Generating matrices of C–nomial coefficients and their spectra

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April 20, 2010

Abstract

In this paper, we consider a generalization of binomial coefficients, called C–nomial coefficients, dependent upon a sequence $\{u_n\}_n$, with indices in arithmetic progressions. We obtain a general recurrence relation and a generating matrix, and point out some new relationships between these coefficients and the generalized Pascal matrices. Further, we obtain generating functions, combinatorial representations, and many new interesting identities and properties of these coefficients.

1 Introduction

Given that binomial coefficients are very important tools in combinatorics, much effort has been devoted to generalize these objects. We mention here the work of Bachmann [1], Carmichael [3], Fontené [6], Ward [25], as well as the more recent [7, 9, 13, 19, 20, 22, 23].

We define the generalized *C–nomial coefficients* formed with the terms of a sequence $C = \{C_n\}_n$: for $n \geq m \geq 1$

$$\begin{bmatrix} n \\ m \end{bmatrix}_C = \frac{C_1 C_2 \ldots C_n}{(C_1 C_2 \ldots C_{n-m})\,(C_1 C_2 \ldots C_m)}$$

with $\begin{bmatrix} n \\ 0 \end{bmatrix}_C = \begin{bmatrix} n \\ n \end{bmatrix}_C = 1$. One can define the nth C-factorial by $C_n! = C_1 C_2 \ldots C_n$. When $C_n = F_n$ is the Fibonacci sequence, the numbers $\begin{bmatrix} m \\ k \end{bmatrix}_F = \frac{F_n!}{F_m!\,F_{n-m}!}$ are called *Fibonomials*, and if $C_n = (q^n - 1)/(q-1)$, the numbers $\begin{bmatrix} m \\ k \end{bmatrix}_q = \frac{(q^m-1)\cdots(q^{m-k+1}-1)}{(q-1)\cdots(q^k-1)}$ are called *q-binomial*, or *Gaussian coefficients*.

Bachmann [1, p. 81], Carmichael [3, p. 40], and Jarden and Motzkin [13], all showed that if $u = \{u_n\}_n$ is a Lucas sequence; that is, if $u_0 = 0,\ u_1 = 1$ and satisfies the recurrence $u_{n+2} = Au_{n+1} + Bu_n$ for all nonnegative integers n with some nonzero integers A and B such that the quadratic equation $x^2 - Ax - B = 0$ has two distinct roots α and β whose ratio is not a root of unity, then all the C–nomial coefficients are integers (by abuse of notation, we will call all such coefficients C–nomials, if the sequence is not special, but understood from context). The Fibonomial coefficients are particular cases of this instance with $A = B = 1$. When $A = q + 1$ and $B = -q$, where $q > 1$ is some fixed integer, the C–nomial coefficients become the q-binomial coefficients.

The Fibonomial coefficients satisfy the relation

$$\begin{bmatrix} n \\ m \end{bmatrix}_F = F_{m+1} \begin{bmatrix} n-1 \\ m \end{bmatrix}_F + F_{n-m-1} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_F.$$

We now take the case of a Lucas sequences with $A \in \mathbb{Z}, B = 1$, say u_n and its companion v_n

$$\begin{aligned} u_n &= Au_{n-1} + u_{n-2} \\ v_n &= Av_{n-1} + v_{n-2} \end{aligned} \tag{1}$$

where $u_0 = 0,\ u_1 = 1$ and $v_0 = 2,\ v_1 = A$, respectively, for all $n \geq 2$.

There are some relationships between the Fibonomial coefficients (for more details see [2, 4, 10, 21, 17]), and the generalized Pascal matrix P_n, which is the $n \times n$ right-justified matrix whose (i, j) entry is given by

$$(P_n)_{ij} = \binom{j-1}{j+i-n-1} A^{i+j-n-1}.$$

In [9], Hoggatt considers the C–nomial coefficients with indices in an arithmetic progression. For A fixed, he defined the numbers (for $m \geq n$)

$$\begin{Bmatrix} n \\ m \end{Bmatrix}_{u,k} = \frac{u_k u_{2k} \dots u_{kn}}{\left(u_k u_{2k} \dots u_{k(n-m)}\right)\left(u_k u_{2k} \dots u_{km}\right)},$$

which we call $k : C$*–nomial coefficients* (or generalized C–nomial coefficients, if k is not specified). It is straightforward to show that they satisfy the following recurrences:

$$\begin{Bmatrix} n \\ m \end{Bmatrix}_{u,k} = u_{km+1} \begin{Bmatrix} n-1 \\ m \end{Bmatrix}_{u,k} + u_{k(n-m)-1} \begin{Bmatrix} n-1 \\ m-1 \end{Bmatrix}_{u,k}$$

and

$$\begin{Bmatrix} n \\ m \end{Bmatrix}_{u,k} = u_{km-1} \begin{Bmatrix} n-1 \\ m \end{Bmatrix}_{u,k} + u_{k(n-m)+1} \begin{Bmatrix} n-1 \\ m-1 \end{Bmatrix}_{u,k}.$$

When the sequence $\{u_n\}_n$ is understood from the context, we will write $\left\{ {n \atop m} \right\}_k$ instead of $\left\{ {n \atop m} \right\}_{u,k}$. When $k = 1$, the coefficient $\left\{ {n \atop m} \right\}_{u,1}$ is reduced to $\left[{n \atop m} \right]_u$.

Let α, β be the roots of the associated equation $x^2 - Ax - 1 = 0$. As a special case of [17], we have that for $r \geq 1$ and $n > 1$,

$$\begin{aligned} u_{rn} &= v_r u_{r(n-1)} + (-1)^{r+1} u_{r(n-2)} \\ v_{rn} &= v_r v_{r(n-1)} + (-1)^{r+1} v_{r(n-2)}. \end{aligned} \tag{2}$$

Using the sequence $\{v_r\}$, the authors of [17] define the $n \times n$ generalized Pascal matrix $P_n(v_r)$ as follows:

$$P_n(v_r) = \left(v_r^{i+j-n-1} \, (-1)^{(r+1)(n-j)} \, \binom{i-1}{n-j} \right)_{1 \le i,j \le n}$$

and then show that for all $m > 0$, the trace

$$tr\left(P_n^m(v_r)\right) = \frac{u_{rnm}}{u_r}.$$

Observe that $P_n(v_1) = P_n$. As an example,

$$P_6(v_r) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & (-1)^{r+1} & v_r \\ 0 & 0 & 0 & 1 & 2(-1)^{r+1} v_r & v_r^2 \\ 0 & 0 & (-1)^{r+1} & 3v_r & 3(-1)^{r+1} v_r^2 & v_r^3 \\ 0 & 1 & 4(-1)^{r+1} v_r & 6v_r^2 & 4(-1)^{r+1} v_r^3 & v_r^4 \\ (-1)^{r+1} & 5v_r & 10(-1)^{r+1} v_r^2 & 10v_r^3 & 5(-1)^{r+1} v_r^4 & v_r^5 \end{pmatrix}.$$

Moreover, they derived that the eigenvalues and the characteristic polynomial of $P_{n+1}(v_r)$ are

$$\alpha^{rn}, \alpha^{r(n-1)}\beta^r, \ldots, \alpha^r \beta^{r(n-1)}, \beta^{rn}. \tag{3}$$

and

$$P_{r,n}(x) = \prod_{j=0}^{n-1} \left(x - \alpha^{jr} \beta^{(n-j-1)r} \right) = \sum_{i=0}^{n} (-1)^i \, (-1)^{ri(i-1)/2} \left\{ {n \atop i} \right\}_r x^{n-i} \tag{4}$$

where the coefficient $\left\{ {n \atop i} \right\}_r$ is our generalized $r : C$–nomial coefficients.

In [18], we defined the recursive analogue of the entries of the Lehmer matrix and then formulated its determinant via the Fibonomial factorials (or F–factorials).

In this paper, our purpose is to find generating matrices for generalized $r : C$–onomial coefficients $\left\{ {n \atop m} \right\}_r$ and derive a linear recurrence relation for the generalized $r : C$–nomial coefficients. We obtain new general identities, generating functions, combinatorial representations for them and their sums by matrix methods.

2 Generalized C–nomial coefficients

In this section, we fix a Lucas sequence u_n as in (1) and derive a recurrence relation and generating matrix for the generalized C–nomial coefficients. The case $k = 2$ and $r = 1$, was investigated in [14, 15].

First, we define the sign function

$$s(i,r) = \begin{cases} (-1)^{(i-1)(i-2)/2} & \text{if } r \text{ is odd,} \\ (-1)^{i+1} & \text{if } r \text{ is even,} \end{cases} \tag{5}$$

and extend the generalized C–nomial coefficients by

$$\left\{ \begin{matrix} n \\ m \end{matrix} \right\}_r = \begin{cases} 0 & \text{if } m > n \text{ and } n \geq 0, \\ s(m,r) & \text{if } m > n \text{ and } n < 0, \\ \frac{u_r u_{2r} \dots u_{nr}}{(u_r u_{2r} \dots u_{mr})(u_r u_{2r} \dots u_{(n-m)r})} & m \leq n. \end{cases}$$

For the sake of compactness, we shall use the following notations, for fixed k with $1 \leq i \leq k+1$:

$$a_{n,i} = s(i,r) \left\{ \begin{matrix} n+k \\ k-i+1 \end{matrix} \right\}_r \left\{ \begin{matrix} n+i-2 \\ i-1 \end{matrix} \right\}_r . \tag{6}$$

As we will be using it later, recall the following identity from [24, p. 176]

$$F_{n+m} = F_{m-1} F_n + F_m F_{n+1},$$

which generalizes (for the sequence $\{u_n\}$), for any integers k, m, n and positive integer $r > 0$

$$u_{rk} u_{r(n+m)} = u_{rm} u_{r(n+k)} + (-1)^{rm} u_{rn} u_{r(k-m)}. \tag{7}$$

Now we give the following result.

Lemma 1. *For $n > 0$, $1 \leq i \leq k$ and for odd r,*

$$a_{1,i} a_{n,1} + (-1)^{i-1} a_{n,i+1} = a_{n+1,i},$$

and for even r,

$$a_{1,i} a_{n,1} - a_{n,i+1} = a_{n+1,i},$$

where $a_{n,i}$ be as before.

Proof. The proof for r odd can be found in [14]. Assume r is even. We simplify the equality $a_{1,i} a_{n,1} - a_{n,i+1} = a_{n+1,i}$, reducing it to

$$u_{r(k+1)} u_{r(n+i)} + u_{rn} u_{r(k-i+1)} = u_{ri} u_{r(n+k+1)}.$$

By taking $k \to k+1$ in (7), we obtain the second claim of our lemma. □

For $k, r \geq 1$, we define the $(k+1) \times (k+1)$ companion matrix $G_{r,k}$ and the matrix $H_{n,r,k}$ as follows:

$$G_{r,k} = \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,k+1} \\ 1 & & & \\ & \ddots & & \vdots \\ 0 & & 1 & 0 \end{bmatrix} \text{ and}$$
$$H_{n,r,k} = \begin{bmatrix} a_{n,1} & a_{n,2} & \dots & a_{n,k+1} \\ a_{n-1,1} & a_{n-1,2} & \dots & a_{n-1,k+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-k,1} & a_{n-k,2} & \dots & a_{n-k,k+1} \end{bmatrix}. \tag{8}$$

We say that the matrix $G_{r,k}$ is the *generalized C–nomial* matrix (with indices in the arithmetic progression $\equiv 0 \pmod{k}$).

Now we give our main result of this section.

Theorem 2. *For all $n, r > 0$,*

$$G_{r,k}^{n} = H_{n,r,k}.$$

Proof. We use induction on n. The case of $n = 1$ follows from the definitions of the matrix $H_{n,r,k}$ and $k : C$–nomial coefficients. Suppose that the equation holds for $n \geq 1$. Now we show that the equation holds for $n + 1$. First, we write

$$G_{r,k}^{n+1} = G_{r,k} G_{r,k}^{n} = G_{r,k} H_{n,r,k}.$$

From Lemma 1 and matrix multiplication, we get

$$G_{r,k}^{n+1} = G_{r,k} H_{n,r,k} = H_{n+1,r,k},$$

and the theorem is shown. □

It is valuable to note that when $k = 1$, we obtain the following fact (see [16]):

$$G_{1,1} = \begin{bmatrix} v_r & (-1)^{r+1} \\ 1 & 0 \end{bmatrix} \text{ and } H_{n,1,1} = \begin{bmatrix} u_{r(n+1)} & u_{rn} \\ u_{rn} & u_{r(n-1)} \end{bmatrix}.$$

When $A = r = 1$, we get the following well known fact:

$$G_{1,1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } H_{n,1,1} = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}.$$

When $A = 1$, $r = 3, k = 2$, we get the matrix $G_{3,2}$ and its nth power as

$$G_{3,2} = \begin{bmatrix} \frac{F_9}{F_3} & \frac{F_9}{F_3} & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } H_{n,3,2} = \begin{bmatrix} \frac{F_{3(n+1)}F_{3(n+2)}}{F_3F_6} & \frac{F_{3n}F_{3(n+2)}}{F_3F_3} & -\frac{F_{3n}F_{3(n+1)}}{F_3F_6} \\ \frac{F_{3n}F_{3(n+1)}}{F_3F_6} & \frac{F_{3(n-1)}F_{3(n+1)}}{F_3F_3} & -\frac{F_{3(n-1)}F_{3n}}{F_3F_6} \\ \frac{F_{3(n-1)}F_{3n}}{F_3F_6} & \frac{F_{3(n-2)}F_{3n}}{F_3F_3} & -\frac{F_{3(n-2)}F_{3(n-1)}}{F_3F_6} \end{bmatrix}$$

The case $A = 1, r = 1$ and $k = 2$ can be found in [15].

From the companion matrix $G_{r,k}$, we give a linear recurrence relation for the C–nomial coefficients.

Corollary 3. *For $n, k > 0$, the C–nomial coefficients satisfy the following order-$(k+1)$ linear recursion*

$$a_{n+1,1} = \sum_{i=1}^{k+1} a_{1,i} a_{n-i+1,1}$$

or, equivalently,

$$\begin{aligned} \left\{ {n+k+1 \atop k} \right\}_r &= s(1,r) \left\{ {k+1 \atop k} \right\}_r \left\{ {n+k \atop k} \right\}_r + s(2,r) \left\{ {k+1 \atop k-1} \right\}_r \left\{ {n+k-1 \atop k} \right\}_r \\ &\quad + \cdots + s(k,r) \left\{ {k+1 \atop 1} \right\}_r \left\{ {n+1 \atop k} \right\}_r + s(k+1,r) \left\{ {n \atop k} \right\}_r. \end{aligned}$$

Proof. Since $a_{n,1} = \left\{ {n+k \atop k} \right\}_r$ and using matrix multiplication, by equating the $(1,1)$ entries in the equation $H_{1,r,k} H_{n,r,k} = H_{n+1,r,k}$, the claim follows. □

All identities in the next corollary can be obtained from a property of the matrix multiplication in $H_{n+1,r,k} = H_{n,r,k}H_{1,r,k}$, $H_{n+m,r,k} = H_{n,r,k}H_{m,r,k}$ and $H_{n+t,r,k} = H_{n+m,r,k}H_{t-m,r,k}$ for $n, m > 0$ and $t > q$.

Corollary 4. *For $n, r > 0$, the following identities hold:*

$$
\begin{aligned}
&a_{n-1,1} = a_{n,k+1},\\
&a_{m+n+1-i,j} = \textstyle\sum_{t=1}^{k+1} a_{n+1-i,t}a_{m+1-t,j} \text{ for all } m > 0,\\
&a_{n+t+1-i,j} = \textstyle\sum_{m=1}^{k+1} a_{n+q+1-i,m}a_{t-q+1-i,j} \text{ for } t > 0 \text{ and } t > q,\\
&a_{n+1,1} = a_{1,1}a_{n,1} + a_{n,2},\\
&a_{n+1,k+1} = a_{n,1}a_{1,k+1},\\
&a_{n+1,i} = a_{1,i}a_{n,1} + a_{n,i+1} \text{ for } 2 \le i \le k.
\end{aligned}
$$

3 The eigenvalues of the matrix $G_{r,k}$

In this section we determine the eigenvalues of the matrix $G_{r,k}$. Since $G_{r,k}$ is a companion matrix, its characteristic polynomial, $f_{r,k}(x)$, can be easily derived.

Thus we have the following result, whose proof is immediate.

Proposition 5. *For $n, k, r > 0$,*

$$f_{r,k}(x) = \sum_{t=0}^{k+1} (-s(t,r)) \left\{ {k+1 \atop t} \right\}_r x^{k+1-t}$$

where the sign function $s(t,r)$ is as in (5).

Here clearly

$$-s(t,r) = \begin{cases} (-1)^{i(i+1)/2} & \text{if } r \text{ is odd},\\ (-1)^i & \text{if } r \text{ is even}, \end{cases}$$

and for odd $r > 0$ and $n, k > 0$,

$$f_{r,k}(x) = \sum_{t=0}^{k+1} (-1)^{i(i+1)/2} \left\{ {k+1 \atop t} \right\}_r x^{k+1-t}$$

and even $r > 0$,

$$f_{r,k}(x) = \sum_{t=0}^{k+1} (-1)^{i} \left\{ {k+1 \atop t} \right\}_r x^{k+1-t}.$$

According to [5, 8, 9, 12], the nth powers of Fibonacci numbers satisfy the following auxiliary polynomial

$$C_n(x) = \sum_{i=0}^{n} (-1)^{i(i+1)/2} \left\{ {n \atop i} \right\}_{F,1} x^{n-i}.$$

In [2, 5], the authors show that the characteristic polynomial of the Pascal matrix $H_n(p)$ is also equal to the polynomial $C_n(x) = f_{1,k}(x)$ and so $C_n(x) = h_{1,n}(x) = f_{1,k}(x)$.

Consequently, we have the following result.

Proposition 6. *Let* $\alpha, \beta = \left(p \pm \sqrt{p^2+4}\right)/2$. *The characteristic roots of* $C_{m+1}(x) = f_{1,m}(x)$ *are:*

$$\begin{cases} \left\{(-1)^j \alpha^{m-2j},\ (-1)^j \beta^{m-2j}\right\}_{j=0,1,\ldots,k-1} & \text{if } m = 2k-1, \\ \left\{(-1)^k,\ (-1)^j \alpha^{m-2j},\ (-1)^j \beta^{m-2j}\right\}_{j=0,1,\ldots,k-1} & \text{if } m = 2k. \end{cases}$$

As a general case of Proposition 6, in [17] we considered the generalized Pascal matrix $P_n(V_r)$ and then gave its characteristic polynomial $h_{r,n}(x)$ as in (3). It is clear that the characteristic polynomials of the generalized $k : C$–nomial matrix $G_{r,k}$ and the generalized Pascal matrix $P_{k+1}(v_r)$ are the same. Also we know that the all roots of the polynomial from [17], that is, the eigenvalues (clearly, distinct) of the matrix $G_{r,k}$ are

$$\alpha^{rk}, \alpha^{r(k-1)}\beta^r, \ldots, \alpha^r \beta^{r(k-1)}, \beta^{rk}.$$

Alternatively, we may give the following proposition.

Proposition 7. *Let* $\alpha, \beta = \left(p \pm \sqrt{p^2+4}\right)/2$. *The eigenvalues of* $G_{r,k}$ *are:*

$$\begin{cases} \left\{(-1)^{jr} \alpha^{r(k-2j)},\ (-1)^{jr} \beta^{r(k-2j)}\right\}_{j=0,1,\ldots,t-1} & \text{if } k = 2t-1, \\ \left\{(-1)^{tr},\ (-1)^{jr} \alpha^{r(k-2j)},\ (-1)^{jr} \beta^{r(k-2j)}\right\}_{j=0,1,\ldots,t-1} & \text{if } k = 2t. \end{cases}$$

As an example, when $k = r = 4$, we have

$$G_{4,4} = \begin{bmatrix} a_1 & -b_1 & c_1 & -d_1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \text{ and } H_{n,4,4} = \begin{bmatrix} a_n & -b_n & c_n & -d_n & a_{n-1} \\ a_{n-1} & -b_{n-1} & c_{n-1} & -d_{n-1} & a_{n-2} \\ a_{n-2} & -b_{n-2} & c_{n-2} & -d_{n-2} & a_{n-3} \\ a_{n-3} & -b_{n-3} & c_{n-3} & -d_{n-3} & a_{n-4} \\ a_{n-4} & -b_{n-4} & c_{n-4} & -d_{n-4} & a_{n-5} \end{bmatrix}$$

where $a_n = \left\{{n+4 \atop 4}\right\}_4, b_n = \left\{{n+4 \atop 3}\right\}_4 \left\{{n \atop 1}\right\}_4, c_n = \left\{{n+4 \atop 2}\right\}_4 \left\{{n+1 \atop 2}\right\}_4, d_n = \left\{{n+4 \atop 1}\right\}_4 \left\{{n+2 \atop 3}\right\}_4$.

The characteristic polynomial and the roots of $G_{4,4}$ are given by

$$f_{4,4}(x) = \sum_{i=0}^{5} (-1)^i \left\{{5 \atop i}\right\}_4 x^{5-i}$$

and $\lambda_5 = \alpha^{16}, \lambda_4 = \beta^{16}, \lambda_3 = \alpha^8, \lambda_2 = \beta^8, \lambda_1 = 1$ where $\alpha, \beta = \left(p \pm \sqrt{p^2+4}\right)/2$, and so, the matrix $H_5(v_4)$ has the same eigenvalues as the matrix $G_{4,4}$.

If we take $k = 4$ and $r = 3$, then

$$G_{3,4} = \begin{bmatrix} a_1 & b_1 & -c_1 & -d_1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \text{ and } H_{n,3,4} = \begin{bmatrix} a_n & b_n & -c_n & -d_n & a_{n-1} \\ a_{n-1} & b_{n-1} & -c_{n-1} & -d_{n-1} & a_{n-2} \\ a_{n-2} & b_{n-2} & -c_{n-2} & -d_{n-2} & a_{n-3} \\ a_{n-3} & b_{n-3} & -c_{n-3} & -d_{n-3} & a_{n-4} \\ a_{n-4} & b_{n-4} & -c_{n-4} & -d_{n-4} & a_{n-5} \end{bmatrix}$$

where $a_n = \left\{ {n+4 \atop 4} \right\}_3$, $b_n = \left\{ {n+4 \atop 3} \right\}_3 \left\{ {n \atop 1} \right\}_3$, $c_n = \left\{ {n+4 \atop 2} \right\}_3 \left\{ {n+1 \atop 2} \right\}_3$, $d_n = \left\{ {n+4 \atop 1} \right\}_3 \left\{ {n+2 \atop 3} \right\}_3$. The characteristic polynomial and the roots of $G_{3,4}$ are given by

$$f_{3,4}(x) = \sum_{i=0}^{5} (-1)^{i(i+1)/2} \left\{ {5 \atop i} \right\}_3 x^{5-i}$$

and $\lambda_5 = \alpha^{12}, \lambda_4 = \beta^{12}, \lambda_3 = -\alpha^6, \lambda_2 = -\beta^6, \lambda_1 = 1$.

Generalizing this discussion, we can give the following result.

Corollary 8. *For $n, k, r > 0$,*

$$\prod_{i=1}^{k+1} (x - \lambda_i) = \sum_{t=0}^{k+1} (-s(t-1)) \left\{ {k+1 \atop t} \right\}_r x^{k+1-t}.$$

In [4], Cooper and Kennedy show that

$$\operatorname{tr}(P_k(v_1)) = \frac{u_{kn}}{u_n},$$

where $H_n(p)$ is the generalized Pascal matrix. This was extended in one direction by us (along with G.N. Stănică) in [17] to show

$$\operatorname{tr}(P_k^n(v_r)) = \frac{u_{rkn}}{u_k}.$$

Since the matrices $H_{n,r,k}$ and $P_{k-1}^n(v_r)$ have the same eigenvalues, we also get

$$\operatorname{tr}(H_{n,r,k-1}) = \frac{u_{rkn}}{u_r},$$

which easily implies the next result.

Theorem 9. *For $n > 0$,*

$$\operatorname{tr}(H_{n,r,k}) = \sum_{i=0}^{\lfloor k-1/2 \rfloor} (-1)^{inr} v_{(k-2i)nr} + \frac{1}{2}\left(1 + (-1)^k\right).$$

From Proposition 7, we explicitly find the eigenvalues of the generalized $k : C$–nomial matrix.

4 The diagonalization of $G_{r,k}$ and the generalized Binet formula

In this section we diagonalize the matrix $G_{r,k}$ and then derive the Binet formula for the $r : C$–nomial coefficients $\left\{ {k \atop m} \right\}_r$. Let $\lambda_1, \lambda_2, \ldots, \lambda_{k+1}$ be the eigenvalues of $G_{r,k}$. Since the eigenvalues are all distinct, we can diagonalize $G_{r,k}$.

We define the $(k+1)\times(k+1)$ Vandermonde matrix V and diagonal matrix $D = diag(\lambda_1, \lambda_2, \ldots, \lambda_{k+1})$ as shown:

$$V = \begin{bmatrix} \lambda_1^k & \lambda_2^k & \ldots & \lambda_{k+1}^k \\ \vdots & \vdots & & \vdots \\ \lambda_1^2 & \lambda_2^2 & \ldots & \lambda_{k+1}^2 \\ \lambda_1 & \lambda_2 & \ldots & \lambda_{k+1} \\ 1 & 1 & 1 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_{k+1} \end{bmatrix}.$$

Since $\lambda_i \neq \lambda_j$ for $1 \leq i, j \leq k+1$, $\det V \neq 0$.

Let $V_j^{(i)}$ is the $(k+1)\times(k+1)$ matrix obtained from the transpose V^T by replacing the jth column of V by w_i where

$$w_i = \left[\ \lambda_1^{n-i+k+1} \quad \lambda_2^{n-i+k+1} \quad \ldots \quad \lambda_{k+1}^{n-i+k+1}\ \right]^T.$$

Now we give the Binet formula for the generalized $r : C$–nomial coefficients. The case $r = 1$ is given in [14]. The cases $r > 1$ can be easily obtained similar to the case $r = 1$.

Theorem 10. *For $n, r, k > 0$, and $a_{r,s}$ as in (6), then*

$$a_{n-i+1,j} = \frac{\det\left(V_j^{(i)}\right)}{\det(V)}.$$

Let $V_j^{(e_i)}$ be a $(k+1)\times(k+1)$ matrix obtained from the Vandermonde matrix V by replacing the jth column of V by e_i where V is defined as before and e_i is the ith element of the natural basis for $\mathbb{R}^n$, that is,

$$V_j^{(e_i)} = \begin{bmatrix} \lambda_1^k & \ldots & \lambda_{j-1}^k & 0 & \lambda_{j+1}^k & \ldots & \lambda_{k+1}^k \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \lambda_1^{k-i+1} & \ldots & \lambda_{j-1}^{k-i+1} & 0 & \lambda_{j+1}^{k-i+1} & \ldots & \lambda_{k+1}^{k-i+1} \\ \lambda_1^{k-i} & \ldots & \lambda_{j-1}^{k-i} & 1 & \lambda_{j+1}^{k-i} & \ldots & \lambda_{k+1}^{k-i} \\ \lambda_1^{k-i-1} & \ldots & \lambda_{j-1}^{k-i-1} & 0 & \lambda_{j+1}^{k-i-1} & \ldots & \lambda_{k+1}^{k-i-1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \lambda_1 & \ldots & \lambda_{j-1} & 0 & \lambda_{j+1} & \ldots & \lambda_{k+1} \\ 1 & \ldots & 1 & \underset{\substack{\downarrow \\ e_i}}{0} & 1 & \ldots & 1 \end{bmatrix}.$$

Let $q_j^{(i)} = \frac{\left|V_j^{(e_i)}\right|}{|V|}$ where the $(k+1)\times(k+1)$ matrices $V_j^{(e_i)}$ and V are defined as before.

We give the following theorem, whose proof is straightforward (the case $r = 1$ is given in [14], and the remaining cases, $r > 1$ can be similarly obtained).

Theorem 11. *Let $\lambda_1, \lambda_2, \ldots, \lambda_{k+1}$ be the distinct roots of $x^{k+1} - a_{1,1}x^k - a_{1,2}x^{k-1} - \cdots - a_{1,k}x - a_{1,k+1} = 0$. For any integer n and $1 \leq i \leq k+1$,*

$$a_{n,i} = \sum_{j=1}^{k+1} q_j^{(i)} \lambda_j^{n+k}.$$

For example, when $A=1$, we get $\alpha,\beta=\left(1\pm\sqrt{5}\right)/2$. For $k=2, r=3$ and $A=1$, the roots of $x^3-\frac{F_9}{F_3}x^2-\frac{F_9}{F_3}x+1=0$ are $\gamma_1=\alpha^6, \gamma_2=\beta^6, \gamma_3=-1$. By some computations, we get

$$\begin{aligned}
q_1^{(1)} &= \tfrac{1}{(\gamma_1-\gamma_3)(\gamma_1-\gamma_2)},\ q_2^{(1)}=\tfrac{1}{(\gamma_2-\gamma_3)(\gamma_2-\gamma_1)},\ q_3^{(1)}=\tfrac{1}{(\gamma_2-\gamma_3)(\gamma_1-\gamma_3)},\\
q_1^{(2)} &= -\tfrac{\gamma_2+\gamma_3}{(\gamma_1-\gamma_2)(\gamma_1-\gamma_3)},\ q_2^{(2)}=\tfrac{\gamma_1+\gamma_3}{(\gamma_2-\gamma_3)(\gamma_1-\gamma_2)},\ q_3^{(2)}=-\tfrac{\gamma_1+\gamma_2}{(\gamma_2-\gamma_3)(\gamma_1-\gamma_3)},\\
q_1^{(3)} &= \tfrac{\gamma_2\gamma_3}{(\gamma_1-\gamma_3)(\gamma_1-\gamma_2)},\ q_2^{(3)}=-\tfrac{\gamma_1\gamma_3}{(\gamma_1-\gamma_2)(\gamma_2-\gamma_3)},\ q_3^{(3)}=\tfrac{\gamma_1\gamma_2}{(\gamma_2-\gamma_3)(\gamma_1-\gamma_3)}.
\end{aligned}$$

Thus, by Theorem 11 and some arrangements, we get

$$\begin{aligned}
F_{3(n+1)}F_{3(n+2)} &= F_3F_6\left(\tfrac{\gamma_1^{(n+2)}}{(\gamma_1-\gamma_3)(\gamma_1-\gamma_2)}+\tfrac{\gamma_2^{(n+2)}}{(\gamma_2-\gamma_3)(\gamma_2-\gamma_1)}+\tfrac{(-1)^{n+2}}{(\gamma_2-\gamma_3)(\gamma_1-\gamma_3)}\right)\\
&= \frac{F_{6n+12}+F_{6n+6}+F_6\left(-1\right)^n}{10},
\end{aligned}$$

and

$$\begin{aligned}
F_{3n}F_{3(n+2)} &= F_3F_3\left(-\tfrac{(\gamma_2+\gamma_3)\gamma_1^{n+2}}{(\gamma_1-\gamma_2)(\gamma_1-\gamma_3)}+\tfrac{(\gamma_1+\gamma_3)\gamma_2^{n+2}}{(\gamma_2-\gamma_3)(\gamma_1-\gamma_2)}-\tfrac{(\gamma_1+\gamma_2)\gamma_3^{n+2}}{(\gamma_2-\gamma_3)(\gamma_1-\gamma_3)}\right)\\
&= \frac{F_{6(n+2)}-F_{6n}-F_{12}\left(-1\right)^n}{40}.
\end{aligned}$$

5 Sums and a generating function for the generalized $C-$nomial coefficients

In this section, we consider the sum of the generalized $r:C$–nomial coefficients by matrix methods. For this sum, we define a new matrix by extending $G_{r,k}$ (8). Take the $(k+2)\times(k+2)$ matrices

$$T_{r,k}=\begin{bmatrix} 1 & 0 & \dots & 0 \\ 1 & & & \\ 0 & & G_{r,k} & \\ \vdots & & & \\ 0 & & & \end{bmatrix} \text{ and } W_{n,r,k}=\begin{bmatrix} 1 & 0 & \dots & 0 \\ S_{n,r} & & & \\ \vdots & & H_{n,r,k} & \\ S_{n-k,r} & & & \end{bmatrix}$$

where $S_{n,r}$ is given by

$$S_{n,r}=\sum_{i=0}^{n-1}a_{i,1}=\sum_{i=0}^{n-1}\left\{ {k+i \atop k} \right\}_r.$$

Then we have the following result.

Theorem 12. *For $n,r,k>0$,*

$$T_{r,k}^n=W_{n,r,k}.$$

Proof. Since $S_{n+1,r} = a_{n,1} + S_{n,r}$ and by Theorem 2, we write the matrix recurrence relation $W_{n,r,k} = W_{n-1,r,k}T_{r,k}$. By the induction method, we write $W_{n,k} = W_{1,k}T_k^{n-1}$. From the definition of $W_{n,k}$, we obtain $W_{1,r,k} = T_{r,k}^1$ and so $W_{n,r,k} = T_{r,k}^n$, and the proof is complete. □

From Proposition 7, we know that the polynomial $f_{r,k}$ has root 1 for even r and $k \equiv 0 \pmod 4$. Expanding $\det(\lambda I_{k+2} - T_{r,k})$ with respect to the first row, it is easily seen that the matrix $T_{r,k}$ has also the eigenvalue 1. Thus the matrix $T_{r,k}$ has a double eigenvalue for even r and $k \equiv 0 \pmod 4$. For odd r and $k \not\equiv 0 \pmod 4$, we can diagonalize the matrix $T_{r,k}$ and so we derive an explicit formula for the sum.

Define a $(k+2) \times (k+2)$ matrix M as shown:

$$M = \begin{bmatrix} 1 & 0 & \dots & 0 \\ \delta & & & \\ \vdots & & V & \\ \delta & & & \end{bmatrix}$$

where $\delta = \left(1 - \sum_{i=1}^{k+1} a_{1,i}\right)^{-1}$ and the Vandermonde matrix V is defined as before.

It is easy to see that $T_{r,k}M = MD_1$, where $D_1 = diag(1, \lambda_1, \dots, \lambda_{k+1})$. By the Vandermonde matrix V, computing $\det M$ with respect to the first row shows $\det M = \det V$.

Theorem 13. *For $n, k > 0$, $k \not\equiv 0 \pmod 4$ and odd $r > 0$*

$$S_{n,r} = \frac{a_{n,1} + a_{n,2} + \cdots + a_{n,k+1} - 1}{\sum_{i=1}^{k+1} a_{1,i} - 1}.$$

Proof. Since M is invertible, $M^{-1}T_{r,k}M = D_1$, that is, $T_{r,k}$ is similar to D_1. Thus we write $T_k^n M = MD_1^n$. By Theorem 12, $W_{n,r,k}M = MD_1^n$. Equating the $(2,1)$th elements of $W_{n,r,k}M = MD_1^n$ and from a matrix multiplication, the proof follows. □

From Theorem 13, for $r = k = 3$ and $n \geq 0$, one may obtain

$$\sum_{i=0}^{n} \left\{ {3+i \atop 3} \right\}_3 = \frac{\left\{ {n+3 \atop 3} \right\}_3 + \left\{ {n+3 \atop 2} \right\}_3 \left\{ {n \atop 1} \right\}_3 - \left\{ {n+3 \atop 1} \right\}_3 \left\{ {n+1 \atop 2} \right\}_3 - \left\{ {n+2 \atop 3} \right\}_3 - 1}{\left\{ {4 \atop 3} \right\}_3 + \left\{ {4 \atop 2} \right\}_3 - \left\{ {4 \atop 1} \right\}_3 - \left\{ {4 \atop 0} \right\}_3 - 1}.$$

Next, we display a generating function for the generalized C–nomial coefficients. Define

$$g(i,x) = a_{0,i} + a_{1,i}x + a_{2,i}x^2 + a_{3,i}x^3 + \cdots + a_{n,i}x^n + \cdots$$

where $a_{n,i}$ is defined as before. The proof of the following proposition is rather straightforward and it is left as an exercise to the reader.

Proposition 14. *For $1 \leq i \leq k+1$ and $1 \leq t \leq k$*

$$g(i,x) = \frac{a_{0,i} + \left(a_{t,i} - \sum_{m=1}^{t} a_{1,m}a_{t-m,i}\right)x^t}{1 - a_{1,1}x - a_{1,2}x^2 - \cdots - a_{1,k+1}x^{k+1}}.$$

For example, when $i=1$ in Proposition 14, we get, for odd r,

$$\begin{aligned}&\sum_{n=0}^{\infty}\left\{{n+k \atop k}\right\}_r x^n\\ = &\frac{1}{1-\left\{{k+1 \atop k}\right\}_r x-\left\{{k+1 \atop k-1}\right\}_r x^2+\cdots+(-1)^{k(k+1)/2}\left\{{k+1 \atop 1}\right\}_r x^k+(-1)^{(k+1)(k+2)/2}x^{k+1}}\end{aligned}$$

and for even r

$$\begin{aligned}&\sum_{n=0}^{\infty}\left\{{n+k \atop k}\right\}_r x^n\\ = &\frac{1}{1-\left\{{k+1 \atop k}\right\}_r x+\left\{{k+1 \atop k-1}\right\}_r x^2+\cdots+(-1)^{k}\left\{{k+1 \atop 1}\right\}_r x^k+(-1)^{(k+1)}x^{k+1}}.\end{aligned}$$

For $k=r=3$ and $i=1$, we get

$$\sum_{n=0}^{\infty}\left\{{n+3 \atop 3}\right\}_3 x^n=\frac{1}{1-\left\{{4 \atop 3}\right\}_3 x-\left\{{4 \atop 2}\right\}_3 x^2+\left\{{4 \atop 1}\right\}_3 x^3+x^4}.$$

Similarly, one can obtain a plethora of other identities by taking other values of the involved parameters.

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