

QUANTUM STATES ASSOCIATED TO MIXED GRAPHS AND THEIR ALGEBRAIC CHARACTERIZATION

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ABSTRACT. Graph states are present in quantum information and found applications ranging from quantum network protocols (like secret sharing) to measurement based quantum computing. In this paper, we extend the notion of graph states, which can be regarded as pure quantum graph states, or as homogeneous quadratic Boolean functions associated to simple undirected graphs, to quantum states based on mixed graphs (graphs which allow both directed and undirected edges), obtaining mixed quantum states, which are defined by matrices associated to the measurement of homogeneous quadratic Boolean functions in some (ancillary) variables. In our main result, we describe the extended graph state as the sum of terms of a commutative subgroup of the stabilizer group of the corresponding mixed graph with the edges' directions reversed.

1. INTRODUCTION

Graph states [5] are very important in quantum computation and quantum error-correcting, and are the base of measurement-based quantum computation. There are several generalizations of graph states to various types of graphs, like hypergraphs (a graph in which an edge can join any number of vertices) and we refer here to [3, 7, 9] and the references therein. Certainly, a natural question that arises would be how to extend the concept of graph states to quantum states associated with *mixed graphs*, which are graphs where some or all of the edges may be directed (and no multiple edges or loops are allowed). It is the purpose of our paper to do just that, obtaining mixed quantum states, or matrices associated to the measurement of graph states in some (ancillary) variables. In our main result, we describe the extended graph state as the sum of terms of a commutative subgroup of the

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stabilizer group of the corresponding graph with the arrows reversed (which we call below, the companion graph).

Structure of the paper: In this section, we introduce the notation and concepts needed in the paper. In Section 2, we present the stabilizer basis associated with a mixed graph, and discuss the definition of a quantum state associated to a mixed graph, as well as the non-unicity of this state, via an example. In Section 3, we generalize the definition of graph states, and give a definition of the quantum states associated with a mixed graph, which we call a extended graph state. In Section 4, we discuss the minimal amount of ancillary states needed to extend a non-commutative group of stabilizers to a commutative group. In Section 5, we state and prove the main results of the paper. In Section 6, we discuss how to find extended graph states via maximal commutative subgroups of the stabilizer group of the corresponding mixed graph with the edges' directions reversed, which in turn gives a way to describe the density matrix of the mixed states associated to it. Finally, in Section 7, we summarize the paper. We will have many examples throughout to make the paper easier to read. While we draw our motivation from quantum information theory, the paper is very mathematical in nature.

For a positive integer n , we let \mathbb{F}_2 , \mathbb{F}_2^n , \mathbb{Z}_n , \mathbb{C} , be the two-element field, the vector space of dimension n over \mathbb{F}_2 , the ring of integers modulo n , and the complex numbers set, respectively. The operations in all of these algebraic structures will be denoted by \cdot , $+$, and will be understood from the context. When there is a danger of confusion we shall use \oplus for the binary operation.

A qubit (or qu-bit) [10] can be described as a column vector $|\psi\rangle = (a, b)^T \in \mathbb{C}^2$, where T indicates the transpose, $|a|^2$ is the probability of observing the value 0 when we measure the qubit, and $|b|^2$ is the probability of observing 1 (hence the sum of the two probabilities is 1). If both a and b are nonzero, the qubit has both the value 0 and 1 at the same time, and we call this a *superposition*. Once we have measured the qubit, however, the superposition collapses, and we are left with a classical state that is either 0 or 1 with certainty. A pure quantum state of n qubits is represented by a normalized complex vector with 2^n elements. We define $\langle\psi|$ as the conjugate transpose of $|\psi\rangle$. This notation is known as the *bra-ket* notation. A *mixed quantum state* corresponds to a probabilistic mixture of pure states, and it cannot be expressed as a vector, but only as a matrix $\rho \in \mathbb{C}^{2^n \times 2^n}$ which is positive semidefinite, Hermitian, and has trace 1¹; this matrix is known as the *density matrix* of the quantum state. A pure state $|\psi\rangle$ can also be described as a density matrix, given by $\rho = |\psi\rangle\langle\psi|$. We can distinguish the density matrix of a pure state from that of a mixed state by the following property: if $\text{Tr}(\rho^2) = 1$, the state is pure; if $\text{Tr}(\rho^2) < 1$, the state is mixed.

The *Pauli matrices* are:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, Y = iXZ,$$

where $i = \sqrt{-1}$. Note that all Pauli matrices are unitary, that is, if $U \in \{I, X, Z, Y\}$, then $UU^\dagger = I$, where U^\dagger is the conjugate transpose of the matrix. Note also that X, Z and Y are pairwise anti-commuting, and that X and Z are self-inverse. These properties are used frequently throughout the paper.

The Pauli matrices form a finite group up to global constants $\{\pm 1, \pm i\}$, in the sense that, for instance, $Y = iXZ$, $XY = iZ$, $YX = -iZ$, and similarly for the rest

¹A matrix describes a quantum state if and only if it fulfills these conditions.

of the relevant operations (see [10]). The n -tensor products of these matrices also form a finite group, up to global constants, known as the *Pauli group* in n qubits.

Let G be a simple, undirected graph with n nodes. We define the following set of pairwise commuting matrices (see [5]) in $\mathbb{C}^{2^n \times 2^n}$: For every node a of G , let

$$K_a = X_a Z_{\mathcal{N}_a} = X_a \prod_{b \in \mathcal{N}_a} Z_b,$$

where $A_a = I \otimes I \otimes \cdots \otimes \overset{a}{A} \otimes I \otimes \cdots \otimes I$, and \mathcal{N}_a are the neighbors of node a in G . The choice of matrices is motivated by quantum error-correcting codes. The matrices K_a have a unique common eigenvector corresponding to the eigenvalue $+1$ such that its first entry is $+2^{-\frac{n}{2}}$. This is a quantum state, and, since it can be represented by a simple graph, G , it is known as a *graph state* (see [5, 13]). Graph states are present in quantum information and found applications ranging from quantum network protocols (like secret sharing) to measurement based quantum computing [2, 8].

It is clear that, given two matrices, K_a, K_b , associated to a graph, which have $|\psi\rangle$ as a common eigenvector corresponding to the eigenvalue $+1$, the product $K_a K_b$ also has $|\psi\rangle$ as an eigenvector corresponding to eigenvalue $+1$. These matrices (known as the *stabilizers* of $|\psi\rangle$) generate a finite group (up to global constants), known as the *stabilizer group* of $|\psi\rangle$.

Let G be a simple, undirected graph, with associated graph state $|\psi\rangle$. We can associate to G a homogeneous quadratic Boolean function f in n variables, defined as the Boolean function with algebraic normal form (ANF) $f(x_0, \dots, x_{n-1}) = \sum_{a < b} \Gamma_{ab} x_a x_b$, where $\Gamma = (\Gamma_{ab})_{a,b}$ is the adjacency matrix of the graph. We define the signature of the function f , $(-1)^f$, as the length 2^n bipolar vector $\mathbf{s} = (s_{00\dots 0}, s_{00\dots 1}, \dots, s_{11\dots 1})$ such that $s_{\mathbf{i}} = (-1)^{f(\mathbf{i})}$. Then, it can be proven (see [15]) that $|\psi\rangle = 2^{-\frac{n}{2}} (-1)^f$, that is, the normalized signature of the function f .

Example: For example, let G be the simple, undirected line on nodes $\{0, 1, 2\}$ with edges $01, 12$ (we customarily, write ab for an edge between the vertices a, b). Then, the matrices K_a are defined as follows:

$$\begin{aligned} K_0 &= X_0 \prod_{b \in \{1\}} Z_b = X_0 Z_1 = X \otimes Z \otimes I, \\ K_1 &= X_1 \prod_{b \in \{0,2\}} Z_b = X_1 Z_0 Z_2 = Z \otimes X \otimes Z, \\ K_2 &= X_2 \prod_{b \in \{1\}} Z_b = X_2 Z_1 = I \otimes Z \otimes X. \end{aligned}$$

Note that these matrices pairwise commute. It can be easily checked that a common eigenvector with eigenvalue $+1$ is $|\psi\rangle = \frac{1}{2^{3/2}} (1, 1, 1, -1, 1, 1, -1, 1)^T$. This vector is the associated graph state to the graph G . The associated Boolean function is $f(x_0, x_1, x_2) = x_0 x_1 + x_1 x_2$, and consequently, $|\psi\rangle = \frac{1}{2^{3/2}} (-1)^f$.

2. STABILIZERS OF A MIXED GRAPH

Similarly to the case of undirected simple graphs, we can define stabilizers for mixed graphs:

Definition 2.1. Let G be a mixed graph G with set of vertices V and set of edges E , we can define the *stabilizer basis* associated to G as:

$$K_v^G = X_v Z_{\vec{\mathcal{N}}_v} = X_v \prod_{b \in \vec{\mathcal{N}}_v} Z_b,$$

where $A_v = I \otimes I \otimes \cdots \otimes \overset{v}{A} \otimes I \otimes \cdots \otimes I$, and $\vec{\mathcal{N}}_v = \{u : (v, u) \in E\}$ are the out-neighbors of v in G . When G is clear from context, we will drop the superscript.

This could be interpreted as an interaction pattern in a similar way as for graph states: for simple undirected graphs, whenever two particles, originally spin-1/2 systems, have interacted via a certain (Ising) interaction, the graph connecting the two associated vertices has an edge (see [5]). In the case of a mixed graph, one could interpret a directed edge as a one-way interaction, while undirected edges are two-way interactions. However, in this paper we will concentrate on the stabilizer interpretation of the states, due to its possible applications in quantum error-correcting codes.

In the case of a mixed graph, the matrices K_a that arise from a mixed graph are not necessarily pairwise commuting, so, in general, they will not have a common eigenvector. In fact, two square matrices A, B of size n have a common eigenvector

if and only if $\bigcap_{k, \ell=1}^{n-1} \ker[A^k, B^\ell] \neq \{0\}$, where $[U, V] = UV - VU$ is the commutator

(see [14]). However, since in our case, we deal with matrices which are self-inverse, the result simplifies to the matrices commuting with each other.

For mixed graphs, we then need to slightly modify/extend the context, in order to find an equivalent for this common eigenvector. Below, we display our approach. Let U be any stabilizer of a quantum graph state $|\psi\rangle$. Let $\rho = |\psi\rangle\langle\psi|$ be the density matrix of the state. Since $U|\psi\rangle = |\psi\rangle$, then $U\rho U^\dagger = \rho$, or, equivalently, $U\rho = \rho U$. Note that, then, ρ is an element of the *normalizer* of the stabilizer group (see [6] for more on group theoretical concepts), seen as a subgroup of the group generated by $\{I, X, Y, Z\}^{\otimes n}$.

We go now back to the question of mixed graphs. Even though, in this case, the stabilizers will not have a common eigenvector, since they are not in general pairwise commuting (though, they are either commuting $AB = BA$, or anti-commuting, that is, $AB = -BA$), we can, however, say that a density matrix ρ such that $K_a \rho K_a^\dagger = \rho$ (or, equivalently, $K_a \rho = \rho K_a$ for all a) is stabilized by the stabilizers.

Note that, if $s\rho s^\dagger = \rho$, then ρ is stabilized by any αs , where $|\alpha| = 1$ (that is, $(\alpha s)\rho(\alpha s)^\dagger = \rho$), which is why we do not have to concern ourselves with global phase constants in members of the stabilizer group S , and we will choose a representative where the global constant is 1.

Examples: 1) For instance, given the mixed graph G_1 over the set of nodes $\{0, 1, 2\}$ comprising an undirected edge 12, and a directed edge $0 \rightarrow 1$ (customarily, we shall represent the edges in our graphs by using arrows for directed edges between nodes, and, as before, concatenation, for undirected edges), we obtain the stabilizer basis $K_0^{G_1} = X \otimes Z \otimes I$, $K_1^{G_1} = I \otimes X \otimes Z$, $K_2^{G_1} = I \otimes Z \otimes X$. Note that, while $K_0^{G_1}$ and $K_2^{G_1}$, respectively, $K_1^{G_1}$ and $K_2^{G_1}$, commute, $K_0^{G_1}$ and $K_1^{G_1}$ anti-commute, so there is no common eigenvector $|\psi\rangle$ for all $K_0^{G_1}$, $K_1^{G_1}$, $K_2^{G_1}$. The group of matrices generated by $K_0^{G_1}$, $K_1^{G_1}$ and $K_2^{G_1}$ is then (up to global constants) the stabilizer

group S^{G_1} is

$$\{I \otimes I \otimes I, X \otimes Z \otimes I, I \otimes X \otimes Z, I \otimes Z \otimes X, X \otimes Y \otimes Z, X \otimes I \otimes X, I \otimes Y \otimes Y, X \otimes X \otimes Y\}.$$

We can however associate S^{G_1} to the quantum state whose density matrix (we will discuss in Section 3 how to obtain this matrix) is

$$\rho_0^{G_1} = \frac{1}{8} \begin{pmatrix} 1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & 1 \end{pmatrix}.$$

It is easy to check that

$$K_0^{G_1} \rho_0^{G_1} (K_0^{G_1})^\dagger = K_1^{G_1} \rho_0^{G_1} (K_1^{G_1})^\dagger = K_2^{G_1} \rho_0^{G_1} (K_2^{G_1})^\dagger = \rho_0^{G_1}.$$

2) We now take G_2 be the directed triangle defined by its directed edges $0 \rightarrow 1, 1 \rightarrow 2, 2 \rightarrow 0$. The stabilizer group S^{G_2} is generated by

$$K_0^{G_2} = X \otimes Z \otimes I, K_1^{G_2} = I \otimes X \otimes Z, K_2^{G_2} = Z \otimes I \otimes X.$$

Therefore, S^{G_2} is

$$\{I \otimes I \otimes I, X \otimes Z \otimes I, I \otimes X \otimes Z, Z \otimes I \otimes X, X \otimes Y \otimes Z, Y \otimes Z \otimes X, Z \otimes X \otimes Y, Y \otimes Y \otimes Y\}.$$

Because these generators anti-commute, there is no common eigenvector $|\psi\rangle$ for all $K_0^{G_2}, K_1^{G_2}, K_2^{G_2}$. Instead we can associate S^{G_2} to the quantum state whose density matrix (we will discuss in Section 3 how to obtain this matrix) is

$$\rho_0^{G_2} = \frac{1}{8} \begin{pmatrix} 1 & 0 & 0 & i & 1 & 0 & 0 & -i \\ 0 & 1 & i & 0 & 0 & -1 & i & 0 \\ 0 & -i & 1 & 0 & 0 & i & 1 & 0 \\ -i & 0 & 0 & 1 & -i & 0 & 0 & -1 \\ 1 & 0 & 0 & i & 1 & 0 & 0 & -i \\ 0 & -1 & -i & 0 & 0 & 1 & -i & 0 \\ 0 & -i & 1 & 0 & 0 & i & 1 & 0 \\ i & 0 & 0 & -1 & i & 0 & 0 & 1 \end{pmatrix}.$$

It is easy to check that

$$K_0^{G_2} \rho_0^{G_2} (K_0^{G_2})^\dagger = K_1^{G_2} \rho_0^{G_2} (K_1^{G_2})^\dagger = K_2^{G_2} \rho_0^{G_2} (K_2^{G_2})^\dagger = \rho_0^{G_2}.$$

In both examples, the density matrix displayed does not have the property $\text{Tr}(\rho_0^2) = 1$, and it corresponds therefore to a mixed quantum state. We will see, however, that ρ_0^G is not a unique mixed quantum state associated to S . For both examples, There are other density matrices ρ_j such that the above identities hold. Namely, we can find two other matrices² that are not *locally equivalent* (that is, one cannot be transformed into the other by multiplication by a n -fold tensor product of 2×2 unitary matrices) and three more matrices which are locally equivalent to the above ones, for a total of six. These six states form a family, the family of extended graph states associated to this graph.³

²As we will see, we only consider matrices of the same size (determined uniquely by the graph) in our extension of graph states.

³If one wishes to define a unique state associated to the graph, it could also be natural to define the extended graph state associated to the graph as $\rho = \sum_{i=0}^5 c_i \rho_i$, where $c_j \geq 0$, $\sum_{j=0}^5 c_j = 1$, i.e., as the convex sum of these density matrices. Note that, although only three of them are locally inequivalent, we include them all in the general sum ρ , since changing ρ_j for a locally equivalent ρ'_j does not necessarily give local equivalence in ρ .

3. EXTENDED GRAPH STATES

Throughout the remaining sections we define the pure state vectors using a Boolean function notation [12]. We need a slight generalization of the one in previous sections, since, as we shall see later in the section, we extend the matrix of stabilizers to a fully (row) commuting matrix, which does not always have X in the diagonal, and is thus not the usual stabilizer set of a graph state. A natural generalization of the stabilizer basis for a graph state (associated to a mixed graph G) is given by:

Definition 3.1. Let G be a simple undirected graph with weighted nodes, where the weight can be 0 or 1. For reasons of easy reference, we will refer to the nodes of weight 0 as white nodes, and to the nodes of weight 1 as red nodes. We define the stabilizers associated to G as follows:

$$K_v^G = \sigma_v Z_{\mathcal{N}_v} = \sigma_v \prod_{w \in \mathcal{N}_v} Z_w,$$

where $\sigma_v = X$ if v is a white node, and $\sigma_v = Y$ if v is a red node. When G is clear, we will drop the superscript.

For the construction of extended graph states that we use in this paper we require this extra generality.

Definition 3.2. Let G be a simple undirected graphs with red and white nodes as described above. Let A be the modified adjacency matrix of our graph state with elements A_{jk} , defined as $A_{jk} = 1$ if $k \in \mathcal{N}_j$, $A_{jj} = 0$ if j is a white node, and $A_{jj} = 1$ if j is a red node. We define the *generalized graph state* associated to G , $|\psi\rangle$, as $|\psi\rangle = 2^{-\frac{n}{2}} i^p$, where $p : \mathbb{F}_2^n \rightarrow \mathbb{Z}_4$, defined by $p(x_0, x_1, \dots, x_{n-1}) = \sum_{j < k} 2A_{jk}x_jx_k + \sum_j A_{jj}x_j$.

This is a natural generalization of the concept of graph state. The proof for the next lemma is straightforward, so we omit it.

Lemma 3.3. *Let G be a simple undirected graphs with red and white nodes as described above, and let K_v^G and $|\psi\rangle$ defined as above. Then, for any node v , $K_v^G |\psi\rangle = |\psi\rangle$.*

For simplicity of notation, we will associate to the set of stabilizers K_0, \dots, K_{n-1} a matrix \mathcal{A} defined by $\mathcal{A}_{uu} = \sigma_u$ and $\mathcal{A}_{uv} = \begin{cases} Z, & v \in \mathcal{N}_u \\ I, & v \notin \mathcal{N}_u \end{cases}$, and we will talk about the multiplication of its rows as the result of the multiplication of the corresponding stabilizers. In the same spirit, we refer to the commutativity or non-commutativity of the rows of the matrix \mathcal{A} , meaning the commutativity or non-commutativity of the corresponding stabilizers.

In Example 2) of Section 2, for instance, we obtain the matrix

$$\mathcal{A} = \left(\begin{array}{ccc} X & Z & I \\ I & X & Z \\ Z & I & X \end{array} \right) \longleftrightarrow \left. \begin{array}{l} K_0 = X \otimes Z \otimes I \\ K_1 = I \otimes X \otimes Z \\ K_2 = Z \otimes I \otimes X \end{array} \right\},$$

which has rows that are not pairwise commuting, but can still be interpreted as a quantum object by making it part of a larger fully commuting matrix, where we choose the environment appropriately. This will imply that our quantum object is a mixed state, which we will call an *extended graph state* (in this paper, we limit our study to the addition of environmental qubits as opposed to, more generally, qudits).

Definition 3.4. Let G be a mixed graph, with associated matrix \mathcal{A} , and let e be the (minimum) number of columns, e , to \mathcal{A} so as to make all rows pairwise commute. Let \mathcal{B} be any symmetric matrix obtained by extending \mathcal{A} by e environmental columns and rows. We will refer to the graph state associated to \mathcal{B} as a *parent* graph state of G . We then denote the marginal state given by tracing out the environmental qubits as its *child*, in the following way: Let $|\phi\rangle$ be the parent graph state. Then, $|\phi\rangle = 2^{-\frac{n}{2}} i^{p(x_0, \dots, x_{n-1}, \dots, x_{n-1+e})}$. Let $(a_n, \dots, a_{n-1+e}) \in \mathbb{F}_2^e$. We define $|\phi_{a_n, \dots, a_{n-1+e}}\rangle = 2^{-\frac{n}{2}} i^{p(x_0, \dots, x_{n-1}, x_n=a_n, \dots, x_{n-1+e}=a_{n-1+e})}$, the result of measuring the added variable(s) in $x_k = a_k$, $n \leq k \leq n-1+e$. Their respective density matrices are $\rho_{a_n, \dots, a_{n-1+e}} = |\phi_{a_n, \dots, a_{n-1+e}}\rangle \langle \phi_{a_n, \dots, a_{n-1+e}}|$. Then,

$$\rho = \frac{1}{2^e} \sum_{(a_n, \dots, a_{n-1+e}) \in \mathbb{F}_2^e} \rho_{a_n, \dots, a_{n-1+e}}$$

is the child of the parent $|\phi\rangle$. Note that the child is, in general, a mixed quantum state, being pure if and only if G is undirected (in which case, it is the usual graph state).

NB: We can easily check that ρ is stabilized by the matrix $K_a = X_a \prod_{b \in \mathcal{N}_a} Z_b$ associated to the mixed graph G : observe that if $p : \mathbb{F}_2^n \rightarrow \mathbb{F}_4$ is an arbitrary quadratic generalized Boolean function where all quadratic terms are Boolean, then, X_a will change x_a for $x_a + 1$, and each Z_b will add the linear term x_b , so that $K_a i^p = i^{p(x_0, \dots, x_a+1, \dots, x_{n-1}) + 2 \sum_{b \in \mathcal{N}_a} x_b}$.

Definition 3.5. We define then the *family of extended graph states* associated to G as the set of all children of parent graph states of G .⁴

We are therefore looking to append a (minimum) number of columns, e , to \mathcal{A} so as to make all rows pairwise commute. We can then add the same number, e , of rows to the matrix to make it square, making sure that all rows pairwise commute. We will come back to this minimal append parameter e , later in the section.

NB: Note that one could extend by more than one column/row and still obtain a fully commuting matrix. By stipulating that that we only extend by the minimum possible number e of columns/rows, we take the most natural generalization of the concept of graph state, being the most compatible with the pure graph state formulation as $e = 0$ forces the state to be pure.

The following example illustrates the concepts of parent and child:

Example: Let $K_0 = X \otimes Z \otimes I$, $K_1 = I \otimes X \otimes Z$ and $K_2 = I \otimes Z \otimes X$, so that $\mathcal{A} = \begin{pmatrix} X & Z & I \\ I & X & Z \\ I & Z & X \end{pmatrix}$ be the matrix associated to G_1 in Section 2. We can extend \mathcal{A}

by $e = 1$ column/row (surely, here, e is minimal, since our original matrix has non-commuting rows) to get $\mathcal{A}' = \begin{pmatrix} & & Z & \\ & \mathcal{A} & X & \\ & & I & \\ Z & I & I & X \end{pmatrix} = \begin{pmatrix} X & Z & I & Z \\ I & X & Z & X \\ I & Z & X & I \\ Z & I & I & X \end{pmatrix}$, which is

fully row commuting. Finally, as this matrix is fully commuting we can, by suitable row multiplications, recover its graph form (meaning that X or Y are on the main diagonal, and only Z and I are outside the diagonal) with $\mathcal{B} = \begin{pmatrix} X & Z & I & Z \\ Z & X & Z & I \\ I & Z & X & I \\ Z & I & I & X \end{pmatrix}$.

⁴As mentioned in the examples in Section 2, we could alternatively define a unique *extended graph state* as the convex sum of all children of parents graph states of G , which is a mixed quantum state.

This can be interpreted (via the stabilizers inferred from the rows) as a graph, and, because it is symmetric, as an undirected graph.

We consider the pure graph state described by \mathcal{B} to be a *parent* graph state, being a parent of the extended graph state described by \mathcal{A} . We then denote the marginal state given by tracing out the environmental qubits as its *child*.

Here $e = 1$, and the parent graph state is $|\phi\rangle = 2^{-\frac{3}{2}}(-1)^{x_0(x_1+x_3)+x_1x_2}$. We measure x_3 at 0 and 1, and obtain $|\phi_0\rangle = 2^{-\frac{3}{2}}(-1)^{x_0x_1+x_1x_2} = 2^{-\frac{3}{2}}(1, 1, 1, -1, 1, 1, -1, 1)^T$ and $|\phi_1\rangle = 2^{-\frac{3}{2}}(-1)^{x_0(x_1+1)+x_1x_2} = 2^{-\frac{3}{2}}(1, 1, 1, -1, -1, -1, 1, -1)^T$. Note that $K_0|\phi_0\rangle = |\phi_0\rangle = K_2|\phi_0\rangle$, while $K_1|\phi_0\rangle = |\phi_1\rangle$, and that $K_0|\phi_1\rangle = -|\phi_1\rangle = -K_2|\phi_1\rangle$, while $K_1|\phi_1\rangle = |\phi_0\rangle$. From these two states, we obtain the density matrices $\rho_0 = |\phi_0\rangle\langle\phi_0|$, $\rho_1 = |\phi_1\rangle\langle\phi_1|$. By the argument above, $K_0\rho_0K_0^\dagger = \rho_0 = K_2\rho_0K_2^\dagger$, while $K_1\rho_0K_1^\dagger = \rho_1$, and $K_0\rho_1K_0^\dagger = \rho_1 = K_2\rho_1K_2^\dagger$, while $K_1\rho_1K_1^\dagger = \rho_0$. The child graph state is then given by $\rho = \frac{1}{2}(\rho_0 + \rho_1) = \rho_0^{G^1}$.

Note that this matrix is stabilized by the operators K_0, K_1, K_2 and that this is a density matrix, since it has trace 1, it is Hermitian (a complex square matrix that is equal to its own conjugate transpose) and positive semi-definite (indeed, it is a convex sum of density matrices, thus, a density matrix [11]). This is, therefore, a state stabilized by the associated stabilizers for the graph on three vertices with edges $0 \rightarrow 1, 12$.

NB: A parent graph state, and consequently a child, can be associated to different mixed graphs (this is a natural consequence of Theorem 5.4, since the density matrix of a child is, as we shall see there, a sum of a maximal subset of stabilizers of the graph with the arrows reversed, and such a subset occurs in the stabilizer of different graphs. However, the family of children, i.e. the family of extended graph states, is associated with a unique mixed graph.

4. THE MINIMAL APPEND PARAMETER e

We now go back to the minimal append parameter e , i.e. the number of columns/rows that need to be added to the adjacency matrix of the graph to obtain, via linear operations, a symmetric matrix. Let G be the mixed graph defined by the adjacency matrix A , and G_b be the undirected graph that has adjacency matrix $\Gamma = A + A^T$. Thus, G_b is the simple graph obtained from G by erasing all undirected edges of G and making all directed edges of G undirected. We recall the following result, which is a consequence of [1].

Lemma 4.1. *The minimum number, e , of columns and rows required to be added to \mathcal{A} to make its rows pairwise fully commuting is given by*

$$e = \frac{1}{2}\text{rank}(\Gamma).$$

(We refer to e as the mixed rank).

NB: Note that the rank (we need here the rank over the binary field, though the result is more general) of a symmetric binary matrix with zero diagonal, such as Γ , is always even [4].

Lemma 4.2. *The extension columns and rows do not depend on the direction of the arrows. (Note however that the parent graphs, and therefore the children states, do depend on the direction of the arrows.)*

Proof. Note that from the previous lemma, it is easy to see that Γ is the same regardless of the direction of the arrows, consequently, the number of extension columns and rows is independent of the arrow direction. Furthermore, how we can extend the columns/rows depends exclusively on the commutativity or anti-commutativity of the stabilizer basis, which is not dependent on arrow direction. \square

For example 1) in Section 2, where $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, and so, $\mathcal{A} = \begin{pmatrix} X & Z & I \\ I & X & Z \\ I & Z & X \end{pmatrix}$, the minimum number of columns and rows necessary to add to \mathcal{A} to make it fully commuting is

$$e = \frac{1}{2} \text{rank}(\Gamma) = \frac{1}{2} \text{rank} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 1,$$

which is consistent to what we have previously observed. We extended to a 4-qubit pure graph parent state where the 4th qubit is part of the environment. This pure graph state can be written using the Boolean function notation as

$$|\psi\rangle_e = \frac{1}{4} (-1)^{x_0x_1+x_0x_3+x_1x_2} = \frac{1}{4} (1, 1, 1, -1, 1, 1, -1, 1, 1, -1, 1, 1, -1, -1, -1)^T.$$

Tracing out the 4th qubit, i.e., summing the projections obtained by fixing $x_3 = 0$ and 1, respectively, we obtain the projections $|\phi_0\rangle\langle\phi_0|$ and $|\phi_1\rangle\langle\phi_1|$, where $|\phi_0\rangle = \frac{1}{\sqrt{8}} (-1)^{x_0x_1+x_1x_2}$, $|\phi_1\rangle = \frac{1}{\sqrt{8}} (-1)^{x_0x_1+x_1x_2+x_0}$, and then $\rho = \frac{1}{2} (|\phi_0\rangle\langle\phi_0| + |\phi_1\rangle\langle\phi_1|)$. As we have seen, we can interpret the stabilizer matrix \mathcal{A} as representing the mixed quantum state ρ . To obtain one more member of the family, observe that it is

equally valid to extend \mathcal{A} to $\mathcal{A}' = \begin{pmatrix} & X & & \\ & Z & & \\ \mathcal{A} & & & \\ I & Z & I & X \end{pmatrix} = \begin{pmatrix} X & Z & I & X \\ I & X & Z & Z \\ I & Z & X & I \\ I & Z & I & X \end{pmatrix}$, which,

by multiplicative row operations, renders $\mathcal{A}^e = \begin{pmatrix} X & I & I & I \\ I & X & Z & Z \\ I & Z & X & I \\ I & Z & I & X \end{pmatrix}$. It is clear

(since the former is a connected graph and the latter is not) that the parent graph state described by this \mathcal{A}^e is locally inequivalent to the previous parent. Therefore the resultant mixed state, $\rho = |\phi_0\rangle\langle\phi_0| + |\phi_1\rangle\langle\phi_1|$, obtained by tracing out the 4th qubit, is different from the previous mixed state.

We shall see in later sections (see our main Theorem 5.4) that these density matrices are given in terms of the Pauli group by:

$$\rho = \sum_{j \in J} \sigma_j,$$

where $\pm 1, \pm i\{\sigma_j : j \in J\}$ is a maximal commutative subgroup of the Pauli group generated by the stabilizer of the graph with reversed edges' direction⁵. The following sections will therefore include a discussion over this type of subgroup. We will also see a simple formula to compute the size of a family (see Corollary 3).

We next define the *codespace* (that is, the stabilizer group S) of the $n \times n$ matrix \mathcal{A} to be the 2^n stabilizers formed by products of one or more of the rows of \mathcal{A} (recall that, as some of the rows of \mathcal{A} anti-commute, some of the members of S are only defined up to a global multiplicative constant of ± 1). Likewise, the codespace of \mathcal{A}^e

⁵Here we can also see how this is a natural generalization of the concept of graph states; in the case of an undirected graph the graph with reversed arrows is itself, and the only maximal commutative subgroup is the whole group of stabilizers.

comprises 2^{n+e} stabilizers, but now all rows of \mathcal{A}^e commute, so the global constant is always 1.

We now introduce the well known *Hadamard matrix*, $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ as well as the *negahadamard matrix* $N = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$, both being unitary.

Proposition 4.3. *The Pauli group, $\{I, X, Z, Y\}$, is conjugated (up to a factor of ± 1) by the group generated by $\{I, H, N\}$. Specifically,*

	I	H	N	N^2	NH	HN
X	X	Z	$-Y$	Z	X	Y
Z	Z	X	X	$-Y$	$-Y$	Z
Y	Y	$-Y$	$-Z$	$-X$	Z	$-X$

where the table is read in the following way: the entry matrix C at the intersection of the row corresponding to the matrix A and column matrix B , means that $C = BAB^\dagger$. For instance, $X = HZH^\dagger$, $-Z = NYN^\dagger$, and $-X = N^2Y(N^2)^\dagger$.

Proof. The proof follows by straightforward computation. \square

Proposition 4.3 implies that, by conjugation, to within a factor of ± 1 , we can permute column-wise (there are $3! = 6$ such permutations) the elements of the stabilizer matrix.

The following proposition is well known [10] and so, we omit its proof.

Proposition 4.4. *Consider a quantum system comprising a local system, L , possibly entangled with an environment, E , and let ρ_L be the density matrix that defines the local system. Then, the unitary conjugation, $U_E(L \times E)U_E^\dagger$ on the environment leaves ρ_L unchanged.*

By Propositions 4.3 and 4.4, we can always perform the extension to a fully commutative stabilizer set by taking X as elements jj , $j = 1, \dots, n + e - 1$.

5. CHILDREN OF PURE GRAPH STATE PARENTS

In this section, we will state and prove the main theorem of the paper, which describes how an extended graph state can be expressed as the sum of terms of a commutative subgroup of the stabilizer group of the companion graph. The different maximal commutative subgroups correspond to different extensions of the stabilizer set. We also express the coefficients in this sum in terms of the particular extension that gives rise to a child.

Certainly, if Γ is the adjacency matrix of a graph G , then Γ^T is the adjacency matrix of the graph with the arrows reversed (the companion graph). In terms of the matrix given by the stabilizer, \mathcal{A} , the corresponding stabilizer group is defined by \mathcal{A}^T .

Let $\mathcal{A} = (\sigma_{ij})$, where $\sigma_{ij} \in \{X, Z, I\}$, be, as before, the stabilizer matrix. We let

$$\mathcal{A}_k^T = \sigma_{0k} \otimes \sigma_{1k} \otimes \cdots \otimes \sigma_{(n-1)k}.$$

Given $K = \{k_1 < k_2 < \dots < k_s\} \subseteq \{0, 1, \dots, n - 1\}$, we define $\mathcal{A}_K^T = \prod_{i=1}^s \mathcal{A}_{k_i}^T$ (surely, another ordering is just as valid, and would differ from our defined \mathcal{A}_K^T by a possible global multiplicative factor of -1 , if the factors anti-commute). We can now let $S^T = \{\mathcal{A}_K^T : K \subseteq \{0, 1, \dots, n - 1\}\}$. Note that this is, up to multiplication with a global constant $\pm i$, the stabilizer group of the graph with the arrows reversed.

In this section, given a binary string of length n , $j = j_0 \cdots j_{n-1}$, we define the matrices $\overleftarrow{s}_j := \mathcal{A}_K^T$, where $K = \text{supp}(j) = \{i : j_i = 1\}$ (the arrow above is set to indicate that these are associated to the graph with the arrows reversed). By convention, $\overleftarrow{s}_{0\dots 00} = I_n$. Note that the order of multiplication situation carries over to the \overleftarrow{s}_j . In our main Theorem 5.4, the coefficients b_j are therefore only defined up to ± 1 . Before we state and prove our theorem, we will need several lemmas. By the support of a matrix $M = \{M_{ij}\}_{i,j}$ of dimension $t \times t$, denoted, by abuse of notation, $\text{supp}(M)$, we mean the set of nonzero element positions of M , that is, $\text{supp}(M) = \{(i, j) \in \{0, \dots, t-1\}^2 : M_{ij} \neq 0\}$ ⁶. We also let $(a, b) + \text{supp}(M) = \{(i+a, j+b) : (i, j) \in \text{supp}(M)\}$. We denote by R_A the set of indices k such that the matrix in position k is A .

Lemma 5.1. *Let $\overleftarrow{s}_j \in S^\perp$, the stabilizer group of the graph with the arrows reversed. Then, for any $(0, k) \in \text{supp}(\overleftarrow{s}_j)$, with $k = \sum_{\ell=0}^{n-1} a_\ell 2^\ell$, and $a_\ell = 1$ if and only if $(n-\ell, n-\ell) \in R_X \cup R_Y$.*

Proof. If M be a $t \times t$ matrix, then,

$$I \otimes M = \begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix} \text{ and } X \otimes M = \begin{pmatrix} 0 & M \\ M & 0 \end{pmatrix}.$$

Note that X has the same support as Y , and Z has the same support as I , so we only have to consider these two cases. First, observe that

$$\begin{aligned} \text{supp}(I \otimes M) &= \text{supp}(M) \cup ((t, t) + \text{supp}(M)), \text{ and} \\ \text{supp}(X \otimes M) &= ((0, t) + \text{supp}(M)) \cup ((t, 0) + \text{supp}(M)). \end{aligned}$$

Let $\overleftarrow{s}_j = \otimes_{R_X} X \otimes_{R_Y} Y \otimes_{R_Z} Z \otimes_{R_I} I$. From the above observations, $(0, k) \in \text{supp}(\overleftarrow{s}_j)$, with $k = \sum_{\ell=0}^{n-1} a_\ell 2^\ell$, and $a_\ell = 1$ if and only if $(n-\ell, n-\ell) \in R_X \cup R_Y$. \square

Lemma 5.2. *The set of length n Pauli words (stabilizers) that commute with all rows of \mathcal{A} forms a multiplicative group S^\perp of size 2^n , identifying with \mathcal{A}^T (up to ± 1 constants).*

Proof. Certainly, every row of \mathcal{A}^T commutes with every row of \mathcal{A} . This set is maximal, since half of the possible length n Pauli words commute with a given row. The (independent) n rows of \mathcal{A}^T jointly commute with $4^n/2^n = 2^n$ Pauli words. However, the rows of \mathcal{A}^T generate 2^n distinct words and consequently these are the only ones occurring. \square

Lemma 5.3. *The elements of the stabilizer group S^\perp of a mixed graph have non-intersecting support.*

Proof. First, we observe that the Pauli matrices $\{X, Y\}$ have intersecting support, which is disjoint from the common intersecting support of $\{I, Z\}$. If a tensor product of Pauli matrices differ in at least one tensor position, one from $\{X, Y\}$, and another from $\{I, Z\}$, then these tensor products also have non-intersecting support. As observed, the matrix \mathcal{A} only has X and/or Y on the diagonal and I and/or Z on the off diagonal. We let \mathcal{A} have rows numbered 0 to $n-1$. Let $R, R' \subseteq \{0, \dots, n-1\}$, and consider the matrices $\tilde{\sigma}_{k_R}$ and $\tilde{\sigma}_{k_{R'}}$, being the product of the rows of \mathcal{A} indexed by R, R' , respectively. Then $\tilde{\sigma}_{k_R}$ and $\tilde{\sigma}_{k_{R'}}$ have X or Y at tensor positions indexed by R, R' , respectively, and I or Z at the other positions. So, unless R, R' are equal,

⁶For ease of notation in the results above, we index from 0 to $t-1$, instead of the more conventional from 1 to t .

they must be non-intersecting in at least one tensor position, so therefore $\tilde{\sigma}_{k_R}$ and $\tilde{\sigma}_{k_{R'}}$ have non-intersecting support. The elements of the stabilizer group of the mixed graph are obtained by ranging R over all stabilizer codewords, therefore any two of them must be non-intersecting. \square

We now state and prove our main theorem.

Theorem 5.4. *Let $L_m = \{v : (\mathcal{A}^e)_{v,n+m} = Z \text{ or } Y\}$, $\mathcal{I}(x) = \prod_{m=0}^{e-1} \left(1 + \sum_{k \in L_m} x_k\right) = \prod_{m=0}^{e-1} \mathcal{I}_m(x)$, and $J = \{x \in \mathbb{F}_2^n : \mathcal{I}(x) = 1\}$. Given a directed graph associated with \mathcal{A} , whose symmetric extension is \mathcal{A}^e , the child ρ of the pure graph state associated with \mathcal{A}^e is then given by*

$$\rho = \frac{1}{2^n} \sum_{j \in J} b_j \overleftarrow{s}_j, \quad b_j \in \{\pm 1, \pm i\}, \quad \forall j \in J.$$

Furthermore, all \overleftarrow{s}_j , $j \in J$, pairwise commute, and if $\overleftarrow{s}_j, \overleftarrow{s}_{j'}$ are present in the sum, so is $\overleftarrow{s}_j \overleftarrow{s}_{j'} = \overleftarrow{s}_{j+j'}$, therefore, $\{\overleftarrow{s}_j : j \in J\}$ is a commutative subgroup of S^\perp .

Proof. The qubits present in each L_m will be the ones connected with the environmental qubit $n+m$, implying that the difference between the measurement in $x_{n+m} = 0$ and $x_{n+m} = 1$ will be the Boolean function $\sum_{k \in L_m} x_k$. The support of ρ will be equal to the binary vectors such that these Boolean functions are 0 for all m . Certainly, ρ can be expressed as the sum of some of the matrices in S^\perp .

By Lemma 5.3, all the matrices in S^\perp have non-intersecting support: ρ will be equal to the sum of the matrices in S^\perp whose support intersects the support of ρ ; therefore, it is sufficient to find the nonzero entries of the first row of the matrix. It is easy to check that any matrix in S^\perp has only one nonzero entry on the first row. In other words, ρ will be equal to the sum of the matrices whose support intersects $\{(0, k), k \in K\}$, where $K = \left\{ \sum_{m=0}^{n-1} 2^m b_m, \forall b = b_0 b_1 \dots b_{n-1} \in J \right\}$,

and J is the indicator of the Boolean function $\prod_{m=0}^{e-1} \left(\sum_{k \in L_m} x_k + 1 \right)$. Surely, by

Lemma 5.1, these matrices are equal to \overleftarrow{s}_j , where $j \in J$. To complete the proof of our theorem, it remains to show that all \overleftarrow{s}_j , $j \in J$, pairwise commute, and if $\overleftarrow{s}_j, \overleftarrow{s}_k$ are present in the sum, so is $\overleftarrow{s}_j \overleftarrow{s}_k$ (this last claim follows from the fact that the indicator J is a linear space, since $\overleftarrow{s}_j \overleftarrow{s}_k$ corresponds to the sum of the Boolean vectors). Now we shall show the commutative property. Let $j, k \in J$.

We can write $\overleftarrow{s}_j = \prod_{a \in A} \mathcal{A}_a^T$, and $\overleftarrow{s}_k = \prod_{b \in B} \mathcal{A}_b^T$, where $\mathcal{A}_a^T, \mathcal{A}_b^T$ are in the basis of S^\perp . The corresponding $\mathcal{A}_a \in S, a \in A$, have an even number of Y and Z in each extension column, and similarly for B . Therefore, both $\overleftarrow{s}_j, \overleftarrow{s}_k$ have either X, I in all extension columns, and therefore they commute. \square

Let $j^\alpha \in \mathbb{F}_2^n$ and $j^{\alpha, \beta} \in \mathbb{F}_2^n$, $0 \leq \alpha, \beta < n$ be the weight-one and weight-two binary vectors of support α , respectively, $\{\alpha, \beta\}$. The following, also important, result discusses the sign of b_j .

Theorem 5.5. *Given a directed graph associated with \mathcal{A} , whose symmetric extension is \mathcal{A}^e , the coefficients b_j from Theorem 5.4 of the child ρ of the pure graph state associated with \mathcal{A}^e , can be described as follows:*

(A) *Case $e = 1$:*

- (1) *If $\overleftarrow{s}_{j\alpha}$ is present in the sum, then $b_{j\alpha} = +1$.*
- (2) *If $\overleftarrow{s}_{j\alpha,\beta}$ is present in the sum, and:*
 - (i) *if $\overleftarrow{s}_{j\alpha}$ and $\overleftarrow{s}_{j\beta}$ anti-commute, then $\exists a, b \in \{\alpha, \beta\}$ such that $(\mathcal{A}^e)_{a,n} = Z$ and $(\mathcal{A}^e)_{b,n} = Y$; furthermore, the corresponding term in the sum will be equal to $\pm i \mathcal{A}_{\{\alpha,\beta\}}^T = i \mathcal{A}_a^T \mathcal{A}_b^T$, so $b_{j\alpha,\beta} = \pm i$;*
 - (ii) *if \mathcal{A}_α^T and \mathcal{A}_β^T commute, the corresponding term in the sum will be equal to $\mathcal{A}_{\{\alpha,\beta\}}^T$, and $b_{j\alpha,\beta} = 1$.*
- (3) *Any other matrices, \mathcal{A}_K^T , will be products of $\mathcal{A}_{K_j}^T$ also present in the sum, with K_j of size 1 and size 2. Furthermore, their coefficients are given by the multiplication of the coefficients of the terms into which they are decomposed.*

(B) *Case $e > 1$: The coefficient of \mathcal{A}_K^T , with K of size 1 or 2 is given by multiplying together the coefficients, one from each extension column. Here, there might be nondecomposable terms, however, the coefficient of a term will be giving by multiplying the coefficients obtained by taking each column separately.*

Furthermore, for any e , if we change the sign of a term \mathcal{A}_k^T , $0 \leq k < n$, then we add the Boolean linear term x_k to the quadratic Boolean representation of the pure parent graph state, or if we change the sign of \mathcal{A}_K^T , for a set K of size > 1 present in the sum, we add any of the Boolean linear terms x_k for each k such that $(\mathcal{A}^e)_{n+j,k} = Y$ or $(\mathcal{A}^e)_{n+j,k} = Z$.

Proof. Since all parent Boolean functions have no constant terms and no linear terms involving the environmental qubits, the first entry in a truth table for each measurement $|\psi_m\rangle$ will always be $+1$. The first row of the density matrix for each measurement $|\psi_m\rangle$ will therefore be equal to $\frac{1}{2^{n/2}} \langle \psi_m |$. Also, note that the final ρ will be nonzero only where the entries for all $|\psi_m\rangle$ are equal, and will then be equal to any of them; it is therefore enough to look at $\langle \psi_{0\dots 0} |$.

Case $e = 1$:

- *Size 1: By the proof of Theorem 5.4, any term, $\pm \frac{1}{2^n} \mathcal{A}_j^T$, $0 \leq j \leq n-1$ is such that $(\mathcal{A}^e)_{j,n} = X$ or I . Since row j of the stabilizer of the parent graph state, \mathcal{A}^e , is equal to $\mathcal{A} \otimes (\text{extension entries})$, there is no linear term x_j in the parent graph, so that the entry $0 \dots \overset{u\downarrow}{1} \dots 0$ of $|\psi_{0\dots 0}\rangle$ (and therefore of $\langle \psi_{0\dots 0} |$) of the parent is $+\frac{1}{2^{n/2}}$. As the first row of \mathcal{A}_j^T has $+\frac{1}{2^{n/2}}$ in the same position, the coefficient of \overleftarrow{s}_j is $+\frac{1}{2^n}$.*
- *Size 2: Suppose that the matrix $\mathcal{A}_{\{j,k\}}^T$, $0 \leq j, k \leq n-1$, is present in the sum. If \mathcal{A}_j^T and \mathcal{A}_k^T anti-commute, the term in the sum will be $\pm \frac{i}{2^n} \mathcal{A}_{\{j,k\}}^T$, $0 \leq j, k \leq n-1$, since otherwise $\pm \frac{i}{2^n} \mathcal{A}_j^T \mathcal{A}_k^T$, $0 \leq j, k \leq n-1$, would not be Hermitian. Furthermore, neither \mathcal{A}_j^T or \mathcal{A}_k^T are present in the sum, because by Theorem 5.4 this would imply that both would be present and would therefore be commuting. This implies that $\exists a, b \in \{j, k\}$ such that $(\mathcal{A}^e)_{a,n} = Y$ and $(\mathcal{A}^e)_{b,n} = Z$. The first row of $\pm \frac{i}{2^n} \mathcal{A}_j^T \mathcal{A}_k^T$ has its only nonzero entry where $x_j = x_k = 1$, $x_u = 0 \forall u \neq j, k$. Therefore, the entry in $|\psi_0\rangle$ will be given by the*

presence or absence of $x_j x_k, x_j, x_k$, in the ANF of the parent graph. Since \mathcal{A}_j^T and \mathcal{A}_k^T anti-commute, the restriction of $\mathcal{A}_j, \mathcal{A}_k$ (that also anti-commute) to the pair j, k will give $X_\alpha Z_\beta$ and $I_\alpha X_\beta$ for $\alpha, \beta \in \{j, k\}$. This implies that the restriction of \mathcal{A}_α^T and \mathcal{A}_β^T are, respectively, $X_\alpha I_\beta$ and $Z_\alpha X_\beta$, so the restriction of $i\mathcal{A}_\alpha^T \mathcal{A}_\beta^T$ is $Y_\alpha X_\beta$. Note that we only need to look at the restriction, since any further Kronecker product will be by X, Z or I , and these will not change the first non-zero entry.

- If the extension column gives $(\mathcal{A}^e)_{\alpha,n} = Z$ and $(\mathcal{A}^e)_{\beta,n} = Y$, we get the term $2(x_\alpha x_\beta + x_\beta) + x_\beta$ in the parent graph, so $|\psi_0\rangle$ has entry i for $x_j = x_k = 1, x_u = 0, \forall u \neq j, k$, which implies that $\langle \psi_0 |$ has entry $-i$, the same as the entry in the first row of the matrix given by the order of multiplication $\mathcal{A}_\alpha^T \mathcal{A}_\beta^T$.
- If the extension column gives $(\mathcal{A}^e)_{\alpha,n} = Y$ and $(\mathcal{A}^e)_{\beta,n} = Z$, we get the term $2x_\alpha + x_\alpha$ in the parent graph, so $|\psi_0\rangle$ has entry $-i$ (so $\langle \psi_0 |$ has entry i) for $x_j = x_k = 1, x_u = 0, \forall u \neq j, k$, the same as the entry in the first row of the matrix given by the order of multiplication $\mathcal{A}_\beta^T \mathcal{A}_\alpha^T$.

Suppose \mathcal{A}_j^T and \mathcal{A}_k^T commute. Then, the term $\pm \frac{1}{2^n} \mathcal{A}_j^T \mathcal{A}_k^T, 0 \leq j, k \leq n-1$, is present in the sum, and the matrices are commuting, since $\mathcal{A}_j^T \mathcal{A}_k^T$ is Hermitian. By inspection, on the pair $\mathcal{A}_j, \mathcal{A}_k \in S, X_\alpha Z_\beta$ and $Z_\alpha X_\beta$ give entry -1 , regardless of extension (note that they have the commuting entry in the extension column), and the same is true for $\mathcal{A}_j^T \mathcal{A}_k^T$. Since $X_\alpha I_\beta$ and $I_\alpha X_\beta$ give entry $+1$, regardless of extension, the same is true for $\mathcal{A}_j^T \mathcal{A}_k^T$.

- By the proof of Theorem 5.4, $\mathcal{A}_K^T, K \subseteq \{0, \dots, n-1\}$, is present in the sum if and only if the corresponding stabilizer \mathcal{A}_K^T has either X or I in the extension column. Therefore, there exists a (not necessarily unique) decomposition in size 1 and size 2 terms that have either X or I in the extension column (since $YZ = iX$), and, as the entry $x_j = 1, \forall j \in A, x_j = 0, \forall j \notin A$ will depend on the sum of the terms for the size 1 and size 2 cases, as the Boolean function has degree at most 2, this entry will depend on the entry of the size 1 and size 2 cases, and therefore the coefficient will be the multiplication of the size 1 and size 2 cases (for any given decomposition).

Case $e > 1$. Since the case $e = 1$ was independent of the actual stabilizers (it only depends on the extension), each new column will modify the Boolean expression, accordingly. Therefore, the coefficient of any term is given by the multiplication of the coefficient resultant of each extension column.

Change of sign: Let us first assume that j has weight 1, with support in k . Note that each \mathcal{A}_j^T has support in $(0, 2^k)$. In terms of $\langle \psi_{0\dots 0} |$, this means that changing the sign of \mathcal{A}_j^T changes the sign in the j th element in $|\psi_{0\dots 0}\rangle$ (and, therefore in $\langle \psi_{0\dots 0} |$). This is equivalent to adding a linear term x_k as long as we change the sign all elements in $|\psi_{0\dots 0}\rangle$ (and, therefore in $\langle \psi_{0\dots 0} |$) with $x_k = 1$. Suppose now that j has weight > 1 . If we change the sign in j , we have to change also the signs of an odd number of its decomposing terms. Consider therefore the smallest terms present in the group. If the sign changed is of weight 1, we refer to our previous discussion. If the weight is $t > 1$, with support in k_1, \dots, k_t , then any of the \mathcal{A}_{k_i} are not elements in the subgroup. Adding any linear term x_{k_i} will give the desired matrix, since it is zero in the places were it might differ. \square

Corollary 1. *The commutative subgroup corresponding to a child is maximal.*

Proof. Each column added produces an affine linear Boolean function f_i , which gives a constraint to the indicator J of $f = \prod f_i$. Each constraint f_i is nontrivial (that is, it is not a constant), because if this were the case, then the column would only have X and I , and would therefore be superfluous, yielding a contradiction. The f_i 's are all independent, otherwise we get redundant columns, yielding a contradiction with e being minimal. Furthermore, any new independent linear constraint reduces by half the size of the indicator, which means that this size is equal to 2^{n-e} , so it is a maximal commutative subgroup. \square

Example: Let the directed triangle G_2 Section 2 be defined by the stabilizer basis $\mathcal{A}_0 = X \otimes Z \otimes I$, $\mathcal{A}_1 = I \otimes X \otimes Z$, $\mathcal{A}_2 = Z \otimes I \otimes X$. The basis of S^\perp is given by reversing the arrows: $\mathcal{A}_0^T = X \otimes I \otimes Z$, $\mathcal{A}_1^T = Z \otimes X \otimes I$, $\mathcal{A}_2^T = I \otimes Z \otimes X$, so $S^\perp = \{\overleftarrow{s}_{000} = I \otimes I \otimes I, \overleftarrow{s}_{100} = X \otimes I \otimes Z, \overleftarrow{s}_{010} = Z \otimes X \otimes I, \overleftarrow{s}_{110} = -iY \otimes X \otimes Z, \overleftarrow{s}_{001} = I \otimes Z \otimes X, \overleftarrow{s}_{101} = iX \otimes Z \otimes Y, \overleftarrow{s}_{011} = -iZ \otimes Y \otimes X, \overleftarrow{s}_{111} = -iY \otimes Y \otimes Y\}$. One of the two parents is formed by adding the column $(X, Z, Y)^T$ to \mathcal{A} , rendering $i^{2(x_0x_2+x_1x_2+x_1x_3+x_2x_3+x_2)} = i^{2(x_0x_2+x_1x_2+x_2+(\mathcal{I}(x)+1)x_3)+x_2}$. Here $L_3 = \{1, 2\}$, and $\mathcal{I}(x) = x_1 + x_2 + 1$, so $J = \langle 100, 011 \rangle = \{000, 100, 011, 111\}$.⁷ By tracing over the environmental qubit x_3 we get

$$\begin{aligned} \rho &= \frac{1}{8} (I \otimes I \otimes I + X \otimes I \otimes Z + Z \otimes Y \otimes X + Y \otimes Y \otimes Y) \\ &= \frac{1}{8} (\overleftarrow{s}_{000} + \overleftarrow{s}_{100} + i\overleftarrow{s}_{011} + i\overleftarrow{s}_{111}). \end{aligned}$$

Note that $X \otimes I \otimes Z$ and $Z \otimes Y \otimes X$ commute, and that $(X \otimes I \otimes Z) \cdot (Z \otimes Y \otimes X) = Y \otimes Y \otimes Y$. Another parent is formed by adding the column $(X, Y, Z)^T$ to \mathcal{A} , rendering $i^{2(x_0x_2+x_1+x_1x_3+x_2x_3)+x_1} = i^{2(x_0x_2+x_1+(\mathcal{I}(x)+1)x_3)+x_1}$. Once again $L_3 = \{1, 2\}$, $\mathcal{I}(x) = x_1 + x_2 + 1$, and $J = \{000, 100, 011, 111\}$. By tracing over the environmental qubit x_3 we get

$$\begin{aligned} \rho &= \frac{1}{8} (I \otimes I \otimes I + X \otimes I \otimes Z - Z \otimes Y \otimes X - Y \otimes Y \otimes Y) \\ &= \frac{1}{8} (\overleftarrow{s}_{000} + \overleftarrow{s}_{100} - i\overleftarrow{s}_{011} - i\overleftarrow{s}_{111}) = \rho_0^{G_2}. \end{aligned}$$

H and G are unchanged. Observe that the second parent is obtained from the first by swapping the positions of Y and Z in rows 1 and 2 of the extra column (column 3). In terms of the Boolean function representation of the parents, this translates to adding the Boolean quadratic terms $x_1x_2 + x_1 + x_2$, and removing x_2 and adding x_1 , \mathbb{Z}_4 -linear terms. Observe that the coefficients of the Pauli basis terms $Z \otimes Y \otimes X$ and $Y \otimes Y \otimes Y$ are multiplied by -1 . This is because both terms are generated from rows 1 and 2 of \mathcal{A}^T which are the rows where Y and Z are swapped in \mathcal{A}^e .

For each parent we can also consider how the addition of binary linear terms (in the lab) affects the resultant child density matrix. We do not currently consider the addition of binary linear terms in the environment (x_3 for this example). For instance, for the addition of column $(X, Z, Y)^T$ then $i^{2(x_0x_2+x_1x_2+x_1x_3+x_2x_3+x_0+x_2)+x_2}$, that is, the addition of x_0 , simply flips the signs of $X \otimes I \otimes Z$ and $Y \otimes Y \otimes Y$ as both terms have row 0 of \mathcal{A}^T as a factor. However, the addition of x_1 or x_2 has the same effect as swapping Y and Z in rows 1 and 2, so swaps between the two

⁷We can reinterpret $\mathcal{I}(x)$ and J as parity and generator matrices, H and G , respectively, where $H = (011)$ and $G = \begin{pmatrix} 100 \\ 011 \end{pmatrix}$, and where G generates the binary linear code with codewords in set J .

parents. Thus, the addition of $\bar{\mathcal{I}}(x) = \mathcal{I}(x) + 1$ fixes the child density matrix. So, we have the following maps for $a \in \{0, 1\}$ (in the left column, we insert only the signs $(\epsilon_1, \epsilon_2, \epsilon_3)$ from $I \otimes I \otimes I + \epsilon_1 X \otimes I \otimes Z + \epsilon_2 Z \otimes Y \otimes X + \epsilon_3 Y \otimes Y \otimes Y$):

child = 8ρ	parent
(+1, +1, +1)	$2(x_0x_2 + x_1x_2 + x_2 + \bar{\mathcal{I}}(x)(x_3 + a)) + x_2,$ $2(x_0x_2 + x_1 + x_2 + \bar{\mathcal{I}}(x)(x_3 + a)) + x_1,$
(-1, +1, -1)	$2(x_0x_2 + x_1x_2 + x_0 + x_2 + \bar{\mathcal{I}}(x)(x_3 + a)) + x_2,$ $2(x_0x_2 + x_0 + x_1 + x_2 + \bar{\mathcal{I}}(x)(x_3 + a)) + x_1,$
(+1, -1, -1)	$2(x_0x_2 + x_1 + \bar{\mathcal{I}}(x)(x_3 + a)) + x_1,$ $2(x_0x_2 + x_1x_2 + x_1 + x_2 + \bar{\mathcal{I}}(x)(x_3 + a)) + x_2,$
(-1, -1, +1)	$2(x_0x_2 + x_0 + x_1 + \bar{\mathcal{I}}(x)(x_3 + a)) + x_1,$ $2(x_0x_2 + x_1x_2 + x_0 + x_1 + x_2 + \bar{\mathcal{I}}(x)(x_3 + a)) + x_2.$

Other children given by the extensions $(Z, X, Y)^T$ and $(Z, Y, X)^T$ are, respectively,

$$\rho_1 = \frac{1}{8}(I \otimes I \otimes I + aZ \otimes X \otimes I + bX \otimes Z \otimes Y + cY \otimes Y \otimes Y) = \frac{1}{8}(\bar{s}_{000} + a\bar{s}_{010} - b\bar{i}\bar{s}_{101} + c\bar{i}\bar{s}_{111}),$$

$$\rho_2 = \frac{1}{8}(I \otimes I \otimes I + aY \otimes X \otimes Z + bI \otimes Z \otimes X + cY \otimes Y \otimes Y) = \frac{1}{8}(\bar{s}_{000} + a\bar{i}\bar{s}_{110} + b\bar{s}_{001} + c\bar{i}\bar{s}_{111}),$$

with condition $c = ab$, $a, b \in \{1, -1\}$ in both cases.

Example: Let a mixed 6-clique graph be defined by the stabilizer basis

$$\mathcal{A} = \begin{pmatrix} X & Z & Z & Z & Z & Z \\ I & X & Z & Z & Z & Z \\ I & I & X & Z & Z & Z \\ I & I & I & X & Z & Z \\ I & I & I & I & X & Z \\ I & I & I & I & I & X \end{pmatrix}.$$

Then the basis of S^\perp is obtained from \mathcal{A}^T . The parents are obtained by adding $e = 3$ columns to \mathcal{A} , with subsequent addition of $e = 3$ rows. For instance, one parent is given by

$$\mathcal{A}^e = \begin{pmatrix} X & Z & Z & Z & Z & Z & X & I & X \\ I & Y & Z & Z & Z & Z & Y & I & X \\ I & I & X & Z & Z & Z & Z & I & X \\ I & I & I & X & Z & Z & I & X & Y \\ I & I & I & I & X & Z & I & Y & Y \\ I & I & I & I & I & X & I & Z & Y \\ I & Z & Z & I & I & I & X & I & I \\ I & I & I & I & Z & Z & I & X & I \\ I & I & I & Z & Z & Z & I & I & X \end{pmatrix}$$

$$\equiv \begin{pmatrix} X & I & I & I & I & I & I & I & I \\ I & Y & I & I & I & I & Z & I & I \\ I & I & X & I & I & I & Z & I & I \\ I & I & I & Y & Z & Z & I & I & Z \\ I & I & I & Z & X & Z & I & Z & Z \\ I & I & I & Z & Z & Y & I & Z & Z \\ I & Z & Z & I & I & I & X & I & I \\ I & I & I & I & Z & Z & I & X & I \\ I & I & I & Z & Z & Z & I & I & X \end{pmatrix},$$

which represents

$$i^2(x_1x_6 + x_2x_6 + x_3x_4 + x_3x_5 + x_3x_8 + x_4x_5 + x_4x_7 + x_4x_8 + x_5x_7 + x_5x_8 + x_1 + x_3 + x_4 + x_5) + x_1 + x_3 + x_5.$$

Here $L_3 = \{1, 2\}$, $L_4 = \{4, 5\}$, $L_5 = \{3, 4, 5\}$, and $\mathcal{I}(x) = (x_1 + x_2 + 1)(x_4 + x_5 + 1)(x_3 + x_4 + x_5 + 1)$, so

$$J = \langle 100000, 011000, 000011 \rangle$$

$$= \{000000, 100000, 011000, 111000, 000011, 100011, 011011, 111011\}.$$

We can re-interpret $\mathcal{I}(x)$ and J as parity and generator matrices, H and G , respectively, where $H = \begin{pmatrix} 011000 \\ 000011 \\ 000111 \end{pmatrix}$ and $G = \begin{pmatrix} 100000 \\ 011000 \\ 000011 \end{pmatrix}$, and where G generates the binary linear code with codewords in set J . By tracing over the environmental qubits, x_6, x_7, x_8 , we get

$$\begin{aligned} 8\rho = & I \otimes I \otimes I \otimes I \otimes I \otimes I + X \otimes Z \otimes Z \otimes Z \otimes Z \otimes Z - I \otimes X \otimes Y \otimes I \otimes I \otimes I \\ & - X \otimes Y \otimes X \otimes Z \otimes Z \otimes Z - I \otimes I \otimes I \otimes I \otimes X \otimes Y - X \otimes Z \otimes Z \otimes Z \otimes Y \otimes X \\ & + I \otimes X \otimes Y \otimes I \otimes X \otimes Y + X \otimes Y \otimes X \otimes Z \otimes Y \otimes X. \end{aligned}$$

Remark 1. Note that allowing superfluous constraints f_i will give commutative subgroups that are in general not maximal. In this way, we could also extend graph states in a natural way: the density matrix for the graph state is given by the sum of all the elements of the stabilizer (since it is self-dual), and of course any consistent sign changes that give linear terms (eigenvalue -1). We can however define more density matrices that are stabilized by the stabilizer of the graph state, by allowing also smaller commutative subgroups. For instance, a density matrix associated to the undirected line from 0 to 1 could be then $\rho = a_0(I \otimes I \pm X \otimes Z \pm Z \otimes X \pm Y \otimes Y) + a_1(I \otimes I \pm X \otimes Z) + a_2(I \otimes I \pm Z \otimes X) + a_3(I \otimes I \pm Y \otimes Y) + a_4 I \otimes I$, where $\sum a_i = 1$. Note that in the first term, the signs have to be consistent, changing for instance $X \otimes Z$ to $-X \otimes Z$ forces the change $Y \otimes Y$ to $-Y \otimes Y$.

6. FINDING COMMUTATIVE SUBGROUPS OF S^\perp DIRECTLY FROM THE GRAPH

Given a mixed graph G and one of its parents, we know how to define the density matrix of its child in terms of the Pauli group, by Theorem 5.4, and that the terms of the sum form a commutative subgroup of S^\perp ; we will show in this section that we can find all commutative subgroups directly from the graph G , or alternatively from the adjacency matrix of G_b (the simple undirected graph with adjacency matrix $\Gamma = \mathcal{A} + \mathcal{A}^T$). We start with a lemma.

Lemma 6.1. *Let s_j be row j in S , and let \overleftarrow{s}_j be the corresponding row in S^\perp . Then, \overleftarrow{s}_j commutes with $\prod_{k \in K} \overleftarrow{s}_k$ if and only if $|K \cap \mathcal{N}_j| \equiv 0 \pmod{2}$. Furthermore, $\prod_{j \in A} \overleftarrow{s}_j$ commutes with $\prod_{k \in K} \overleftarrow{s}_k$ if and only if the sum over A of the number of elements of $K \cap \mathcal{N}_j$ is even.*

Proof. First, we shall prove that \overleftarrow{s}_j commutes with $\prod_{k \in K} \overleftarrow{s}_k$ if and only if the number of elements of $K \cap \mathcal{N}_j$ is even. Observe that \overleftarrow{s}_j will anti-commute with any \overleftarrow{s}_k such that $k \in K \cap \mathcal{N}_j$, that is, any k in the neighborhood. If there are an even number of these, the (possible) minus sign will cancel each other out, so that \overleftarrow{s}_j commutes with $\prod_{k \in K} \overleftarrow{s}_k$. If this number is odd, then there will be a minus sign left, and \overleftarrow{s}_j will anti-commute with $\prod_{k \in K} \overleftarrow{s}_k$.

Now, we shall prove that $\prod_{j \in A} \overleftarrow{s}_j$ commutes with $\prod_{k \in K} \overleftarrow{s}_k$ if and only if the sum over A of the number of elements of $K \cap \mathcal{N}_j$ is even. By the first part of the proof, we know that each of the \overleftarrow{s}_j will commute with $\prod_{k \in K} \overleftarrow{s}_k$ if and only if the number of elements of $K \cap \mathcal{N}_j$ is even. If this number is odd, \overleftarrow{s}_j will anti-commute with $\prod_{k \in K} \overleftarrow{s}_k$. However, if the sum of the number of elements for each $j \in A$ is even, the (possible) minus signs will cancel each other out, so that $\prod_{j \in A} \overleftarrow{s}_j$

commutes with $\prod_{k \in K} \overleftarrow{s}_k$. If this number is odd, then there will be a minus sign left, and $\prod_{j \in A} \overleftarrow{s}_j$ will anti-commute with $\prod_{k \in K} \overleftarrow{s}_k$. \square

Example: Let G_b be the simple graph 01,04,12,23,34,14. Then, \overleftarrow{s}_0 commutes with \overleftarrow{s}_3 (since they are independent, so the intersection is empty), and both commute with $\overleftarrow{s}_4 \overleftarrow{s}_2 \overleftarrow{s}_1$, since 0 has 1 and 4 as neighbors, and 3 has 2 and 4 as neighbors, so both have two elements in the intersection with $K = \{1, 2, 4\}$. Note, however, that $\overleftarrow{s}_0(\overleftarrow{s}_4 \overleftarrow{s}_2 \overleftarrow{s}_1) = -\overleftarrow{s}_4 \overleftarrow{s}_0 \overleftarrow{s}_2 \overleftarrow{s}_1 = -\overleftarrow{s}_4 \overleftarrow{s}_2 \overleftarrow{s}_0 \overleftarrow{s}_1 = +\overleftarrow{s}_4 \overleftarrow{s}_2 \overleftarrow{s}_1 \overleftarrow{s}_0$. We also have that $\overleftarrow{s}_3 \overleftarrow{s}_2$ commutes with $\overleftarrow{s}_4 \overleftarrow{s}_1 \overleftarrow{s}_0$, since 3 has 1 neighbor in $K = \{0, 1, 4\}$, and 2 has 1 neighbor in K , so the sum over $A = \{2, 3\}$ is 2, which is even.

In general, in terms of the adjacency matrix of G_b , we have the following proposition.

Proposition 6.2. *Let s_j be row j in S , and let \overleftarrow{s}_j be the corresponding row in S^\perp . Then, \overleftarrow{s}_j commutes with $\prod_{k \in K} \overleftarrow{s}_k$ if and only if $\bigoplus_K r_{k,j} = 0$, where \oplus is the binary sum. Furthermore, $\prod_{j \in A} \overleftarrow{s}_j$ commutes with $\prod_{k \in K} \overleftarrow{s}_k$ if and only if $\bigoplus_{j \in A, k \in K} r_{k,j} = 0$. (Note that due to the symmetry of the adjacency matrix, we get $\bigoplus_{j \in A} \bigoplus_{k \in K} r_{k,j} = \bigoplus_{k \in K} \bigoplus_{j \in A} r_{j,k}$, which ensures that commutativity is a symmetric relationship.)*

Proof. First, we are going to prove that \overleftarrow{s}_j commutes with $\prod_{k \in K} \overleftarrow{s}_k$ if and only if $\bigoplus_K r_{k,j} = 0$. By Lemma 6.1, \overleftarrow{s}_j commutes with $\prod_{k \in K} \overleftarrow{s}_k \Leftrightarrow$ the number of elements of $K \cap \mathcal{N}_j$ is even. But this is equivalent with there being an even number of rows $k \in K$ such that $c_{k,j} = 1$, which is equivalent with $\bigoplus_K r_{k,j} = 0$.

Now, we shall prove that $\prod_{j \in A} \overleftarrow{s}_j$ commutes with $\prod_{k \in K} \overleftarrow{s}_k$ if and only if $\bigoplus_{j \in A, k \in K} r_{k,j} = 0$. By Lemma 6.1, $\prod_{j \in A} \overleftarrow{s}_j$ commutes with $\prod_{k \in K} \overleftarrow{s}_k$ if and only if the sum over A of the number of elements of $K \cap \mathcal{N}_j$ is even. But this is equivalent with there being an even number of rows $k \in K$ such that the amount of $c_{k,j} = 1$ is even, which is equivalent with $\bigoplus_{j \in A, k \in K} r_{k,j} = 0$. \square

Example: In the previous example, $\Gamma_{G_b} = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}$. We can see that

\overleftarrow{s}_0 commutes with \overleftarrow{s}_3 , since $r_{3,0} = 0$, and both commute with $\overleftarrow{s}_4 \overleftarrow{s}_2 \overleftarrow{s}_1$, since $\bigoplus_{\{1,2,4\}} c_{k,0} = 1 \oplus 0 \oplus 1 = 0$ and $\bigoplus_{\{1,2,4\}} c_{k,3} = 0 \oplus 1 \oplus 1 = 0$. We also observe that $\overleftarrow{s}_3 \overleftarrow{s}_2$ commutes with $\overleftarrow{s}_4 \overleftarrow{s}_1 \overleftarrow{s}_0$, since $\bigoplus_{j \in \{2,3\}} \bigoplus_{k \in \{0,1,4\}} r_{k,j} = (0 \oplus 1 \oplus 0) \oplus (0 \oplus 0 \oplus 1) = 0$.

Remark 2. Note that if $\prod_{j \in A} c_j$ commutes with $\prod_{k \in K} c_k$ and $\prod_{j \in A'} c_j$ commutes with $\prod_{k \in K} c_k$, and $A \cap A' = \emptyset$, then $\prod_{j \in A, A'} c_j$ commutes with $\prod_{k \in K} c_k$. Similarly, if $\prod_{j \in A} c_j$ anti-commutes with $\prod_{k \in K} c_k$ and $\prod_{j \in A'} c_j$ anti-commutes with $\prod_{k \in K} c_k$, and $A \cap A' = \emptyset$, then $\prod_{j \in A, A'} c_j$ commutes with $\prod_{k \in K} c_k$. If one of them commutes and the other anti-commutes, we get anti-commutativity. This implies that subgroups can be combined to generate other subgroups in an easy way, so that we only have to look at the commutativity of each of the c_j . Note also that by multiplying two elements that commute with another one, we get a smaller subgroup than the one that has those elements and the one they commute with, so these should be discarded when searching for maximal subgroups.

Example: Let G be the graph defined by the stabilizer basis $\{X \otimes Z \otimes Z \otimes Z, I \otimes X \otimes I \otimes I, I \otimes Z \otimes X \otimes Z, I \otimes I \otimes Z \otimes X\}$. We want to find all maximal commutative subgroups. The adjacency matrix of G_b is $\Gamma_{G_b} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$. The binary linear code generated by this matrix is $C = \{0000, 0111, 1010, 1100, 1000, 1101, 1011, 1111, 0110, 0010, 0100, 0001, 0011, 0101, 1110, 1001\} = \mathbb{F}_2^4$ (this is obvious since Γ_{G_b} has full rank).

The size of a maximal commutative subgroup will be $2^{n-e} = 2^2 = 4$. Now, \overleftarrow{s}_j commutes with $\prod_{k \in K} \overleftarrow{s}_k$ if and only if the corresponding element in the code, $\bigoplus_{k \in K} r_k$, where r_k denotes row k in Γ_{G_b} , has 0 in position j .

- Thus, \overleftarrow{s}_0 commutes (trivially) with the identity and itself, and with $\overleftarrow{s}_1 \overleftarrow{s}_2$, $\overleftarrow{s}_1 \overleftarrow{s}_3$, $\overleftarrow{s}_2 \overleftarrow{s}_3$ of size 2, and with $\overleftarrow{s}_0 \overleftarrow{s}_1 \overleftarrow{s}_2$, $\overleftarrow{s}_0 \overleftarrow{s}_2 \overleftarrow{s}_3$, $\overleftarrow{s}_0 \overleftarrow{s}_1 \overleftarrow{s}_3$ of size 3. We can take a basis for each subgroup, for instance we consider the set $\{\overleftarrow{s}_1 \overleftarrow{s}_2, \overleftarrow{s}_1 \overleftarrow{s}_3, \overleftarrow{s}_2 \overleftarrow{s}_3\}$. Each of these size 2 elements, together with \overleftarrow{s}_0 , will generate a maximal commutative subgroup.
- \overleftarrow{s}_1 commutes with \overleftarrow{s}_3 , $\overleftarrow{s}_0 \overleftarrow{s}_2$, $\overleftarrow{s}_0 \overleftarrow{s}_2 \overleftarrow{s}_3$.
- \overleftarrow{s}_2 commutes with \overleftarrow{s}_3 , $\overleftarrow{s}_0 \overleftarrow{s}_1$, $\overleftarrow{s}_0 \overleftarrow{s}_1 \overleftarrow{s}_3$.
- \overleftarrow{s}_3 commutes with \overleftarrow{s}_1 , \overleftarrow{s}_2 , $\overleftarrow{s}_1 \overleftarrow{s}_2$.

Note that the subgroups generated by \overleftarrow{s}_1 , \overleftarrow{s}_3 , and \overleftarrow{s}_2 , \overleftarrow{s}_3 are counted twice here. One should avoid repetitions by inspecting in lexicographic order. Note also that if \overleftarrow{s}_k and \overleftarrow{s}_v commute with \overleftarrow{s}_j , then $\overleftarrow{s}_k \overleftarrow{s}_v$ commute with \overleftarrow{s}_j . These are all containing at least one element of size 1. Containing at least one element of size 2 (discounting those already found):

- $\overleftarrow{s}_0 \overleftarrow{s}_1$ commutes with $\overleftarrow{s}_0 \overleftarrow{s}_3$, $\overleftarrow{s}_0 \overleftarrow{s}_2 \overleftarrow{s}_3$.
- $\overleftarrow{s}_0 \overleftarrow{s}_2$ commutes with $\overleftarrow{s}_0 \overleftarrow{s}_3$, $\overleftarrow{s}_0 \overleftarrow{s}_1 \overleftarrow{s}_3$.
- $\overleftarrow{s}_0 \overleftarrow{s}_3$ commutes with $\overleftarrow{s}_1 \overleftarrow{s}_2$.

There are no more independent maximal commutative subgroups. Note that each element is in 3 maximal commutative subgroups, and that there are in total 15 maximal commutative subgroups.

In general, the subgroup structure is only dependent on G_b , which is the same regardless of arrow direction, and so we have the following proposition.

Proposition 6.3. *The subgroup structure is independent of the direction of the arrows, that is, graphs with the same G_b will have the same maximal commutative subgroup structure.*

Proposition 6.4. *Suppose G_b has no isolated nodes. Then, the group C generated by the rows of Γ_{G_b} (counting repeated elements) is isomorphic to $n - 2e + 1$ copies of \mathbb{F}_2^{2e} . Note that \overleftarrow{s}_j will commute with any copy of itself in C , so that any copy will be added to all maximal commutative subgroups. Therefore, a copy of \mathbb{F}_2^{2e} means that the basis cardinality will be larger by $n - 2e$.*

Proof. By definition, $\text{rank}(\Gamma_{G_b}) = 2e$. Therefore, there are $2e$ linearly independent elements amongst the rows of the matrix. Let H be a group generated by these $2e$ elements. Since all other rows are linear combinations of these, each of these will yield a copy of H .

It remains now to prove that $H \simeq \mathbb{F}_2^{2e}$. We define $f : H \rightarrow \mathbb{F}_2^{2e}$ by assigning to each element of the basis an element of the standard basis of \mathbb{F}_2^{2e} . The group operations are therefore maintained, so that $f(c_i + c_j) = e_i + e_j = f(c_i) + f(c_j)$. Note

that all elements in both H and \mathbb{F}_2^{2e} are of order 2, so that the order is preserved by f . \square

The first corollary is immediate.

Corollary 2. *For counting arguments, it is enough to show a statement for the fully arrowed $2t$ -clique, which always has $e = t = \frac{n}{2}$, and multiply the number by the number of copies, except for the elements giving $00\dots 0$ in C (which are in any commutative subgroup).*

For instance, the size of subgroups in a fully arrowed clique is 2^e . The size is doubled for every copy of \mathbb{F}_2^{2e} ; that is, we multiply by 2^{n-2e} , and obtain $2^e \cdot 2^{n-2e} = 2^{n-e}$, which is the size of maximal commutative subgroups.

Corollary 3. *The number of maximal commutative subgroups is equal to*

$$\frac{(2^n - 1)a}{2^{n-e} - 1},$$

where $a = \frac{1}{(e-1)!} \prod_{m=1}^{e-1} \binom{n-(2m-1)}{2}$, and consequently, depends solely on n and e .

Proof. By the previous proposition, we consider the clique of even size $n = 2t$. Here, $e = t = \frac{n}{2}$, and $C \simeq \mathbb{F}_2^n$. Any \overleftarrow{s}_j will commute with all even products of \overleftarrow{s}_k , $k \neq j$. These products will commute with each other if the corresponding edges have an even number of nodes in common. Therefore, we can get a base consisting of \overleftarrow{s}_j and size 2 products, $\overleftarrow{s}_k \overleftarrow{s}_{k'}$, for each subgroup. We get thus for \overleftarrow{s}_j the choice of 2 elements out of $n - 1$, then 2 elements out of $n - 3$, and successively up to 2 out of 3. However, we have to divide by 2 for every possible new pair, since for instance $\overleftarrow{s}_0, \overleftarrow{s}_1 \overleftarrow{s}_2, \overleftarrow{s}_3 \overleftarrow{s}_4$ will give the same subgroup as $\overleftarrow{s}_0, \overleftarrow{s}_3 \overleftarrow{s}_4, \overleftarrow{s}_1 \overleftarrow{s}_2$. Therefore, there are $a = \frac{1}{(e-1)!} \prod_{m=1}^{e-1} \binom{n-(2m-1)}{2}$ maximal commutative subgroups that contain \overleftarrow{s}_j .

To find the total number of maximal commutative subgroups, we note that all elements but the identity anti-commute with some other element: there are therefore $2^n - 1$ such elements. Then, $(2^n - 1)a = (2^{n-e} - 1) \cdot x$, where x is the number of maximal commutative subgroups. Thus, $x = \frac{(2^n - 1)a}{2^{n-e} - 1}$. \square

7. CONCLUSION

In this paper, we have extended the concept of graph states, from pure graph states associated to a simple undirected graph, to mixed quantum states associated with mixed graph, that is graphs where some or all the edges are directed. The concept of extended graph states includes graph states, so it is a proper generalization. We also express the density matrices associated to a mixed graph as a linear combination of some Clifford group operators which form a maximal commutative subgroup of the stabilizers of the companion graph (the corresponding mixed graph with the edges' directions reversed), and discuss how to find these groups from the graph.

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