Correction to: “On the spacings between C-nomial coefficients” *J. Number Theory* 130 (2010), 82–100.

1 Introduction

There is an oversight in the published version of the proof of Lemma 1 from the paper mentioned in the title. Here, we provide the correct proof. We keep the notations from that paper. In particular,

$$p := \prod_{i\geq 1}\left(1-\left(\frac{\beta}{\alpha}\right)^i\right) = \prod_{i\geq 1}\left(1-(-\alpha^2)^{-i}\right) \sim 1.226742\cdots . \tag{1}$$

Lemma 1 there asserts the following.

Lemma 1 *Assume that $m \geq 2k \geq 2$. We then have*

$$\begin{bmatrix} m \\ k \end{bmatrix}_F = \frac{\alpha^{mk-k^2}}{p}(1+\zeta_{m,k}),$$

where $\zeta_{m,k}$ is a real number satisfying

$$|\zeta_{m,k}| < \frac{2}{\alpha^{2k+1}}. \tag{2}$$

2 The proof of Lemma 1

We have

$$\begin{aligned}
\begin{bmatrix} m \\ k \end{bmatrix}_F &= \frac{F_m F_{m-1}\cdots F_{m-k+1}}{F_1 F_2 \cdots F_k} \\
&= \alpha^{m+(m-1)+\cdots+(m-k+1)-1-2-\cdots-k} \prod_{1\leq i\leq k}\left(1-\left(\frac{\beta}{\alpha}\right)^i\right)^{-1} \prod_{m-k+1\leq i\leq m}\left(1-\left(\frac{\beta}{\alpha}\right)^i\right) \\
&= \alpha^{mk-k^2} \prod_{1\leq i\leq k}\left(1-\left(\frac{\beta}{\alpha}\right)^i\right)^{-1} \prod_{m-k+1\leq i\leq m}\left(1-\left(\frac{\beta}{\alpha}\right)^i\right).
\end{aligned}$$

Now observe that

$$
\begin{aligned}
p^{-1}(1+\zeta_{m,k}) &= \prod_{1\le i\le k}\left(1-\left(\frac{\beta}{\alpha}\right)^i\right)^{-1}\prod_{m-k+1\le i\le m}\left(1-\left(\frac{\beta}{\alpha}\right)^i\right)\\
&= p^{-1}\prod_{i\ge k+1}\left(1-\left(\frac{\beta}{\alpha}\right)^i\right)\prod_{m-k+1\le i\le m}\left(1-\left(\frac{\beta}{\alpha}\right)^i\right).
\end{aligned}
$$

It remains to estimate $\zeta_{m,k}$. We use the inequality

$$
e^z > 1+z > \begin{cases} e^{z/1.8}, & \text{if } \; z\in(0,1/2),\\ e^{1.35z}, & \text{if } z\in(-0.15,0).\end{cases} \tag{3}
$$

Of course, the inequality on the left-hand side holds for all z, only the inequality on the right-hand side holds in the shown ranges. Observe that $m-k+1\ge k+1$. Observe further that

$$
\begin{aligned}
\prod_{i\ge k+1}\left(1-\left(\frac{\beta}{\alpha}\right)^i\right) &< \exp\left(-\sum_{i\ge k+1}\left(\frac{\beta}{\alpha}\right)^i\right) = \exp\left(\frac{(-1)^{k+2}}{\alpha^{2k+2}(1-\beta/\alpha)}\right)\\
&= \exp\left(\frac{(-1)^{k+2}}{\sqrt{5}\alpha^{2k+1}}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
\prod_{m-k+1\le i\le m}\left(1-\left(\frac{\beta}{\alpha}\right)^i\right) &< \exp\left(-\sum_{m-k+1\le i\le m}\left(\frac{\beta}{\alpha}\right)^i\right)\\
&= \exp\left(-\left(\frac{\beta}{\alpha}\right)^{m-k+1}\left(\frac{1-(\beta/\alpha)^k}{1-\beta/\alpha}\right)\right)\\
&= \exp\left(\frac{(-1)^{m-k}}{\sqrt{5}\alpha^{2(m-k)+1}}\left(1-\left(\frac{\beta}{\alpha}\right)^k\right)\right).
\end{aligned}
$$

Note that $1-\left(\frac{\beta}{\alpha}\right)^k \le 1-\left(\frac{\beta}{\alpha}\right) = 1+\frac{1}{\alpha^2}$. Thus, since $m-k\ge k$, so $1/\alpha^{2(m-k)+1}\le 1/\alpha^{2k+1}$, we have

$$
\begin{aligned}
1+\zeta_{m,k} &< \exp\left(\frac{1}{\sqrt{5}\alpha^{2k+1}}+\frac{1}{\sqrt{5}\alpha^{2(m-k)+1}}\left(1+\frac{1}{\alpha^2}\right)\right)\\
&\le \exp\left(\frac{1}{\sqrt{5}\alpha^{2k+1}}+\frac{1}{\sqrt{5}\alpha^{2k+1}}\left(1+\frac{1}{\alpha^2}\right)\right) = \exp\left(\frac{c}{\alpha^{2k+1}}\right),
\end{aligned}
$$

where $c = \frac{2\alpha^2+1}{\sqrt{5}\alpha^2} < 1.1$ (since $m-k \geq k$, we used $1/\alpha^{2(m-k)+1} < 1/\alpha^{2k+1}$ in the last inequality above). Take

$$z := \frac{1.1 \cdot 1.8}{\alpha^{2k+1}} < 0.5 \qquad \text{for all} \quad k \geq 1.$$

Thus, using (3) and since $1.1 \cdot 1.8 < 2$, we have

$$1 + \zeta_{m,k} < \exp\left(\frac{z}{1.8}\right) < 1 + z < 1 + \frac{2}{\alpha^{2k+1}},$$

rendering

$$\zeta_{m,k} < \frac{2}{\alpha^{2k+1}}. \tag{4}$$

We now deal with the lower bound. Observe that if j is odd, then

$$\begin{aligned}\left(1 - \left(\frac{\beta}{\alpha}\right)^j\right)\left(1 - \left(\frac{\beta}{\alpha}\right)^{j+1}\right) &= 1 + \left|\frac{\beta}{\alpha}\right|^j - \left|\frac{\beta}{\alpha}\right|^{j+1} - \left|\frac{\beta}{\alpha}\right|^{2j+1} \\ &= 1 + \frac{1}{\alpha^{2j}} - \frac{1}{\alpha^{2j+2}} - \frac{1}{\alpha^{4j+2}} > 1,\end{aligned}$$

where the last inequality is equivalent to $\alpha^{2j+2} > \alpha^{2j} + 1$, that is, $\alpha^{2j}(\alpha^2 - 1) = \alpha^{2j+1} > 1$, which is true for all $j \geq 1$. In particular, for all odd j and all $s \geq j$ we have that

$$\prod_{j \leq i \leq s}\left(1 - \left(\frac{\beta}{\alpha}\right)^i\right) > 1.$$

Hence, if either $m-k+1$ or $k+1$ is odd, then the argument from Lemma 1 in the paper together with the above remarks shows that

$$\zeta_{m,k} > -\frac{1.35}{\alpha^{2k+1}}.$$

Indeed, let us repeat that argument. Say $m-k+1$ is odd. Then

$$\begin{aligned}1 + \zeta_{m,k} &\geq \prod_{k+1 \leq i}\left(1 - \left|\frac{\beta}{\alpha}\right|^i\right) > \exp\left(-1.35 \sum_{k+1 \leq i}\left|\frac{\beta}{\alpha}\right|^i\right) \\ &= \exp\left(\frac{-1.35}{\alpha^{2k+2}(1 - 1/\alpha^2)}\right) = \exp\left(\frac{-1.35}{\alpha^{2k+1}}\right) > 1 - \frac{1.35}{\alpha^{2k+1}}.\end{aligned}$$

In the above chain of inequalities, we used estimate (3) for

$$z := -\left|\frac{\beta}{\alpha}\right|^i \geq -\frac{1}{\alpha^{2k+2}} \geq -\frac{1}{\alpha^4} > -0.15 \qquad \text{for all} \quad i \geq k+1.$$

A similar argument works when $k+1$ is odd, since then, by the previous argument with $k+1$ replaced by $m-k+1$, we get

$$1+\zeta_{m,k} \geq \prod_{m-k+1\leq i} \left(1-\left|\frac{\beta}{\alpha}\right|^i\right) \geq 1-\frac{1.35}{\alpha^{m-k+1}} \geq 1-\frac{1.35}{\alpha^{k+1}}.$$

So, if either $k+1$ or $m-k+1$ is odd, we then showed that

$$\zeta_{m,k} > -\frac{1.35}{\alpha^{2k+1}}. \tag{5}$$

Finally, suppose that both $k+1$ and $m-k+1$ are even. Observe that if j is even, then

$$\min\left\{1-\left(\frac{\beta}{\alpha}\right)^j, \left(1-\left(\frac{\beta}{\alpha}\right)^j\right)\left(1-\left(\frac{\beta}{\alpha}\right)^{j+1}\right)\right\} \geq 1-\left|\frac{\beta}{\alpha}\right|^j.$$

Indeed, the inequality

$$1-\left(\frac{\beta}{\alpha}\right)^j \geq 1-\left|\frac{\beta}{\alpha}\right|^j$$

is obvious (in fact, it is an equality), while the inequality

$$\left(1-\left(\frac{\beta}{\alpha}\right)^j\right)\left(1-\left(\frac{\beta}{\alpha}\right)^{j+1}\right) > 1-\left|\frac{\beta}{\alpha}\right|^j$$

follows from the previous one and the fact that the second factor on the left is >1. Thus, since both $k+1$ and $m-k+1$ are even, then, using again inequality (3) with $z=-|\beta/\alpha|^i$ for $i \geq k+1 \geq 2$, we get

$$\begin{aligned} 1+\zeta_{m,k} &\geq \prod_{\substack{k+1\leq i \\ i\equiv 0 \pmod 2}} \left(1-\left|\frac{\beta}{\alpha}\right|^i\right)^2 \geq \exp\left(-2\cdot 1.35 \sum_{\substack{k+1\leq i \\ i\equiv 0 \pmod 2}} \left|\frac{\beta}{\alpha}\right|^i\right) \\ &= \exp\left(\frac{-2.7}{\alpha^{2k+2}(1-1/\alpha^4)}\right) > 1-\left(\frac{2.7\alpha^3}{\alpha^4-1}\right)\frac{1}{\alpha^{2k+1}} > 1-\frac{2}{\alpha^{2k+1}}, \end{aligned}$$

where for the final inequality we used the fact that

$$\frac{2.7\alpha^3}{\alpha^4-1} < 2.$$

Thus, we have shown that

$$\zeta_{m,k} > -\frac{2}{\alpha^{2k+1}}. \tag{6}$$

The desired inequality now follows from estimates (4), (5) and (6).

In addition, all sums on Page 90 whose ranges are $n \geq 1$ should in fact have an additional term equal to 1. This does not affect the proofs.

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