

# The Spectrum of Generalized Petersen Graphs

RALUCCA GERA and PANTELIMON STĂNICĂ\*

Department of Applied Mathematics

Naval Postgraduate School

Monterey, CA 93943, USA

{rgera,pstanica}@nps.edu

September 5, 2010

---

\*Also associated to the Institute of Mathematics of the Romanian Academy, Bucharest, Romania.

Corresponding author:

Pantelimon Stănică

<sup>1</sup>

Department of Applied Mathematics

Naval Postgraduate School

Monterey, CA 93943, USA

pstanica@nps.edu

---

<sup>1</sup>Also associated to the Institute of Mathematics of the Romanian Academy, Bucharest, Romania.

### Abstract

In this paper, we completely describe the spectrum of the generalized Petersen graph  $P(n, k)$ , thus adding to the classes of graphs whose spectrum is completely known.

## 1 Introduction and motivation

Let  $G = (V(G), E(G))$  be a simple graph. The spectrum of a graph  $G$  is the multiset of eigenvalues of the adjacency matrix. The graph spectrum is an important tool one can use to find information about the physical properties of a network, such as robustness, diameter, connectivity [3]. In this research we completely describe the spectrum for the class of graphs, defined below.

The *generalized Petersen graph* (GPG)  $P(n, k)$  has vertices, respectively, edges given by

$$\begin{aligned} V(P(n, k)) &= \{a_i, b_i, 0 \leq i \leq n-1\}, \\ E(P(n, k)) &= \{a_i a_{i+1}, a_i b_i, b_i b_{i+k} \mid 0 \leq i \leq n-1\}, \end{aligned}$$

where the subscripts are expressed as integers modulo  $n$  ( $n \geq 5$ ), and  $k$  is the “skip”. Note that  $k \leq \lfloor \frac{n-1}{2} \rfloor$ , because of the obvious isomorphism  $P(n, k) \cong P(n, n-k)$ . Let  $A(n, k)$  (respectively,  $B(n, k)$ ) be the subgraph of  $P(n, k)$  consisting of the vertices  $\{a_i \mid 0 \leq i \leq n-1\}$  (respectively,  $\{b_i \mid 0 \leq i \leq n-1\}$ ) and edges  $\{a_i a_{i+1} \mid 0 \leq i \leq n-1\}$  (respectively,  $\{b_i b_{i+k} \mid 0 \leq i \leq n-1\}$ ). We will call  $A(n, k)$  (respectively,  $B(n, k)$ ) the *outer* (respectively, *inner*) subgraph of  $P(n, k)$ . We display in Figure 1 the graph  $P(12, 3)$ .

For other graph theoretical terminology the reader could refer to [7].

## 2 Eigenvalues of $P(n, k)$

In this section we find our description for the spectrum of generalized Petersen graphs  $P(n, k)$ . We denote the adjacency matrix of the GPG  $P(n, k)$

---

**AMS Mathematics Subject Classification:** 05C70, 05C12, 05C50, 15A42.

**Keywords:** Generalized Petersen Graphs; Eigenvalues; Spectrum.

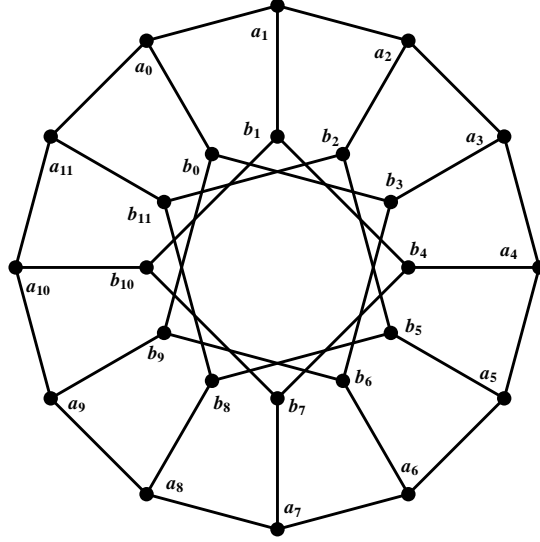


Figure 1: The Generalized Petersen Graph  $P(12, 3)$

by  $A(P(n, k))$ . Let  $\lambda_0 = 3 > \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{2n-1}$  be the sequence of eigenvalues of  $P(n, k)$ .

We call an  $n \times n$  matrix *circulant*, and denote it by  $\text{circ}(a_1, a_2, \dots, a_n)$  if it is of the form

$$\text{circ}(a_1, a_2, \dots, a_n) = \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ a_n & a_1 & a_2 & \dots & a_{n-1} \\ \vdots & & & & \vdots \\ a_2 & a_3 & a_4 & \dots & a_1 \end{pmatrix}.$$

**Lemma 2.1** *The  $(2n) \times (2n)$  adjacency matrix of the GPG  $P(n, k)$  has the block form*

$$A(P(n, k)) = \begin{pmatrix} C_k^n & I_n \\ I_n & C^n \end{pmatrix},$$

where  $I_n$  is the  $n \times n$  identity matrix,  $C^n, C_k^n$  are circulant matrices, with  $C^n = \text{circ}(0, 1, 0, 0, \dots, 0, 1)$  and  $C_k^n = \text{circ}(\underbrace{0, \dots, 0}_{k \text{ times}}, 1, 0, \dots, 0, 1, \underbrace{0, \dots, 0}_{k-1 \text{ times}})$  being the adjacency matrix for  $A(n, k)$  and  $B(n, k)$ , respectively. Thus,  $C^n$

is the adjacency matrix of a cycle graph on  $n$  vertices  $\mathcal{C}_n$ , respectively,  $C_k^n$  is the union of  $d$  cycle graphs  $\mathcal{C}_{n/d}$  on  $n/d$  vertices, where  $d = \gcd(n, k)$ .

**Proof.** The outer subgraph (whose adjacency matrix is  $C^n$ ) of  $P(n, k)$  is the cycle graph  $\mathcal{C}_n$  and the inner subgraph (whose adjacency matrix is  $C_k^n$ ) has  $d$  connected components each isomorphic to  $\mathcal{C}_{n/d}$ . Also, the adjacency matrix (which depends on the labeling) has the claimed form where the labels used on the outer subgraph are consecutively  $1, 2, \dots, n$ , and on the inner subgraph the adjacent labels are  $i, i + k, i + 2k, \dots$  (where  $i + sk$  is understood as  $1 + (i - 1 + sk) \pmod{n}$ ). Note that  $b_0$  is adjacent to vertex  $b_k$  in the subgraph and to vertex labeled  $b_{n-k}$  in  $B(n, k)$ , and so  $C_k^n = \text{circ}(\overbrace{0, \dots, 0}^{k \text{ times}}, 1, 0, \dots, 0, 1, \overbrace{0, \dots, 0}^{k-1 \text{ times}})$   $\square$

We recall the Chebyshev's polynomial of the first kind [5], defined by the identity  $T_n(\cos \theta) = \cos(n\theta)$ , with the generating function  $\sum_{n=0}^{\infty} T_n(x)t^n = \frac{1-xt}{1-2xt+t^2}$ . We now present the eigenvectors and eigenvalues for  $\mathcal{C}_n$  (see [2, p. 53 and pp. 72–73]). Let  $\mathbf{v}^t$  denote the transpose of  $\mathbf{v}$ .

**Lemma 2.2** *The eigenvalues of the cycle graph  $\mathcal{C}_n$  on  $n$  vertices are*

$$\alpha_j = 2 \cos \left( \frac{2\pi j}{n} \right)$$

with a corresponding eigenvector

$$\mathbf{v}_j = (1, \zeta^j, \zeta^{2j}, \dots, \zeta^{(n-1)j})^t,$$

$0 \leq j \leq n - 1$ . The characteristic polynomial of the cycle  $\mathcal{C}_n$  is  $2T_n(x/2) - 2$  where  $T_n$  is the Chebyshev's polynomial of the first kind.

**Corollary 2.3** *The eigenvalues corresponding to the circulant  $C$  in the adjacency matrix  $A(P(n, k))$  are  $\alpha_j = 2 \cos \left( \frac{2\pi j}{n} \right)$  ( $0 \leq j \leq n - 1$ ), and the eigenvalues corresponding to  $C_k$  are  $\beta_j = 2 \cos \left( \frac{2\pi jk}{n} \right)$  ( $0 \leq j \leq n - 1$ ).*

We now state our main theorem which adds to the class of graphs whose spectrum is now known.

**Theorem 2.4** *The eigenvalues of  $P(n, k)$ , say  $\delta_{2j}, \delta_{2j+1}$ , are all roots of the quadratic equation*

$$\delta^2 - (\alpha_j + \beta_j)\delta + \alpha_j\beta_j - 1 = 0, \tag{1}$$

where  $\alpha_j = 2 \cos \left( \frac{2\pi j}{n} \right)$ ,  $\beta_j = 2 \cos \left( \frac{2\pi jk}{n} \right) = 2T_k(\alpha_j/2)$  ( $0 \leq j \leq n - 1$ ) are the eigenvalues of  $C$ , respectively  $C_k$ .

**Proof.** We first consider the case of  $d = \gcd(n, k) = 1$ . Since  $d = 1$ , then  $C_k$  is the adjacency matrix of a cycle graph isomorphic to  $C_n$ , and so it is similar to  $C$ , that is, there exists a permutation matrix  $P$ , such that  $P^{-1}C_kP = C$ . This implies that the two matrices will have the same eigenvalues and eigenvectors. Then  $\alpha_j, \beta_j$  are eigenvalues corresponding to the same eigenvector, say  $\mathbf{v}_j = (1, \zeta_n^j, \dots, \zeta_n^{(n-1)j})^t$ . We are looking for an eigenvector for  $A(P(n, k))$  of the form  $\mathbf{w}_j = (a_j\mathbf{v}_j, \mathbf{v}_j)^t$ , where  $a_j$  will be determined later. If two distinct values for  $a_j$  are to be found, for any  $0 \leq j \leq n-1$ , then we are done with our search for the eigenvectors/eigenvalues.

With this value for  $\mathbf{w}_j$ , we need  $\delta$  (dependent on  $j$ ) such that

$$\begin{pmatrix} C_k & I_n \\ I_n & C \end{pmatrix} \begin{pmatrix} a_j\mathbf{v}_j \\ \mathbf{v}_j \end{pmatrix} = \delta \begin{pmatrix} a_j\mathbf{v}_j \\ \mathbf{v}_j \end{pmatrix}$$

and so, we get the system

$$\begin{cases} a_j C_k \mathbf{v}_j + \mathbf{v}_j = \delta a_j \mathbf{v}_j \\ a_j \mathbf{v}_j + C \mathbf{v}_j = \delta \mathbf{v}_j \end{cases} \iff \begin{cases} a_j \beta_j \mathbf{v}_j + \mathbf{v}_j = \delta a_j \mathbf{v}_j \\ a_j \mathbf{v}_j + \alpha_j \mathbf{v}_j = \delta \mathbf{v}_j \end{cases}$$

which implies

$$\begin{cases} a_j(\delta - \beta_j)\mathbf{v}_j = \mathbf{v}_j \\ (\delta - \alpha_j)\mathbf{v}_j = a_j\mathbf{v}_j \end{cases}$$

and so,  $(\delta - \beta_j)(\delta - \alpha_j) = 1$ , which renders the claim for this case, that is,  $\delta$  must satisfy the equation  $\delta^2 - (\alpha_j + \beta_j)\delta + \alpha_j\beta_j - 1 = 0$ .

The case of  $d > 1$  is treated similarly. The eigenvectors  $\mathbf{w}_j$  must have the form  $\mathbf{w}_j = (a_1\mathbf{v}'_j, a_2\mathbf{v}'_j, \dots, a_d\mathbf{v}'_j, \mathbf{v}_j)$ , with  $\mathbf{v}_j$  as before and  $\mathbf{v}'_j = (1, \zeta_n^j, \dots, \zeta_n^{(n'-1)j})^t$ ,  $n' = n/d$ , for some appropriate multipliers  $a_i$ . A similar system to the one for  $d = 1$  case will be obtained and, interestingly enough, the same polynomial whose roots are the eigenvalues  $\lambda_i$  will be found. The theorem is proved.  $\square$

Using the quadratic formula in (1) and simplifying we get the following corollary.

**Corollary 2.5** *The eigenvalues of  $P(n, k)$  are given by*

$$\cos\left(\frac{2\pi j}{n}\right) + \cos\left(\frac{2\pi jk}{n}\right) \pm \sqrt{\left(\cos\left(\frac{2\pi j}{n}\right) - \cos\left(\frac{2\pi jk}{n}\right)\right)^2 + 1}, \quad 0 \leq j \leq n-1.$$

The largest eigenvalue of  $P(n, k)$ ,  $\lambda_0 = 3$ , is one of the two values obtained for  $j = 0$  in the previous corollary. It is known (see [2, Thm. 3.11]) that if a graph is bipartite, then its spectrum is symmetric with respect to 0. In our case, we have the following result.

**Corollary 2.6** *If  $n$  is even and  $k$  is odd, then the eigenvalues of the bipartite graph  $P(n, k)$  are given by  $\pm 3$  and*

$$\begin{aligned} & \cos(2j\pi/n) + \cos(2jk\pi/n) \pm \sqrt{(\cos(2j\pi/n) - \cos(2jk\pi/n))^2 + 1} \\ & - \cos(2j\pi/n) + (-1)^k \cos(2jk\pi/n) \mp \sqrt{(\cos(2j\pi/n) + (-1)^k \cos(2jk\pi/n))^2 + 1}, \end{aligned}$$

for  $0 \leq j < n/2$ .

### 3 Bounds on the eigenvalues of $P(n, 2)$

In the previous section we found the complete set of eigenvalues of  $P(n, k)$  under no restrictions on  $n$  and  $k$ . Here, we would like to find some bounds on some eigenvalues. Eigenvalue interlacing techniques (see the great survey by Haemers [4] on the topic) will not work easily since there is no visible connection between the various  $P(n, k)$ , and moreover, the technique is not sensitive enough for our purpose. We shall use a different method.

Here, we will take  $k = 2$  and consider  $P(n, 2)$  (this includes the case of the classical Petersen graph  $P(5, 2)$ ). Since the second Chebyshev polynomial of the first kind is  $T_2(x) = 2x^2 - 1$ , we immediately obtain the following:

**Theorem 3.1** *The eigenvalues of  $P(n, 2)$  are (for  $0 \leq j \leq n - 1$ )*

$$2 \cos^2(2j\pi/n) + \cos(2j\pi/n) - 1 \pm \sqrt{(2 \cos^2(2j\pi/n) - \cos(2j\pi/n) - 1)^2 + 1}.$$

To find good bounds on the eigenvalues in this case, we look for the extreme points of the two functions

$$f_{\pm}(x) = 2x^2 + x - 1 \pm \sqrt{(2x^2 - x - 1)^2 + 1}, \quad (2)$$

in the interval  $-1 \leq x \leq 1$ . Certainly, we cannot expect exact or even tight results, in general, since the sequence  $\frac{2j\pi}{n}$ ,  $0 \leq j < n - 1$ , is finite and therefore,  $\cos\left(\frac{2j\pi}{n}\right)$  is not dense in this interval. However, we will have lower

and upper bounds, which is what we are interested in. Since any differentiable function in a compact domain attains its extreme points at either the critical points or on the boundary, we proceed by studying first the functions' critical points:

$$f'_{\pm}(x) = 4x + 1 \pm \frac{(2x^2 - x - 1)(4x - 1)}{\sqrt{(2x^2 - x - 1)^2 + 1}} = 0,$$

has solutions (computed by Mathematica) at  $x_1 \sim -0.41100$  (for  $f_+$ ) and  $x_2 \sim -0.65041, x_3 \sim -0.04610$  (for  $f_-$ ). The values of the corresponding  $f_{\pm}$  at these critical points are

$$\begin{aligned} f_+(-0.41100) &= -0.04210\dots \\ f_-(-0.65041) &= -1.92081\dots \\ f_-(-0.04610) &= -2.42092\dots \end{aligned}$$

Further, we look at the values of  $f_{\pm}$  at  $|x| = 1$ . Thus,  $f_+(1) = 3, f_+(-1) = \sqrt{5}$ , and  $f_-(1) = 1, f_-(-1) = -\sqrt{5}$ . Certainly, the maximum value is 3, and the minimum value is approximately -2.42092. We sketch in Figure 2 the two functions  $f_{\pm}$ , to visualize our analysis from above:

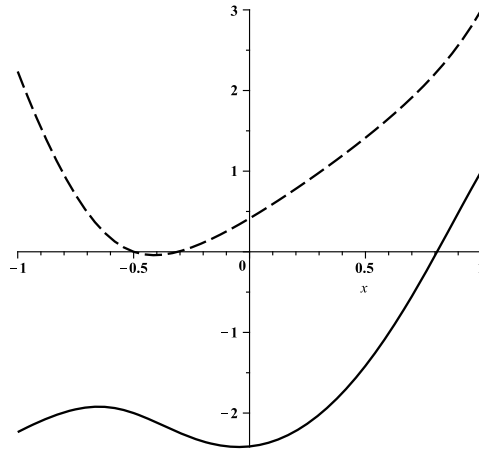


Figure 2: The top function is  $f_+$  and the bottom function is  $f_-$

Every value of  $f_+$  is above every value of  $f_-$ , and so the minimum is attained by  $f_-$  and the maximum is attained by  $f_+$ . Furthermore, we see

that the second largest eigenvalue of  $P(n, 2)$  is

$$\begin{aligned} \lambda_1 &= f_+ \left( \cos \left( \frac{2\pi}{n} \right) \right) \\ &= \cos \left( \frac{2\pi}{n} \right) + \cos \left( \frac{4\pi}{n} \right) + \sqrt{4 \left( 2 \cos \left( \frac{2\pi}{n} \right) + 1 \right)^2 \sin^4 \left( \frac{\pi}{n} \right) + 1}, \end{aligned} \quad (3)$$

which increases as  $n$  increases (shown simply by using Calculus techniques). For instance, for  $3 \leq n \leq 20$  the sequence  $\lambda_1 = \lambda_1(n)$  is

$$0, 0.41421, 1., 1.41421, 1.71083, 1.93185, 2.10199, 2.23607, 2.34356, 2.43091, \\ 2.50268, 2.56224, 2.61211, 2.65421, 2.69002, 2.7207, 2.74716, 2.77011.$$

Since  $\lim_{n \rightarrow \infty} \cos \left( \frac{2\pi}{n} \right) = 1$ , we obtain the next result.

**Theorem 3.2** *The eigenvalues of  $P(n, 2)$  are*

$$\lambda_0 = 3 > \lambda_1 \geq \dots \geq \lambda_{2n-1} \geq -2.42092.$$

Moreover, the second largest eigenvalue satisfies  $\lim_{n \rightarrow \infty} \lambda_1(n) = 3$ .

**Acknowledgement.** The authors are grateful to the referee whose suggestions have led to an improvement in the presentation of this paper. During the preparation of this paper, the authors were partially supported by a RIP grant from NPS.

## 4 Further comments

All of our results for  $P(n, 2)$  can be certainly extended to  $P(n, 3)$ ,  $P(n, 4)$ , etc., but to find sensitive bounds on eigenvalues for arbitrary GPG  $P(n, k)$  does not seem to be easy, since the sequence of the involved Chebyshev's polynomials of the first kind does not have a "controllable" behavior in  $|x| \leq 1$ .

Also, it would be interesting to investigate the number of and distinct values among the eigenvalues of  $P(n, k)$ , and that is presumably doable. We suspect that the methods of this paper can be also applied to the  $I$ -graphs of [1] or the *supergeneralized* Petersen graphs of [6].

## References

- [1] M. Boben, T. Pisanski, and A. Žitnik,  $I$ -graphs and the corresponding configurations, *J. Combinatorial Designs* **13:6** (2005), 406–424.
- [2] D.M. Cvetkovic, M. Doob, and H. Sachs, *Spectra of Graphs*, Academic Press, 1980.
- [3] E.R. van Dam and R.E. Kooij, The minimal spectral radius of graphs with a given diameter, *Linear Algebra Appl.* **423** (2007), 408–419.
- [4] W.H. Haemers, Interlacing eigenvalues and graphs, *Linear Algebra Appl.* **226–228** (1995), 593–616.
- [5] T.J. Rivlin, *Chebyshev Polynomials*, New York: Wiley, 1990.
- [6] M.L. Saražin, W. Paccob, and A. Previtali, Generalizing the generalized Petersen graphs, *Discrete Math.* **307** (2007), 534–543.
- [7] D. West, *Introduction to graph theory*, 2nd Edition, Prentice Hall, Upper Saddle River, NJ, 2001.