# General Approach in Computing Sums of Products of Binary Sequences

E. Kiliç[1], P. Stănică[2]

[1]TOBB Economics and Technology University, Mathematics Department
06560 Sogutozu, Ankara, Turkey; *ekilic@etu.edu.tr*
[2]Naval Postgraduate School, Department of Applied Mathematics
Monterey, CA 93943, USA; *pstanica@nps.edu*

December 8, 2011

**Abstract**

In this paper we find a general approach to find closed forms of sums of products of arbitrary sequences satisfying the same recurrence with different initial conditions. We apply successfully our technique to sums of products of such sequences with indices in (arbitrary) arithmetic progressions. It generalizes many results from literature. We propose also an extension where the sequences satisfy different recurrences.

MSC: 11B37, 11C20
*Keywords:* Second order recurrences; Sums; Products

## 1 Introduction

We consider a generic (nondegenerate, that is, $\delta = \sqrt{p^2 - 4q} \neq 0$) binary recurrence satisfying

$$X_{n+1} = pX_n - qX_{n-1}, n \in \mathbb{Z} \tag{1}$$

with some initial conditions. Let $\alpha, \beta$ be the roots of the equation $x^2 - px + q = 0$, and so, $\alpha + \beta = p, \alpha\beta = q, \delta = \alpha - \beta$. We associate the companion *Lucas* sequence $L_n$ which also satisfies (1) together with $L_0 = 2$, $L_1 = p$, and so $L_n = \alpha^n + \beta^n$.

Let $\{U_n^{(j)}\}_{j=1}^p$ be a set of $p$ binary sequences, all of which will satisfy the recurrence (1) with some initial conditions, such that the Binet formula for these sequences is

$$U_n^{(j)} = A_j\alpha^n + B_j\beta^n, n \in \mathbb{Z},$$

where $A_j = \frac{U_1^{(j)}-U_0^{(j)}\beta}{\delta}$, $B_j = \frac{U_0^{(j)}\alpha-U_1^{(j)}}{\delta}$.

For easy notation, we will denote the recurrence $\{X_n\}$ given by (1) by $\{X_n(p,q,a,b)\}$ where $a = X_0$ and $b = X_1$ are initial conditions of it.

Several authors investigated products of two terms of a sequence or products of two sequences, and also, the sums of these products. As a first example, note that the sum of square terms of Fibonacci numbers [7, 8, 12] is

$$\sum_{i=1}^{n} F_i^2 = F_nF_{n+1}.$$

The sum of products of variable subscripted terms of certain second order recurrences have been considered by several authors. For example (see [11])

$$\begin{aligned}
\sum_{i=1}^{n} F_iF_{i+2} &= F_{2n+1}F_{2n+2} - 1, \\
\sum_{i=1}^{n} F_iF_{i+1} &= F_{2n+1}^2 - 1, \\
\sum_{i=1}^{n} F_{2i-1}F_{2i+3} &= \left(3F_{2n+2}^2 - 2F_{2n+1}^2 + 7n - 1\right)/5.
\end{aligned}$$

Certainly, the classical Fibonacci, $F_n$ and Pell numbers $P_n$ are $F_n = X_n(1,-1,0,1)$ and $P_n = X_n(2,-1,0,1)$. Generalizations of the above sums by taking different recurrences and their variable subscripted terms have also been studied. For example, in [9], the author found $\sum_{i=1}^{n} F_iP_i$. Melham [10] looked at the sum of the squares of the sequence $\{X_n(2,1,0,1)\}$. Recently, in [2, 3, 4, 5, 6], the authors gave several formulas for sums of squares of even and odd Fibonacci, Lucas and Pell-Lucas numbers, and their sums of products of even and odd subscripted terms. Also the authors of [1] established several formulas for sums and alternating sums of products of certain subscripted terms of recurrences $\{X_n(p,q,0,1)\}$ and $\{X_n(p,q,2,p)\}$.

It is our goal in this paper to propose a general approach for the theory of closed forms for sums of products of nondegenerate second-order recurrent sequences, thus generalizing many of these kind of results that the reader can find scattered throughout the literature.

## 2 Main Results

Let $\mathcal{P}(n)$ be the power set of $\{1,2,\ldots,n\}$, that is the set of all subsets of $\{1,2,\ldots,n\}$. Given a sequence of $p$ functions $f_j(i)$, $j=1,\ldots,p$, for all $M\in\mathcal{P}(p)$, we let $F_M(i)=\sum_{\ell\in M} f_\ell(i)$, $F_\emptyset(i)=0$, and for simplicity, $F(i)=F_{\{1,\ldots,p\}}(i)=\sum_{\ell=1}^p f_\ell(i)$. Let us define a set of *twisted product* sequences, indexed by the sets $M\in\mathcal{P}(p)$, in the following way: for a set $M\in\mathcal{P}(p)$, we let $W_n^{(M)}$ be a ($M$*-twisted product*) rational sequence satisfying (1) with the Binet formula

$$W_n^{(M)}=\left(\prod_{j\in M}A_j\prod_{k\notin M}B_k\right)\alpha^n+\left(\prod_{j\in M}B_j\prod_{k\notin M}A_k\right)\beta^n.$$

Further, we use $\bar{M}=\{1,2,\ldots,p\}\setminus M$, for the complement of the set $M$ in $\{1,2,\ldots,p\}$. We shall first show that $W_n^{(M)}$ is a rational sequence, even more precise that $W_n^{(M)}\in\frac{1}{\delta^{2^{p-1}}}\mathbb{Z}$.

**Lemma 1.** *For any integer $n$, the twisted product sequences satisfy*

$$W_n^{(\bar{M})}=q^nW_{-n}^{(M)}. \tag{2}$$

*Proof.* Straightforward using the Binet formula. ∎

**Theorem 1.** *For $p,n\in\mathbb{Z}$, we have*

$$W_n^{(M)}\in\frac{1}{\delta^{2^{p-1}}}\mathbb{Z}.$$

*Proof.* We will prove the claim by induction. First, we let $p=2$, and consider two sequences $U_n=A_1\alpha^n+B_1\beta^n$, $V_n=A_2\alpha^n+B_2\beta^n$ (for simplicity of notations). We write the superscript sets as $\{a,\ldots\}$ instead of $(\{a,\ldots\})$. The associated twisted product sequences are

$$\begin{aligned}
W_n^{\{1,2\}}&=A_1A_2\alpha^n+B_1B_2\beta^n,\\
W_n^{\{1\}}&=A_1B_2\alpha^n+B_1A_2\beta^n,\\
W_n^{\{2\}}&=A_2B_1\alpha^n+A_1B_2\beta^n,\\
W_n^{\emptyset}&=B_1B_2\alpha^n+A_1A_2\beta^n.
\end{aligned}$$

Since our index $n$ runs through the entire set of integers, by Lemma 1, it will be sufficient to consider only the case of $W_n^{\{1,2\}}$, and $W_n^{\{1\}}$.

First, using the expressions for $A_1, A_2, B_1, B_2$ in terms of initial conditions of $U_n, V_n$, and simplifying, we get

$$\begin{aligned}\delta^2(A_1A_2 + B_1B_2) &= 2U_1V_1 - p(U_0V_1 + U_1V_0) + (p^2 - 2q)U_0V_0 \in \mathbb{Z}\\ \delta^2(A_1A_2\alpha + B_1B_2\beta) &= pU_1V_1 - 2q(U_1V_0 + U_0V_1) + pqU_0V_0 \in \mathbb{Z}.\end{aligned}$$

Further,

$$\begin{aligned}\delta^2(A_1B_2 + B_1A_2) &= p(U_1V_0 + V_1U_0) - 2qU_0V_0 - 2U_1V_1 \in \mathbb{Z}\\ \delta^2(A_1B_2\alpha + B_1A_2\beta) &= (p^2 - 2q)U_1V_0 - p(U_0V_0q + U_1V_1) + 2U_0V_1q \in \mathbb{Z}.\end{aligned}$$

Now, let $U_n \in \mathbb{Z}$ and, from the induction step, assume that $V_n \in \frac{1}{\delta^{2^{p-1}}}\mathbb{Z}$. As before, writing $\delta^2 W_0^{\{1,2\}}, \delta^2 W_1^{\{1,2\}}$, $\delta^2 W_0^{\{1\}}, \delta^2 W_1^{\{1\}}$ in terms of $U_0, U_1, V_0, V_1$, we see that each term in these expressions contains only one factor based on either $V_0$, or $V_1 \in \frac{1}{\delta^{2^{p-1}}}\mathbb{Z}$, and therefore $W_i^{\{1,2\}}, W_i^{\{1\}} \in \frac{1}{\delta^{2^p}}\mathbb{Z}$, $i = 0, 1$. Certainly, since the initial terms of the twisted product sequences are in $\frac{1}{\delta^{2^p}}\mathbb{Z}$, so is $W_n^{(M)}$. ∎

We show now our general approach to finding sums of products of recurrences.

**Theorem 2.** *Given a set of $p$ functions $f_j(i)$, $j = 1, \ldots, p$, such that $f_j(i) - f_\ell(i)$ is a function of $j, \ell$ only and it does not depend on $i$, we have*

$$\sum_{i=0}^{n}\prod_{j=1}^{p} U_{f_j(i)}^{(j)} = \frac{1}{2}\sum_{M\in\mathcal{P}(p)}\sum_{i=0}^{n} q^{F(i)-F_M(i)} W_{2F_M(i)-F(i)}^{(M)}.$$

*Proof.* First, we associate to every set $M \in \mathcal{P}(p)$ a bit string $\epsilon$ of length $p$ in the usual manner (a 1 bit appears in the bit string if and only if its corresponding position appears in $M$, otherwise the bit is 0). For $\epsilon \in \mathbb{Z}_2^p$, we let $wt(\epsilon)$ to be the Hamming weight of the bit string $\epsilon$, that is, the number of 1's in its expression, and $supp(\epsilon) = \{i_1 < i_2 < \ldots < i_{wt(\epsilon)}\}$ to be the support of $\epsilon$ (the positions where 1's appear in $\epsilon$). Certainly, $supp(\epsilon) \subseteq \{1, 2, \ldots, p\}$.

Next, we compute the product

$$\begin{aligned}
\prod_{j=1}^{p} U_{f_j(i)}^{(j)} &= \prod_{j=1}^{p}\left(A_j\alpha^{f_j(i)} + B_j\beta^{f_j(i)}\right)\\
&= \sum_{\epsilon=(\epsilon_1,\dots,\epsilon_p)\in\mathbb{Z}_2^p}\prod_{j=1}^{p} A_j^{\epsilon_j}B_j^{1-\epsilon_j}\alpha^{\epsilon_j f_j(i)}\beta^{(1-\epsilon_j)f_j(i)}\\
&= \frac{1}{2}\sum_{\epsilon\in\mathbb{Z}_2^p}\left(\prod_{j\in supp(\epsilon)}\prod_{k\notin supp(\epsilon)} A_jB_k\alpha^{\sum_{j\in supp(\epsilon)} f_j(i)}\beta^{\sum_{k\notin supp(\epsilon)} f_k(i)}\right.\\
&\qquad\left. + \prod_{j\in supp(\epsilon)}\prod_{k\notin supp(\epsilon)} A_kB_j\beta^{\sum_{j\in supp(\epsilon)} f_j(i)}\alpha^{\sum_{k\notin supp(\epsilon)} f_k(i)}\right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}\sum_{M\in\mathcal{P}(p)}(\alpha\beta)^{\sum_{j\notin M} f_j(i)}\left(\prod_{j\in M}\prod_{k\notin M} A_jB_k\alpha^{\sum_{j\in M} f_j(i)-\sum_{j\notin M} f_j(i)}\right.\\
&\quad\left. + \prod_{j\in M}\prod_{k\notin M} A_kB_j\beta^{\sum_{j\in M} f_j(i)-\sum_{j\notin M} f_j(i)}\right)\\
&= \frac{1}{2}\sum_{M\in\mathcal{P}(p)} q^{F(i)-F_M(i)}W_{2F_M(i)-F(i)}^{(M)},
\end{aligned}$$

from which our theorem follows easily. ∎

Obviously, if the sum $\sum_{i=0}^{n} q^{F(i)-F_M(i)}W_{2F_M(i)-F(i)}^{(M)}$ can be simplified, then the previous theorem takes quite an attractive form. The rest of the paper is devoted in finding various functions $f_j$ for which such a sum can be computed. Many papers are investigating sums of products of very few recurrences (mostly two) where the indices are very specific linear functions. We will attack this case in its full generality here and solve it completely, by taking $f_j$ to be arbitrary linear functions.

Let $W_n$ be our generic sequence satisfying (1) such that $W_n = A\alpha^n + B\beta^n$, and recall that $L_n = \alpha^n + \beta^n$ is the companion Lucas sequence.

**Lemma 2.** *For $a, b, c, d \in \mathbb{Z}$, we have the generating function*

$$\sum_{i=0}^{n} x^{a+bi}W_{c+di} = x^a\frac{q^d x^{b(n+2)}W_{c+dn} - x^{b(n+1)}W_{c+d(n+1)} - x^b q^d W_{c-d} + W_c}{x^{2b}q^d - x^b L_d + 1}.$$

*Proof.* Using Binet formula for $W_n$, we obtain

$$\begin{aligned}
\sum_{i=0}^{n} x^{bi} W_{c+di} &= A\alpha^c \sum_{i=0}^{n} (x^b\alpha^d)^i + B\beta^c \sum_{i=0}^{n} (x^b\beta^d)^i \\
&= A\alpha^c \frac{(x^b\alpha^d)^{n+1}-1}{x^b\alpha^d-1} + B\beta^c \frac{(x^b\beta^d)^{n+1}-1}{x^b\beta^d-1} \\
&= \frac{\begin{array}{c} Ax^{b(n+2)}\beta^d\alpha^{c+d(n+1)} - Ax^{b(n+1)}\alpha^{c+d(n+1)} - Ax^b\alpha^c\beta^d + A\alpha^c \\ +Bx^{b(n+2)}\alpha^d\beta^{c+d(n+1)} - Bx^{b(n+1)}\beta^{c+d(n+1)} - Bx^b\beta^c\alpha^d + B\beta^c \end{array}}{x^{2b}(\alpha\beta)^d - x^b(\alpha^d+\beta^d)+1}
\end{aligned}$$

$$\begin{aligned}
&= \frac{\begin{array}{c} q^d x^{b(n+2)}(A\alpha^{c+dn} + B\beta^{c+dn}) - x^{b(n+1)}(A\alpha^{c+d(n+1)} + B\beta^{c+d(n+1)}) \\ -q^d x^b(A\alpha^{c-d} + B\beta^{c-d}) + (A\alpha^c + B\beta^c) \end{array}}{x^{2b}q^d - x^b L_d + 1} \\
&= \frac{q^d x^{b(n+2)} W_{c+dn} - x^{b(n+1)} W_{c+d(n+1)} - x^b q^d W_{c-d} + W_c}{x^{2b}q^d - x^b L_d + 1}.
\end{aligned}$$

∎

Taking $W_n = u_n = X_n\left(p,q,0,1\right)$, we reach at the following result:

$$\sum_{i=0}^{n} (-1)^i u_{r+4i} = (-1)^n \frac{v_{4n+r+2} + u_{r-2}}{v_2}$$

where $v_n = X_n\left(p,q,2,p\right)$. One can also find this result in [1, Lemma 5].

Let $f_j(i) = a_j + b_j i$ be linear functions. Under these conditions,

$$\begin{aligned}
F(i) - F_M(i) &= \sum_{j=1}^{p}(a_j + b_j i) - \sum_{j\in M}(a_j + b_j i) \\
&= \left(\sum_{j\notin M} a_j\right) + \left(\sum_{j\notin M} b_j\right) i = a^{(\bar{M})} + b^{(\bar{M})} i,
\end{aligned}$$

where we use the notations $a^{(\bar{M})} = \sum_{j\notin M} a_j$ and $b^{(\bar{M})} = \sum_{j\notin M} b_j$. We shall also use $a^{(M)} = \sum_{j\in M} a_j$, $b^{(M)} = \sum_{j\in M} b_j$. Further,

$$\begin{aligned}
2F_M(i) - F(i) &= \sum_{j\in M}(a_j + b_j i) - \sum_{j\notin M}(a_j + b_j i) \\
&= \left(a^{(M)} - a^{(\bar{M})}\right) + \left(b^{(M)} - b^{(\bar{M})}\right) i.
\end{aligned}$$

Applying Lemma 2 with $x := q$, $a := a^{(\bar{M})}, b := b^{(\bar{M})}$, $c := a^{(M)} - a^{(\bar{M})}, d := b^{(M)} - b^{(\bar{M})}$, and using Theorem 2 we obtain our next result.

**Theorem 3.** *Given a set of linear functions $f_j(i) = a_j + b_j i$, and binary sequences $U_k^{(j)}$ satisfying (1) with some initial conditions, we have*

$$\sum_{i=0}^{n} \prod_{j=1}^{p} U_{f_j(i)}^{(j)} = \frac{1}{2} \sum_{M \in \mathcal{P}(p)} q^{a^{(M)}} \frac{A(M)}{q^{b^{(M)}} - q^{b^{(\bar{M})}} L_{b^{(M)} - b^{(\bar{M})}} + 1},$$

*where* $A(M) = q^{b^{(M)} + b^{(\bar{M})}(n+1)} W^{(M)}_{a^{(M)} - a^{(\bar{M})} + n(b^{(M)} - b^{(\bar{M})})} - q^{b^{(\bar{M})}(n+1)} W^{(M)}_{a^{(M)} - a^{(\bar{M})} + (n+1)(b^{(M)} - b^{(\bar{M})})} - q^{b^{(M)}} W^{(M)}_{a^{(M)} + a^{(\bar{M})} - b^{(M)} - b^{(\bar{M})}} + W^{(M)}_{a^{(M)} - a^{(\bar{M})}}$.

## 3 A Particular Case

To understand our general result better, we shall consider now a particular case of two binary recurrences, which is the case most often encountered in literature. Let $U_n, V_n$ be two binary recurrent sequences satisfying (1) with some initial conditions. The Binet formula indicates that

$$\begin{aligned} U_n &= A_1 \alpha^n + B_1 \beta^n, \\ V_n &= A_2 \alpha^n + B_2 \beta^n, \end{aligned}$$

where $A_1 = \frac{U_0\beta - U_1}{\beta - \alpha}$, $B_1 = \frac{U_1 - U_0\alpha}{\beta - \alpha}$, $A_2 = \frac{V_0\beta - V_1}{\beta - \alpha}$, $B_2 = \frac{V_1 - V_0\alpha}{\beta - \alpha}$.

As before, we take the twisted products $W_n^{\{1,2\}}, W_n^{\{1\}}$, satisfying (1), with initial conditions $W_0^{\{1,2\}} = A_1A_2 + B_1B_2, W_1^{\{1,2\}} = A_1A_2\alpha + B_1B_2\beta$, $W_0^{\{1\}} = A_1B_2 + B_1A_2, W_1^{\{1\}} = A_1B_2\alpha + B_1A_2\beta$, so that $W_n^{\{1,2\}} = A_1A_2\alpha^n + B_1B_2\beta^n$, and $W_n^{\{1\}} = A_1B_2\alpha^n + B_1A_2\beta^n$. From Theorem 1 we know that $W_n^{\{1\}}, W_n^{\{1,2\}} \in \frac{1}{\delta^2}\mathbb{Z}$. We next consider the example $f_1(i) = r + ki, f_2(i) = s + ki$.

**Theorem 4.** *Let $k, r, s$ be fixed integers. We have*

$$\begin{aligned} \sum_{i=0}^{n} U_{r+ki} V_{s+ki} &= q^s W_{r-s}^{\{1\}} \frac{q^{k(n+1)} - 1}{q^k - 1} \\ &\quad + \frac{q^{2k} W^{\{1,2\}}_{r+s+2kn} - W^{\{1,2\}}_{r+s+2k(n+1)} - q^{2k} W^{\{1,2\}}_{r+s-2k} + W^{\{1,2\}}_{r+s}}{q^{2k} - L_{2k} + 1}. \end{aligned}$$

*Proof.* First,

$$\begin{aligned}
U_{r+ki}V_{s+ki} &= (A_1\alpha^{r+ki}+B_1\beta^{r+ki})(A_2\alpha^{s+ki}+B_2\beta^{s+ki}) \\
&= (A_1A_2\alpha^{r+s+2ki}+B_1B_2\beta^{r+s+2ki}) \\
&\quad +(A_1B_2\alpha^{r+ki}\beta^{s+ki}+A_2B_1\alpha^{s+ki}\beta^{r+ki}) \\
&= W^{\{1,2\}}_{r+s+2ki}+q^{s+ki}(A_1B_2\alpha^{r-s}+A_2B_1\beta^{r-s}) \\
&= W^{\{1,2\}}_{r+s+2ki}+q^{s+ki}W^{\{1\}}_{r-s}. \qquad (3)
\end{aligned}$$

In the notations of Theorem 3, the previous product will be

$$\begin{aligned}
&\frac{1}{2}\Big(q^{f_1(i)+f_2(i)}W^{\emptyset}_{-F(i)}+q^{f_2(i)}W^{\{1\}}_{f_1(i)-f_2(i)}+q^{f_1(i)}W^{\{2\}}_{f_2(i)-f_1(i)} \\
&+W^{\{1,2\}}_{f_1(i)+f_2(i)}\Big)=W^{\{1,2\}}_{f_1(i)}+q^{f_2(i)}W^{\{1\}}_{f_1(i)-f_2(i)}.
\end{aligned}$$

Using (3), we separate the sum $\sum_{i=0}^{n}U_{r+ki}V_{s+ki}$ into two sums. First,

$$\sum_{i=0}^{n}q^{s+ki}W^{\{1\}}_{r-s}=q^sW^{\{1\}}_{r-s}\sum_{i=0}^{n}(q^k)^i=q^sW^{\{1\}}_{r-s}\frac{q^{k(n+1)}-1}{q^k-1}, \qquad (4)$$

(we could have also used Lemma 2 with $x:=q$, $a=s, b=k$ and $c=r-s, d=0$). Next, using Lemma 2 with $x:=q$, $a=b=0$ and $c=r+s, d=2k$, we get

$$\sum_{i=0}^{n}W_{t+\ell i}=\frac{q^{\ell}W_{t+\ell n}-W_{t+\ell(n+1)}-q^{\ell}W_{t-\ell}+W_t}{q^{\ell}-L_{\ell}+1}.$$

and the second sum becomes

$$\sum_{i=0}^{n}W^{\{1,2\}}_{r+s+2ki}=\frac{q^{2k}W^{\{1,2\}}_{r+s+2kn}-W^{\{1,2\}}_{r+s+2k(n+1)}-q^{2k}W^{\{1,2\}}_{r+s-2k}+W^{\{1,2\}}_{r+s}}{q^{2k}-L_{2k}+1},$$

which finishes the proof of our theorem. ∎

If we take $u_n=X_n\left(p,q,0,1\right)$ and $v_n=X_n\left(p,q,2,p\right)$ ($p\neq 0$, $\sqrt{p^2-4q}\neq 0$), then by required arrangements, we obtain for $k=2$

$$\sum_{i=0}^{n}u_{r+2i}v_{s+2i}=\frac{v_{4n+r+s+2}-v_{r+s-2}-p\left(n+1\right)q^ru_{s-r}}{p\left(p^2-4q\right)}$$

which is the main result of [1, Theorem 1].

## 4 Further Extension

Presumably, one can take the same approach as we have done it previously, for sequences determined by different recurrence relations, but the computation becomes more difficult. However, there are instances when one can get similar results. Let $X_n^{(j)}$ ($j = 1, 2, \ldots, p$) be arbitrary sequences satisfying some second-order recurrences (with integer coefficients) with Binet formulas $X_n^{(j)} = A_j\alpha_j^n + B_j\beta_j^n$, where $X_0^{(j)}, X_1^{(j)} \in \mathbb{Z}$ and $\alpha_j + \beta_j, \alpha_j\beta_j \in \mathbb{N}$. As before, we define the *p-twisted product* sequences

$$W_{n_1,n_2,\ldots,n_p}^{(M)} = \left(\prod_{j\in M} A_j\alpha_j^{n_j}\right)\left(\prod_{k\notin M} B_k\beta_k^{n_k}\right) + \left(\prod_{j\in M} B_j\alpha^{n_j}\right)\left(\prod_{k\notin M} A_k\beta^{n_k}\right).$$

Let the associated Lucas sequences be defined by $L_{n_1,n_2,\ldots,n_p} = \alpha_1^{n_1}\alpha_2^{n_2}\cdots\alpha_p^{n_p} + \beta_1^{n_1}\beta_2^{n_2}\cdots\beta_p^{n_p}$.

Certainly, the next theorem can be extended to any sum of products of $p$ sequences, indexed by linear functions, using the above twisted products, but it is obviously more difficult to formalize the result. We shall give an instance of such a result, however we do not include the proof, since it goes along the lines of the proof of Theorem 4. Let $\alpha_1 + \beta_1 = p, \alpha_1\beta_1 = q$, and $\alpha_2 + \beta_2 = r, \alpha_2\beta_2 = s$, where $p, q, r, s \in \mathbb{Z}$, and $p^2 - 4q \neq 0, r^2 - 4s \neq 0$.

**Theorem 5.** *If $f, g : \mathbb{N} \to \mathbb{N}$, then*

$$\sum_{j=1}^{n} X_{f(j)}^{(1)} X_{g(j)}^{(2)} = \sum_{j=1}^{n} \left(W_{f(j),g(j)}^{\{1,2\}} + W_{f(j),g(j)}^{\{1\}}\right).$$

*Further, if $a, b, c, d \in \mathbb{N}$, then*

$$\sum_{j=1}^{n} X_{aj+b}^{(1)} X_{cj+d}^{(2)} = \frac{q^a s^c W_{an+b,cn+d}^{\{1,2\}} - q^a s^c W_{b-a,d-c}^{\{1,2\}} - W_{a(n+1)+b,c(n+1)+d}^{\{1,2\}} + W_{b,d}^{\{1,2\}}}{q^a s^c - L_{a,c} + 1}.$$

Besides the usefulness of the defined twisted products (based on the same recurrence) in the computation of sums of products, one could also ask, independently, about the arithmetic, or primitive primes in the factorization of these products (multiplied by an appropriate power of the discriminant $\delta$, cf. Theorem 1), but we shall investigate that elsewhere.

## References


[1] H. Belbachir and F. Bencherif, *Sums of products of generalized Fibonacci and Lucas numbers*, Arxiv:0708.2347v1, 2009.

[2] Z. Cerin, Sums of products of generalized Fibonacci and Lucas numbers, Demonstratio Mathematica, 42 (2) (2009), 247–258.

[3] Z. Cerin, On Sums of Products of Horadam Numbers, Kyungpook Mathematical Journal, 49 (2009), 483–492.

[4] Z. Cerin and G.M. Gianella, *On sums of squares of Pell-Lucas numbers*, Integers 6 (2006), A15, 16 pp.

[5] Z. Cerin, *Alternating sums of Fibonacci products*, Atti Semin. Mat. Fis. Univ. Modena Reggio Emilia 53 (2) (2005), 331–344.

[6] Z. Cerin, *Some alternating sums of Lucas numbers*, Cent. Eur. J. Math. 3 (1) (2005), 1–13.

[7] E. Kılıç, Sums of the squares of terms of sequence $\{u_n\}$, Proc. Indian Acad. Sci. 118 (1) (2008), 27–41.

[8] T. Koshy, Fibonacci and Lucas numbers with applications. Pure and Applied Mathematics, Wiley-Interscience, New York, 2001.

[9] D.G. Mead, *Problem B*-67, Fibonacci Quart. 3 (4) (1965), 326–327.

[10] R.S. Melham, *On sums of powers of terms in a linear recurrence*, Portugal. Math. 56 (4) (1999), 501–508.

[11] K.S. Rao, *Some properties of Fibonacci numbers*, The Amer. Math. Monthly, 60 (1953), 680–684.

[12] S. Vajda, Fibonacci & Lucas numbers, and the golden section. John Wiley & Sons, Inc., New York, 1989.