largest solution in the statement of the above theorem we mean that if $(n_1, \dots, n_k)$ are distinct positive integers ≥ 3 such that $F_{n_1}\dots F_{n_k}$ is a product of factorials, then $\{n_1, \dots, n_k\} \subset \{3, 4, 5, 6, 8, 10, 12\}$. That being said, a solution to (1) is of the form

$$F_3^{\epsilon_1}F_4^{\epsilon_2}F_5^{\epsilon_3}F_6^{\epsilon_4}F_8^{\epsilon_5}F_{10}^{\epsilon_6}F_{12}^{\epsilon_7} = \prod_i m_i!,$$

where $\epsilon_i \in \{0, 1\}$. For easy writing, we label $[\epsilon_1, \epsilon_2, \dots, \epsilon_7]$ the left hand side of the previous equation. With this notation, a computer program (assuming Theorem 1) revealed the following corollary (we do not write every possible product of factorials; for instance, $(2!)^4(3!)^3$ is written as $3!(4!)^2$, that is, we maximize the involved factorials).

Corollary 2. *The solutions to equation* (1) *are*

$$\begin{array}{lll}
[1,0,0,0,0,0,0] = 2!; & [1,1,0,0,0,0,0] = 3!; & [0,0,0,1,0,0,0] = (2!)^3; \\
[1,0,0,1,0,0,0] = (2!)^4; & [0,1,0,1,0,0,0] = 4!; & [1,1,0,1,0,0,0] = 2!4!; \\
[0,1,1,1,0,0,0] = 5!; & [0,0,0,0,0,0,1] = 3!4!; & [1,1,1,1,0,0,0] = 2!5!; \\
[1,0,0,0,0,0,1] = 2!3!4!; & [0,1,0,0,0,0,1] = 2!(3!)^3; & [0,0,1,0,0,0,1] = 3!5!; \\
[1,1,0,0,0,0,1] = (3!)^2 4!; & [0,0,0,1,0,0,1] = 2!(4!)^2; & [1,0,1,0,0,0,1] = 2!3!5!; \\
[1,0,0,1,0,0,1] = (2!)^2(4!)^2; & [0,1,0,1,0,0,1] = 3!(4!)^2; & [1,1,1,0,0,0,1] = (3!)^2 5!; \\
[1,1,1,1,1,0,0] = 7!; & [0,0,1,1,0,0,1] = 2!4!5!; & [1,1,0,1,0,0,1] = 2!3!(4!)^2; \\
[1,0,1,1,0,0,1] = (2!)^2 4!5!; & [0,1,1,1,0,0,1] = 3!4!5!; & [1,0,1,0,1,0,1] = 3!7!; \\
[1,1,1,1,0,0,1] = 2!3!4!5!; & [0,0,1,1,1,0,1] = 4!7!; & [1,0,1,1,1,0,1] = 2!4!7!; \\
[0,1,1,1,1,0,1] = 2!(3!)^2 7!; & [1,1,1,1,1,0,1] = 3!4!7!; & [1,1,1,1,1,1,1] = 11!.
\end{array}$$

Throughout this paper, we use p, q and r to denote prime numbers. For a positive real number x we use $\log x$ for its natural logarithm. By $p^e || n$ we mean that $p^e | n$, but $p^{e+1} \nmid n$.

3 The Proof

We assume that $3 \leq n_1 < n_2 < \cdots < n_k$ and $2 \leq m_1 \leq \cdots \leq m_t$ are integers satisfying equation (1). We write $N = n_k$ and $M = m_t$. We shall find upper bounds on N and M.

Recall that if m is any nonnegative integer then the identity

$$F_m = \frac{\alpha^m - \beta^m}{\alpha - \beta}$$

holds, where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. We start by recalling the classical argument which leads to a proof of the primitive divisor theorem (see, for example, [3, 9]). We have

$$F_m = \prod_{1\le k<m} (\alpha - e^{\frac{2\pi i k}{m}}\beta).$$

Write

$$\Phi_m = \prod_{\substack{1\le k<m\\ \gcd(k,m)=1}} (\alpha - e^{\frac{2\pi i k}{m}}\beta).$$

By the principle of inclusion and exclusion

$$\Phi_m = \frac{(\alpha^m - \beta^m)}{\prod_{p|m}(\alpha^{\frac{m}{p}} - \beta^{\frac{m}{p}})} \cdot \frac{\prod_{\substack{p<q\\ pq|m}} (\alpha^{\frac{m}{pq}} - \beta^{\frac{m}{pq}})}{\prod_{\substack{p<q<r\\ pqr|m}}(\alpha^{\frac{m}{pqr}} - \beta^{\frac{m}{pqr}})} \cdots. \tag{3}$$

Using now the trivial fact that the inequalities

$$\alpha^\ell - \beta^\ell \ge \alpha^\ell - |\beta|^\ell = (\alpha - |\beta|)(\alpha^{\ell-1} + \alpha^{\ell-2}|\beta| + \cdots + |\beta|^{\ell-1}) \ge \alpha^{\ell-1}$$

and

$$\alpha^\ell - \beta^\ell < 2\alpha^\ell < \alpha^{\ell+2}$$

hold for every positive integer ℓ, we then get, by (3), that the inequality

$$\Phi_m \ge \alpha^{(m - \sum_{p|m} \frac{m}{p} + \sum_{\substack{p<q\\ pq|m}} \frac{m}{pq} - \ldots) - 3\cdot 2^{\omega(m)-1}} = \alpha^{\phi(m) - 3\cdot 2^{\omega(m)-1}} \tag{4}$$

holds, where we use $\phi(m)$ and $\omega(m)$ to denote the Euler function of m and the number of distinct prime factors of m, respectively.

In order to get an upper bound on N, it suffices to assume that N is large. Thus, we assume that $N > 12$. By the *cyclotomic criterion* (see Theorem 2.4 in [1]), it follows that we have a representation

$$\Phi_m = A_m B_m$$

with positive integers A_m and B_m where $A_m \le m$ and every prime factor of B_m is congruent to $\pm 1 \pmod{m}$. Thus,

$$B_m \ge \frac{1}{m} \cdot \alpha^{\phi(m) - 3\cdot 2^{\omega(m)-1}}. \tag{5}$$

We now make the following claim.

Claim 1. *There exists N_0 such that if $N > N_0$ then one of the following holds:*

(i) *$M > N^{\frac{6}{5}}$;*

(ii) *If s is the smallest index in $\{1, \ldots, t\}$ such that $m_s \ge N - 1$, then $t - s + 1 > N^{\frac{1}{5}}$.*

We now prove the above claim and find a suitable value for N_0. Well, assume that M and N are such that the above claim does not hold. In this case, we let p be an arbitrary prime number $\equiv \pm 1 \pmod N$ dividing $\prod_{j=1}^t m_j!$. Clearly, $p \ge N-1$, therefore $p \mid \prod_{j=s}^t m_j! \mid (M!)^{t-s+1}$. We compute an upper bound for the exact order at which p divides $(M!)^{t-s+1}$. The order at which p divides $M!$ equals

$$\left\lfloor \frac{M}{p} \right\rfloor + \left\lfloor \frac{M}{p^2} \right\rfloor + \cdots < \frac{M}{p} + \frac{M}{p^2} + \cdots = \frac{M}{p-1} \le \frac{N^{\frac{6}{5}}}{N-2} < N^{\frac{1}{5}} + 2,$$

where in the above inequality we used the fact that $M \le N^{6/5}$ together with the fact that $N > 12$. This shows that if $p \ge N-1$ and $p^{\alpha_p} || (M!)^{t-s+1}$, then $\alpha_p < (N^{\frac{1}{5}}+2)(t-s+1) \le N^{\frac{1}{5}}(N^{\frac{1}{5}}+2)$.

Hence,

$$\prod_{\substack{p^{\alpha_p} || (M!)^{t-s+1} \\ p \equiv \pm 1 \pmod N}} p^{\alpha_p} \le M^{N^{\frac{1}{5}}(N^{\frac{1}{5}}+2)(\pi(N^{\frac{6}{5}},N,-1)+\pi(N^{\frac{6}{5}},N,1))},$$

where, as usual, we write $\pi(x,k,l)$ for the number of primes $p \le x$ which are congruent to $l \pmod k$. Since clearly $\pi(N^{\frac{6}{5}}, N, \pm 1) \le N^{\frac{1}{5}} + 1$, we get that

$$\prod_{\substack{p^{\alpha_p} || (M!)^{t-s+1} \\ p \equiv \pm 1 \pmod N}} p^{\alpha_p} \le \exp\left(2N^{\frac{1}{5}}(N^{\frac{1}{5}}+2)(N^{\frac{1}{5}}+1)\log\left(N^{\frac{6}{5}}\right)\right). \tag{6}$$

Since B_N obviously divides the number appearing in the left hand side of the above inequality, we get, from (5) and (6), that

$$\frac{1}{N} \cdot \alpha^{\phi(N)-3\cdot 2^{\omega(N)-1}} \le B_N \le \prod_{\substack{p^{\alpha_p} || (M!)^{t-s+1} \\ p \equiv \pm 1 \pmod N}} p^{\alpha_p} \le \exp\left(\frac{12}{5} \cdot N^{\frac{1}{5}}(N^{\frac{1}{5}}+2)(N^{\frac{1}{5}}+1)\log N\right).$$

By taking logarithms of both sides the above inequality becomes

$$\left(\phi(N) - 3 \cdot 2^{\omega(N)-1}\right)\log\alpha - \log N < \frac{12}{5} \cdot N^{\frac{1}{5}}(N^{\frac{1}{5}}+2)(N^{\frac{1}{5}}+1)\log N. \tag{7}$$

We now show that we can choose $N_0 = 5 \cdot 10^7$. Indeed, assume that $N > 5 \cdot 10^7$. In this case, we show that $3 \cdot 2^{\omega(N)-1} < \sqrt{N}$. This inequality holds if $\omega(N) \le 12$ because

$$3 \cdot 2^{\omega(N)-1} \le 6 \cdot 2^{10} < 7 \cdot 10^3 < \sqrt{N}.$$

Assume now that $\omega(N) \ge 13$ and let $p_1 < p_2 < \cdots < p_\ell$ be all the prime factors of N. Here, $\ell = \omega(N)$. Then, $\sqrt{N} \ge \prod_{i=1}^{\ell} \sqrt{p_\ell}$. Since $\sqrt{p_\ell} \ge \sqrt{41} > 6$, $\sqrt{p_{\ell-1}} \ge \sqrt{37} > 4$ and $\sqrt{p_i} > 2$ holds for $i = 3, \ldots, \ell-2$, we get that

$$\sqrt{N} > 2^{\ell-2-3+1} \cdot 4 \cdot 6 = 3 \cdot 2^{\ell-1} = 3 \cdot 2^{\omega(N)-1},$$

which is the desired inequality. Thus, if inequality (7) holds for some $N > 5 \cdot 10^7$, then the inequality

$$\left(\phi(N) - \sqrt{N}\right)\log\alpha - \log N < \frac{12}{5} \cdot N^{\frac{1}{5}}(N^{\frac{1}{5}}+2)(N^{\frac{1}{5}}+1)\log N \tag{8}$$

also holds. By Lemma 4.1 in [9], we know that $\phi(N) > N/\log N$ holds for all $N \ge 2 \cdot 10^9$. Thus, if $N \ge 2 \cdot 10^9$, then inequality (8) leads to

$$\left(\frac{N}{\log N} - \sqrt{N}\right)\log\alpha - \log N < \frac{12}{5} \cdot N^{\frac{1}{5}}(N^{\frac{1}{5}}+2)(N^{\frac{1}{5}}+1)\log N. \tag{9}$$

We used Mathematica and checked that the largest solution of this inequality is $< 1.6 \cdot 10^9$, which is impossible. Thus, $N < 2 \cdot 10^9$. By Lemma 4.2 in [9], we know that in this range $\phi(N) > N/6$. Thus, inequality (8) leads to the inequality

$$\left(\frac{N}{6} - \sqrt{N}\right) \log \alpha - \log N < \frac{12}{5} \cdot N^{\frac{1}{5}} (N^{\frac{1}{5}} + 2)(N^{\frac{1}{5}} + 1) \log N. \tag{10}$$

With Mathematica, we checked that the largest solution N of inequality (10) is $< 7 \cdot 10^6$. This indeed shows that the claim is true with $N_0 = 5 \cdot 10^7$.

We now show that, in fact, $N \le N_0$. Indeed, assume that $N > N_0$. By Claim 1, it follows that either (i) or (ii) holds. If (i) holds, then the exponent at which 2 appears in the right hand side of equation (1) is

$$\ge \left\lfloor \frac{M}{2} \right\rfloor + \left\lfloor \frac{M}{4} \right\rfloor + \cdots \ge M - \frac{\log(M+1)}{\log 2} > N^{\frac{6}{5}} - \frac{\log(N^{\frac{6}{5}} + 1)}{\log 2}, \tag{11}$$

while if (ii) holds, then the exponent at which 2 appears in the right hand side of equation (1) is

$$\ge (t - s + 1) \left(\left\lfloor \frac{N-1}{2} \right\rfloor + \left\lfloor \frac{N-1}{4} \right\rfloor + \ldots \right) > N^{\frac{1}{5}} \left(N - 1 - \frac{\log N}{\log 2} \right). \tag{12}$$

In is easy to check that in our range the right hand side of (12) is smaller than the right hand side of (11). Thus, in either case, the order at which 2 appears in the right hand side of equation (1) is

$$> N^{\frac{1}{5}} \left(N - 1 - \frac{\log N}{\log 2} \right). \tag{13}$$

It is known (see [5]) that if ℓ is a positive integer and $2^\ell || F_n$ then n is an odd multiple of 3 if $\ell = 1$, and $n = 2^{\ell-2} \cdot 3 \cdot m$, where m is coprime to 6 if $\ell \ge 3$ (the instance $\ell = 2$ can never occur). This shows that the exponent at which 2 appears in F_n is

$$\le 2 + \frac{\log(n/3)}{\log 2} = \frac{\log\left(\frac{4n}{3}\right)}{\log 2}.$$

Since

$$F_{n_1} F_{n_2} \ldots F_{n_k} \mid \prod_{n=1}^{N} F_n,$$

it follows that the order at which 2 appears in the left hand side of equation (1) does not exceed

$$\frac{1}{\log 2} \sum_{n=1}^{N} \log\left(\frac{4n}{3}\right) \le N \frac{\log\left(\frac{4N}{3}\right)}{\log 2}. \tag{14}$$

Comparing (13) with (14), we get the inequality

$$N^{\frac{1}{5}} \left(N - 1 - \frac{\log N}{\log 2} \right) < N \frac{\log\left(\frac{4N}{3}\right)}{\log 2}, \tag{15}$$

whose largest solution N is $< 7 \cdot 10^6$. This contradicts the fact that $N \ge N_0$.

In conclusion, any solution of equation (1) has $N \le N_0$. We now show that $M < 10^{14}$. We clearly have that

$$\prod_{n=1}^{N} F_n \ge F_{n_1} \ldots F_{n_k} \ge M! \ge \left(\frac{M}{e}\right)^M,$$

where the last inequality follows from Stirling's formula. Since the inequality $F_n < \alpha^n$ holds for all positive integers n, we get that

$$\left(\frac{M}{e}\right)^M < \alpha^{\sum_{n=1}^N n} = \alpha^{N(N+1)/2},$$

which, after taking logarithms and using the fact that $N \le N_0 = 5 \cdot 10^7$, leads to

$$M \log(M/e) < \frac{N_0(N_0+1)\log\alpha}{2}.$$

This inequality implies that $M < M_0 = 10^{14}$.

It now remains to cover the range $M \le M_0$. Assume first that $M \ge M_1 = 1069$. In this case, 1069 divides the right hand side of equation (1). The *entry point of* 1069 (i.e., the smallest positive integer k such that $1069|F_k$) is 89. However, F_{89} is also divisible with the 16 digit prime 1665088321800481 which exceeds M_0. Thus, $M < M_1$. Assume now that $M \ge M_2 = 73$. In this case, 73 divides the right hand side of equation (1). The entry point of 73 is 37. However, F_{37} is also divisible with the prime 2221 which exceeds M_1. Thus, $M < M_2$. Assume now that $M \ge M_3 = 37$. In this case, 37 divides the right hand side of equation (1). The entry point of 37 is 19. However, F_{19} is also divisible with the prime 113 which exceeds M_2. Thus, $M < M_3$, therefore the largest prime factor of the number appearing in either side of equation (1) is ≤ 31. By the Primitive Divisor Theorem (see [1, 3, 9]), it follows that F_N has a prime factor $\ge N-1$ if $N \ge 12$. Thus, $N \le 32$. A quick computation revealed that the only Fibonacci numbers F_n whose largest prime factor is ≤ 31 are the ones corresponding to $n \in \mathcal{A} = \{3, 4, \ldots, 10, 12, 14, 18, 24\}$. However, if $n_i \in \{9, 14, 18, 24\}$ for some $i = 1, \ldots, k$, then $M \ge 19$. In particular, 5^3 divides the right hand side of equation (1). On the other hand, if $5|F_n$ for some $n \in \mathcal{A}$, then $n \in \{5, 10\}$ and $5||F_n$ in both cases. This shows that $n_i \in \{3, 4, 5, 6, 8, 10, 12\}$ and the product of all the Fibonacci numbers whose indices are in this last set is 11!, which completes the proof of Theorem 1. $\square$

4 Comments

Recall that if r and s are coprime integers with $rs \ne 0$, $\Delta = r^2 + 4s \ne 0$ and such that the roots γ, δ of the quadratic equation

$$x^2 - rx - s = 0$$

have the property that γ/δ is not a root of 1, then the sequences $(u_n)_{n\ge 0}$ and $(v_n)_{n\ge 0}$ of general terms

$$u_n = \frac{\gamma^n - \delta^n}{\gamma - \delta} \qquad \text{and} \qquad v_n = \gamma^n + \delta^n$$

are called *Lucas sequences of the first and second kind*, respectively.

Arguments similar to the ones used in this paper combined with standard arguments from the theory of linear forms in logarithms of algebraic numbers (see [8]) lead to the following generalization of Theorem 1.

Theorem 3. *Let $(w_n)_{n\ge 0}$ be a Lucas sequence of the first or second kind. Then there exists an effectively computable constant c depending only on the sequence $(w_n)_{n\ge 0}$ such that all the solutions of the diophantine equation*

$$w_{n_1} \ldots w_{n_k} = m_1! \ldots m_t!,$$

in positive integer unknowns $1 < n_1 < \cdots < n_k$ and $2 \le m_1 \le \cdots \le m_t$ have $\max\{n_k, m_t\} < c$.

A similar result as the one above holds with the Lucas sequence $(w_n)_{n\geq 0}$ replaced by a classical Lehmer sequence, for the definition of which we refer the reader to the papers [1, 9].

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