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Discrete Applied Mathematics

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# Generalized bent Boolean functions and strongly regular Cayley graphs

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## ARTICLE INFO

### Article history:

Received 10 July 2018

Received in revised form 1 November 2019

Accepted 24 January 2020

Available online xxxx

### Keywords:

(Generalized) Boolean functions

Bent

Generalized bent

Edge-weighted graphs

Strongly regular graphs

## ABSTRACT

In this paper we define the (edge-weighted) Cayley graph associated to a generalized Boolean function, introduce a notion of strong regularity and give several of its properties. We show some connections between this concept and generalized bent functions (gbent), that is, functions with flat generalized Walsh–Hadamard spectrum. In particular, we find a complete characterization of quartic gbent functions in terms of the strong regularity of their associated Cayley graph.

Published by Elsevier B.V.

## 1. Introduction

Let  $\mathbb{V}_n$  be the vector space of dimension  $n$  over the two element field  $\mathbb{F}_2$ , and, for a positive integer  $q$ , let  $\mathbb{Z}_q$  be the ring of integers modulo  $q$ . Let us denote the addition, respectively, product operators over  $\mathbb{F}_2$  by “ $\oplus$ ”, respectively, “ $\cdot$ ”. A Boolean function  $f$  on  $n$  variables is a mapping from  $\mathbb{V}_n$  into  $\mathbb{F}_2$ , that is, a multivariate polynomial over  $\mathbb{F}_2$ ,

$$f(x_1, \dots, x_n) = a_0 \oplus \bigoplus_{i=1}^n a_i x_i \oplus \bigoplus_{1 \leq i < j \leq n} a_{ij} x_i x_j \oplus \dots \oplus a_{12\dots n} x_1 x_2 \dots x_n, \quad (1)$$

where the coefficients  $a_0, a_i, a_{ij}, \dots, a_{12\dots n} \in \mathbb{F}_2$ . This representation of  $f$  is called the *algebraic normal form* (ANF) of  $f$ . The number of variables in the highest order product term with nonzero coefficient is called the *algebraic degree*, or simply the degree of  $f$ .

For a Boolean function on  $\mathbb{V}_n$ , the *Hamming weight* of  $f$ ,  $wt(f)$ , is the cardinality of  $\Omega_f = \{\mathbf{x} \in \mathbb{V}_n : f(\mathbf{x}) = 1\}$  (this is extended to any vector, by taking its weight to be the number of nonzero components of that vector). The *Hamming distance* between two functions  $f, g : \mathbb{V}_n \rightarrow \mathbb{F}_2$  is  $d(f, g) = wt(f \oplus g)$ . A Boolean function  $f(\mathbf{x})$  is called an *affine function* if its algebraic degree is 1. If, in addition,  $a_0 = 0$  in (1), then  $f(\mathbf{x})$  is a *linear function*. In  $\mathbb{V}_n = \mathbb{F}_2^n$ , the vector space of the  $n$ -tuples over  $\mathbb{F}_2$ , we use the conventional dot product  $\mathbf{u} \cdot \mathbf{x} = u_1 x_1 \oplus \dots \oplus u_n x_n$  as an inner product, where  $\mathbf{u} = (u_1, \dots, u_n)$ ,  $\mathbf{x} = (x_1, \dots, x_n)$ .

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There has been quite a bit of research in the area of generalized Boolean functions, that is, functions defined on  $\mathbb{V}_n$  with values in the ring of integers modulo some integer  $q$ , and we mention here just a handful of references [14–17,20–23]. We introduce below some of the concepts from this area, which we will deal with in this paper.

For a generalized Boolean function  $f : \mathbb{V}_n \rightarrow \mathbb{Z}_q$  we define the *generalized Walsh–Hadamard transform* to be the complex valued function

$$\mathcal{H}_f^{(q)}(\mathbf{u}) = \sum_{\mathbf{x} \in \mathbb{V}_n} \zeta_q^{f(\mathbf{x})} (-1)^{\mathbf{u} \cdot \mathbf{x}},$$

where  $\zeta_q = e^{\frac{2\pi i}{q}}$  (we often use  $\zeta$ ,  $\mathcal{H}_f$ , instead of  $\zeta_q$ , respectively,  $\mathcal{H}_f^{(q)}$ , when  $q$  is fixed). The inverse is given by  $\zeta_q^{f(\mathbf{x})} = 2^{-n} \sum_{\mathbf{u}} \mathcal{H}_f(\mathbf{u}) (-1)^{\mathbf{u} \cdot \mathbf{x}}$ . For  $q = 2$ , we obtain the usual *Walsh–Hadamard transform*

$$\mathcal{W}_f(\mathbf{u}) = \sum_{\mathbf{x} \in \mathbb{V}_n} (-1)^{f(\mathbf{x})} (-1)^{\mathbf{u} \cdot \mathbf{x}},$$

which defines the coefficients of character form of  $f$  with respect to the orthonormal basis of the group characters  $\chi_{\mathbf{w}}(\mathbf{x}) = (-1)^{\mathbf{w} \cdot \mathbf{x}}$ . In turn,  $f(\mathbf{x}) = 2^{-n} \sum_{\mathbf{w}} \mathcal{W}_f(\mathbf{w}) (-1)^{\mathbf{u} \cdot \mathbf{x}}$ .

We denote the set of all generalized Boolean functions by  $\mathcal{GB}_n^q$  and when  $q = 2$ , by  $\mathcal{B}_n$ . A function  $f : \mathbb{V}_n \rightarrow \mathbb{Z}_q$  is called *generalized bent (gbent)* if  $|\mathcal{H}_f(\mathbf{u})| = 2^{n/2}$  for all  $\mathbf{u} \in \mathbb{V}_n$ . We recall that a function  $f$  for which  $|\mathcal{W}_f(\mathbf{u})| = 2^{n/2}$  for all  $\mathbf{u} \in \mathbb{V}_n$  is called a *bent function*, which only exists for even  $n$  since  $\mathcal{W}_f(\mathbf{u})$  is an integer. Let  $f \in \mathcal{GB}_n^q$ , where  $2^{k-1} < q \leq 2^k$ , then, based on the binary representation of the elements in the image set  $\mathbb{Z}_{2^k}$ , we can represent  $f$  uniquely as

$$f(\mathbf{x}) = a_0(\mathbf{x}) + 2a_1(\mathbf{x}) + \dots + 2^{k-1}a_{k-1}(\mathbf{x})$$

for some Boolean functions  $a_i$ ,  $0 \leq i \leq k-1$ . For results on Boolean functions and related topics, the reader can consult [6,8,9,11,18,24].

In this paper, motivated by the classical result that a bent function exactly corresponds to a strongly regular Cayley graph with an added condition (the Bernasconi–Codonotti correspondence we will refer to in Section 2), we are defining a concept of strong regularity for generalized bent functions, which naturally leads to weighted regular graphs. We show that the gbent functions with output in  $\mathbb{Z}_4$  exactly corresponds to this strong regularity of the associated generalized Cayley graph with an added condition similar to the classical case. For general  $q = 2^k$ ,  $k \geq 2$ , we are able to find a necessary condition on gbent functions in terms of this strong regularity.

## 2. Unweighted strongly regular graphs

A graph is *regular of degree  $r$*  (or  *$r$ -regular*) if every vertex has degree  $r$  (number of edges incident to it). We say that an  $r$ -regular graph  $G$  is a *strongly regular graph* (srg) with parameters  $(v, r, e, d)$  if there exist nonnegative integers  $e, d$  such that for all vertices  $\mathbf{u}, \mathbf{v}$  the number of vertices adjacent to both  $\mathbf{u}, \mathbf{v}$  is  $e, d$ , if  $\mathbf{u}, \mathbf{v}$  are adjacent, respectively, nonadjacent (see for instance [12]). The *complementary graph*  $\bar{G}$  of the strongly regular graph  $G$  is also strongly regular with parameters  $(v, v-r-1, v-2r+d-2, v-2r+e)$  (see [12]).

Since the objects of this paper are edge-weighted graphs  $G = (V, E, w)$  (with vertices  $V$ , edges  $E$  and weight function  $w$  defined on  $E$  with values in some set, which in our case it will be either the set of integers modulo  $q$ ,  $\mathbb{Z}_q$  with  $q = 2^k$ , or the complex numbers set  $\mathbb{C}$ ), we define the *weighted degree*  $d(v)$  of a vertex  $v$  to be the sum of the weights of its incident edges, that is,  $d(v) = \sum_{u, (u,v) \in E} w(u, v)$  (later, we will introduce yet another degree or strength concept). Certainly, one can also define the *combinatorial degree*  $r(v)$  of a vertex to be the number of such incident edges. For more on graph theory the reader can consult [4,12] or one's favorite graph theory book.

Let  $f$  be a Boolean function on  $\mathbb{V}_n$ . We define the *Cayley graph* of  $f$  to be the graph  $G_f = (\mathbb{V}_n, E_f)$  whose vertex set is  $\mathbb{V}_n$  and the set of edges is defined by

$$E_f = \{(\mathbf{w}, \mathbf{u}) \in \mathbb{V}_n \times \mathbb{V}_n : f(\mathbf{w} \oplus \mathbf{u}) = 1\}.$$

For some fixed (but understood from the context) positive integer  $s$ , let the canonical injection  $\iota : \mathbb{V}_s \rightarrow \mathbb{Z}_{2^s}$  be defined by  $\iota(\mathbf{c}) = \mathbf{c} \cdot (1, 2, \dots, 2^{s-1}) = \sum_{j=0}^{s-1} c_j 2^j$ , where  $\mathbf{c} = (c_0, c_1, \dots, c_{s-1})$ . For easy writing, we denote by  $\mathbf{j} := \iota^{-1}(j)$ .

The adjacency matrix  $A_f$  is the matrix whose entries are  $A_{i,j} = f(\mathbf{i} \oplus \mathbf{j})$  (here  $\iota$  is defined on  $\mathbb{V}_n$ ). It is simple to prove that  $A_f$  has the dyadic property:  $A_{i,j} = A_{i+2^{k-1}, j+2^{k-1}}$ . Also, from its definition, we derive that  $G_f$  is a *regular graph of degree*  $wt(f) = |\Omega_f|$  (see [12, Chapter 3] for further definitions and properties of these graphs).

Given a graph  $f$  and its adjacency matrix  $A$ , the *spectrum*, with notation  $\text{Spec}(G_f)$ , is the set of eigenvalues of  $A$  (called also the eigenvalues of  $G_f$ ). We assume throughout that  $G_f$  is connected (in fact, one can show that all connected components of  $G_f$  are isomorphic).

It is known (see [12, pp. 194–195]) that a connected  $r$ -regular graph is strongly regular iff it has exactly three distinct eigenvalues  $\lambda_0 = r, \lambda_1, \lambda_2$  (so  $e = r + \lambda_1 \lambda_2 + \lambda_1 + \lambda_2, d = r + \lambda_1 \lambda_2$ ).

The following result is known [12, Th. 3.32, p. 103] (the second part follows from a counting argument and is also well known).

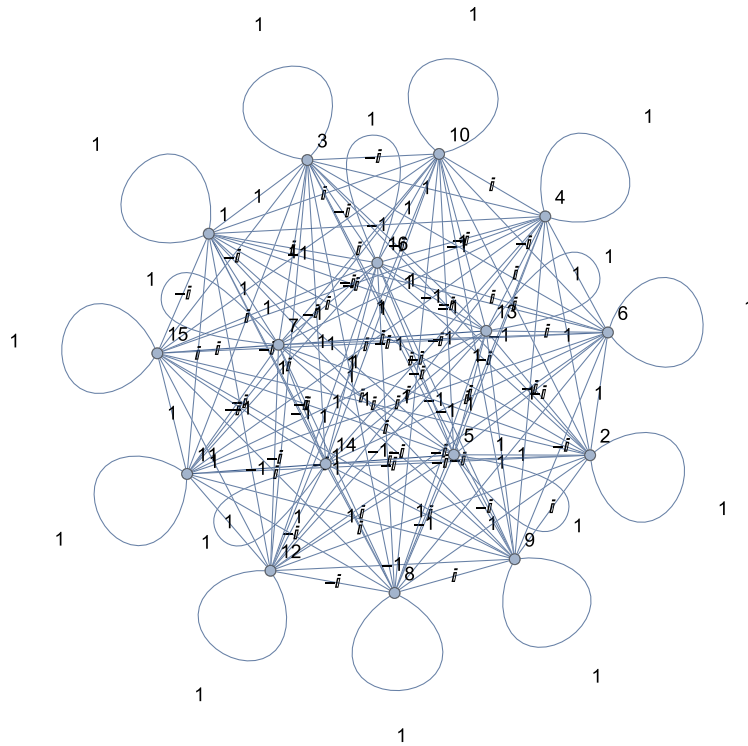


Fig. 1. Cayley graph of the gbent  $f(\mathbf{x}) = x_1 + 2(x_1x_2 \oplus x_3x_4)$ .

**Proposition 1.** The following identity holds for a strongly  $r$ -regular graph:

$$A^2 = (e - d)A + (r - d)I + dJ$$

where  $J$  is the all 1 matrix. Moreover,  $r(r - e - 1) = d(v - r - 1)$ .

In [1,2] it was shown that a Boolean function  $f$  is bent if and only if the Cayley graph  $G_f$  is strongly regular with  $e = d$ . We shall refer to this as the Bernasconi–Codonotti correspondence.

### 3. The Cayley graph of a generalized Boolean function

The applications of edge-weighted graphs are well known and we mention here the book by Bondy and Murty [5] or the readers' favorite book on graph theory and applications.

**Definition 2.** We now let  $f : \mathbb{V}_n \rightarrow \mathbb{Z}_q$  be a generalized Boolean function. We define the (generalized) Cayley graph  $G_f$  to be the graph whose vertices are the elements of  $\mathbb{V}_n$ , and where two vertices  $\mathbf{u}, \mathbf{v}$  are connected by a weighted edge of (multiplicative) weight  $\zeta^{f(\mathbf{u} \oplus \mathbf{v})}$  (alternatively, additive weight  $f(\mathbf{u} \oplus \mathbf{v})$ ).

Certainly, with the multiplicative weight, the underlying unweighted graph is a complete pseudograph (every vertex also has a loop). We sketch in Fig. 1 such an example.

Certainly, one could define a modified (generalized) Cayley graph  $G'_f$  where two vertices are connected if and only if  $f(\mathbf{u} \oplus \mathbf{v}) \neq 0$  with weights given by  $\zeta^{f(\mathbf{u} \oplus \mathbf{v})}$ . We sketch in Fig. 2 such a graph (it is ultimately the above graph with all weight 1 edges removed).

Example 3 illustrates a generalized Cayley graph, and its spectrum.

**Example 3.** Let  $f : \mathbb{V}_4 \rightarrow \mathbb{Z}_4$  defined by  $f(x_1, x_2) = x_1x_2 + 2x_1$ . The truth table is  $(0\ 0\ 2\ 3)^T$  (using the lexicographical order  $x_1, x_2$ ). Then, the adjacency matrix (with multiplicative weights) is

$$A_f = \begin{pmatrix} 1 & 1 & -1 & -i \\ 1 & 1 & -i & -1 \\ -1 & -i & 1 & 1 \\ -i & -1 & 1 & 1 \end{pmatrix}.$$

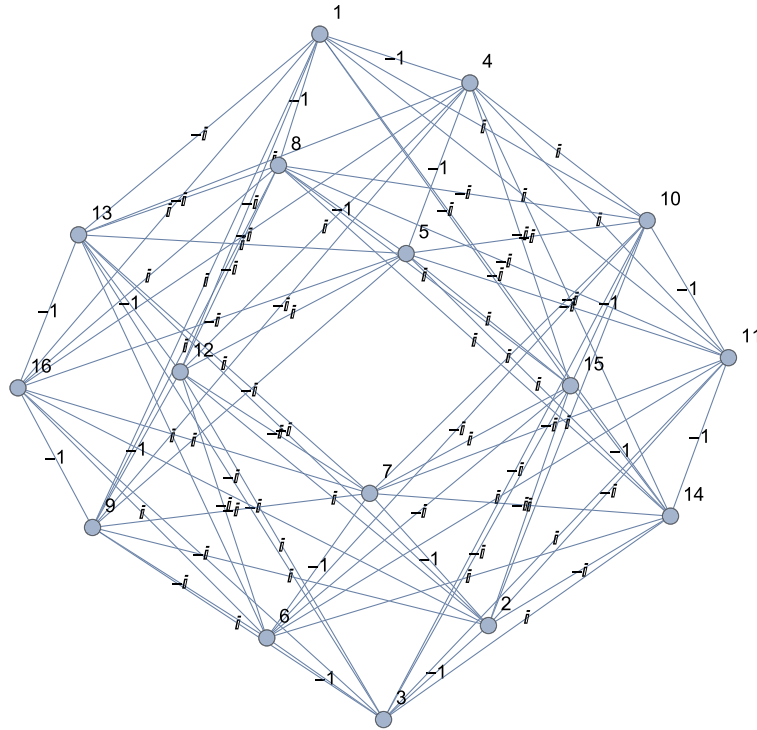


Fig. 2. Modified Cayley graph of the gbent  $f(\mathbf{x}) = x_1 + 2(x_1x_2 \oplus x_3x_4)$ .

A basis for its eigenspace is  $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ , where  $\vec{v}_1 = (1\ 1\ 1\ 1)^T$  with  $\chi_1(\mathbf{x}) = (-1)^0$ ,  $\vec{v}_2 = (1\ -1\ 1\ -1)^T$  with  $\chi_2(\mathbf{x}) = (-1)^{x_2}$ ,  $\vec{v}_3 = (1\ 1\ -1\ -1)^T$  with  $\chi_3(\mathbf{x}) = (-1)^{x_1}$ ,  $\vec{v}_4 = (1\ -1\ -1\ 1)^T$  with  $\chi_4(\mathbf{x}) = (-1)^{x_1+x_2}$ , having respective eigenvalues  $\lambda_0 = 1 - i$ ,  $\lambda_1 = -1 + i$ ,  $\lambda_2 = 3 + i$ ,  $\lambda_3 = 1 - i$ . We can see that the eigenvalues  $A_f$  are

$$\begin{aligned} \lambda_0 &= i^0 \chi_1(00) + i^0 \chi_1(01) + i^2 \chi_1(10) + i^3 \chi_1(11) = 1 + 1 + i^2 + i^3 = 1 - i = \mathcal{H}_f^{(4)}(\mathbf{0}), \\ \lambda_1 &= i^0 \chi_2(00) + i^0 \chi_2(01) + i^2 \chi_2(10) + i^3 \chi_2(11) = 1 - 1 + i^2 - i^3 = -1 + i = \mathcal{H}_f^{(4)}(\mathbf{1}), \\ \lambda_2 &= i^0 \chi_3(00) + i^0 \chi_3(01) + i^2 \chi_3(10) + i^3 \chi_3(11) = 1 + 1 - i^2 - i^3 = 3 + i = \mathcal{H}_f^{(4)}(\mathbf{2}), \\ \lambda_3 &= i^0 \chi_4(00) + i^0 \chi_4(01) + i^2 \chi_4(10) + i^3 \chi_4(11) = 1 - 1 - i^2 + i^3 = 1 - i = \mathcal{H}_f^{(4)}(\mathbf{3}). \end{aligned}$$

1 Although we do not use it in this paper, we define the *strength* of the vertex  $\mathbf{a}$  in the Cayley graph  $G_f$  as the sum of  
 2 the additive weights of incident edges, that is,  $s(\mathbf{a}) = \sum_b f(\mathbf{a} \oplus \mathbf{b})$ .

3 **Remark 4.** If  $f \in \mathcal{GB}_n^q$  and  $G_f$  is its Cayley graph, we observe that all vertices are adjacent of multiplicative (respectively,  
 4 additive) weights in  $\mathbb{U}_q = \{1, \zeta, \zeta^2, \dots, \zeta^{q-1}\}$  (respectively, in  $\mathbb{Z}_q = \{0, 1, \dots, q - 1\}$ ).

5 As it is the case for classical Boolean functions and their associated Cayley graphs, we next show that the eigenvalues  
 6 of the Cayley graph  $G_f$  (with multiplicative weights) are also precisely the (generalized) Walsh–Hadamard coefficients.

7 **Theorem 5.** Let  $f : \mathbb{V}_n \rightarrow \mathbb{Z}_q$ ,  $q = 2^k$ , and let  $\lambda_i$ ,  $0 \leq i \leq 2^n - 1$  be the eigenvalues of its associated (multiplicative)  
 8 edge-weighted graph  $G_f$ . Then,

9 
$$\lambda_i = \mathcal{H}_f^{(q)}(\mathbf{i}) \text{ (recall that } \mathbf{i} = \iota^{-1}(i)\text{)}.$$

10 **Proof.** Let  $\chi : \mathbb{V}_n \rightarrow \mathbb{C}$  be a character of  $\mathbb{V}_n$ , and for each such character, let  $\mathbf{x}_\chi = (x_j)_{0 \leq j \leq 2^n - 1} \in \mathbb{C}^{2^n}$ , where  $x_j = \chi(\mathbf{j})$ . We  
 11 claim (and show) that  $\mathbf{x}_\chi$  is an eigenvector of  $A = A_f$  (for simplicity, we use  $A$  in lieu of  $A_f$  in this proof), with eigenvalue  
 12  $\sum_{k=0}^{q-1} \sum_{\mathbf{s}_k \in S_k} \zeta^k \chi(\mathbf{s}_k)$ , where  $S_k = \{\mathbf{s}_k : f(\mathbf{s}_k) = k\}$ . (Observe that the characters of  $\mathbb{V}_n$  are  $\chi_{\mathbf{w}}(\mathbf{x}) = (-1)^{\mathbf{w} \cdot \mathbf{x}}$ , and thus the  
 13 eigenvalues are exactly the Walsh–Hadamard transform coefficients).

The  $i$ th entry of  $A\mathbf{x}$  is

$$(A\mathbf{x})_i = \sum_j A_{i,j}x_j = \sum_j A_{i,j}\chi(\mathbf{j}) = \sum_{k=0}^{q-1} \sum_{\mathbf{i} \oplus \mathbf{j} \in S_k} \zeta^k \chi(\mathbf{j})$$

If  $\mathbf{i} \oplus \mathbf{j} \in S_k$ , then  $\mathbf{i} \oplus \mathbf{j} = \mathbf{s}_k$ , for some  $\mathbf{s}_k \in S_k$ , and so,  $\mathbf{j} = \mathbf{i} \oplus \mathbf{s}_k$ . Since  $\chi$  is a character,

$$\chi(\mathbf{j}) = \chi(\mathbf{i} \oplus \mathbf{s}_k) = \chi(\mathbf{i})\chi(\mathbf{s}_k) = x_i\chi(\mathbf{s}_k)$$

Then,

$$(A\mathbf{x})_i = \sum_{k=0}^{q-1} \sum_{\mathbf{s}_k \in S_k} \zeta^k x_i \chi(\mathbf{s}_k) = x_i \sum_{k=0}^{q-1} \sum_{\mathbf{s}_k \in S_k} \zeta^k \chi(\mathbf{s}_k),$$

which shows our theorem.  $\square$

#### 4. Generalized bents and their Cayley graphs

We recall that a  $q$ -Butson Hadamard matrix [7] ( $q$ -BH) of dimension  $d$  is a  $d \times d$  matrix  $H$  with all entries  $q$ th roots of unity such that  $HH^* = dI_d$ , where  $H^*$  is the conjugate transpose of  $H$ . When  $q = 2$ ,  $q$ -BH matrices are called Hadamard matrices (where the entries are  $\pm 1$ ). Recall that the *crosscorrelation* function is defined by

$$C_{f,g}(\mathbf{z}) = \sum_{\mathbf{x} \in \mathbb{V}_n} \zeta^{f(\mathbf{x}) - g(\mathbf{x} \oplus \mathbf{z})},$$

and the *autocorrelation* of  $f \in \mathcal{GB}_n^q$  at  $\mathbf{u} \in \mathbb{V}_n$  is  $C_{f,f}(\mathbf{u})$  above, which we denote by  $C_f(\mathbf{u})$ .

**Theorem 6.** Let  $f \in \mathcal{GB}_n^q$ . Then  $f$  is gbent if and only if the adjacency matrix  $A_f$  of the (multiplicative) edge-weighted Cayley graph associated to  $f$  is a  $q$ -Butson Hadamard matrix (of dimension  $2^n$ ).

**Proof.** Let  $A_f = (\zeta^{f(\mathbf{a} \oplus \mathbf{b})})_{\mathbf{a}, \mathbf{b}}$ . Then, the  $(\mathbf{a}, \mathbf{b})$ -entry of  $A_f \cdot \bar{A}_f$  is

$$(A_f \cdot \bar{A}_f)_{\mathbf{a}, \mathbf{b}} = \sum_{\mathbf{c} \in \mathbb{V}_n} \zeta^{f(\mathbf{a} \oplus \mathbf{c})} \zeta^{-\bar{f}(\mathbf{c} \oplus \mathbf{b})} = \sum_{\mathbf{c} \in \mathbb{V}_n} \zeta^{f(\mathbf{a} \oplus \mathbf{c}) - f(\mathbf{c} \oplus \mathbf{b})} = C_f(\mathbf{a} \oplus \mathbf{b}). \quad (2)$$

Now, recall from [21] that if  $f, g \in \mathcal{GB}_n^q$ , then

$$\begin{aligned} \sum_{\mathbf{u} \in \mathbb{V}_n} C_{f,g}(\mathbf{u})(-1)^{\mathbf{u} \cdot \mathbf{x}} &= 2^{-n} \mathcal{H}_f(\mathbf{x}) \overline{\mathcal{H}_g(\mathbf{x})}, \\ C_{f,g}(\mathbf{u}) &= 2^{-n} \sum_{\mathbf{x} \in \mathbb{V}_n} \mathcal{H}_f(\mathbf{x}) \overline{\mathcal{H}_g(\mathbf{x})} (-1)^{\mathbf{u} \cdot \mathbf{x}}. \end{aligned}$$

Thus, Eq. (2) becomes

$$(A_f \cdot \bar{A}_f)_{\mathbf{a}, \mathbf{b}} = 2^{-n} \sum_{\mathbf{x} \in \mathbb{V}_n} \mathcal{H}_f(\mathbf{x}) \overline{\mathcal{H}_f(\mathbf{x})} (-1)^{(\mathbf{a} \oplus \mathbf{b}) \cdot \mathbf{x}} = 2^{-n} \sum_{\mathbf{x} \in \mathbb{V}_n} \|\mathcal{H}_f(\mathbf{x})\|^2 (-1)^{(\mathbf{a} \oplus \mathbf{b}) \cdot \mathbf{x}}.$$

By Parseval's identity, if  $\mathbf{a} = \mathbf{b}$ , then  $(A_f \cdot \bar{A}_f)_{\mathbf{a}, \mathbf{a}} = 2^n$ . Assume now that  $\mathbf{a} \neq \mathbf{b}$  and we shall show that  $(A_f \cdot \bar{A}_f)_{\mathbf{a}, \mathbf{b}} = 0$  for some  $\mathbf{a} \neq \mathbf{b}$  if and only if  $f$  is gbent. Certainly, if  $f$  is gbent then  $\|\mathcal{H}_f(\mathbf{x})\|^2 = 2^n$ , and since  $\sum_{\mathbf{x} \in \mathbb{V}_n} (-1)^{(\mathbf{a} \oplus \mathbf{b}) \cdot \mathbf{x}} = 0$ , for  $\mathbf{a} \oplus \mathbf{b} \neq \mathbf{0}$ , we have that implication. We can certainly show it directly, but the converse follows from [21, Theorem 1 (iv)].  $\square$

For the remaining of the paper, for simplicity, we shall only consider additive weights, namely, our edge-weighted graphs  $(V, E, w)$  will have the weight function  $w : E \rightarrow \mathbb{Z}_q$ ,  $q = 2^k$ .

Next, we say that a weighted graph  $G = (V, E, w)$ ,  $V \subseteq \mathbb{V}_n$ ,  $w : E \rightarrow \mathbb{Z}_q$ ,  $q = 2^k$ , is a *weighted regular graph* (wrg) with parameters  $(v; r_0, r_1, \dots, r_{q-1})$  if every vertex will have exactly  $r_j$  neighbors of edge weight  $j$ . We denote by  $N_f(\mathbf{a})$  the set of all neighbors of a vertex  $\mathbf{a}$  of corresponding edge weight  $j$ .

**Proposition 7.** Given a generalized Boolean function  $f \in \mathcal{GB}_n^q$ , the associated Cayley graph is weighted regular, that is, every vertex will have the same number of incident edges with a fixed weight.

**Proof.** Fix a weight  $j$  and a vertex  $\mathbf{x}_0$ , and consider the equation  $f(\mathbf{x}_0 \oplus \mathbf{y}) = j$  with solutions  $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_t$ , say. For any other vertex  $\mathbf{x}_1$ , the equation  $f(\mathbf{x}_1 \oplus \mathbf{y}) = j$  will have solutions  $\mathbf{y}_1 \oplus \mathbf{x}_1 \oplus \mathbf{x}_0, \mathbf{y}_2 \oplus \mathbf{x}_1 \oplus \mathbf{x}_0, \dots, \mathbf{y}_t \oplus \mathbf{x}_1 \oplus \mathbf{x}_0$ , and the proof is completed.  $\square$

We will define our first concept of strong regularity here. Let  $X, \bar{X}$  be a fixed bisection of the weights  $\mathbb{Z}_q = X \cup \bar{X}, X \cap \bar{X} = \emptyset, |X| = |\bar{X}| = 2^{k-1}$ , and let  $Y \subseteq \mathbb{Z}_q$ . We say that a weighted regular (with parameters  $(v; r_0, r_1, \dots, r_{q-1})$ ) graph  $G = (V, E, w), V \subseteq \mathbb{V}_n, w : E \rightarrow \mathbb{Z}_q, q = 2^k$ , is a (generalized)  $(X; Y)$ -strongly regular (srg) with parameters  $(v; r_0, r_1, \dots, r_{q-1}; e_X, d_X)$  if and only if the number of vertices  $\mathbf{c}$  adjacent to both  $\mathbf{a}, \mathbf{b}$ , with  $w(\mathbf{a}, \mathbf{c}) \in Y, w(\mathbf{b}, \mathbf{c}) \in Y$ , is exactly  $e_X$  if  $w(\mathbf{a}, \mathbf{b}) \in X$ , respectively,  $d_X$  if  $w(\mathbf{a}, \mathbf{b}) \in \bar{X}$ . One can weaken the condition and define a  $(X_1, X_2; Y)$ -srg notion, where  $X_1 \cap X_2 = \emptyset$ , not necessarily a bisection, and require the number of vertices  $\mathbf{c}$  adjacent to both  $\mathbf{a}, \mathbf{b}$ , with  $w(\mathbf{a}, \mathbf{c}) \in Y, w(\mathbf{b}, \mathbf{c}) \in Y$ , to be exactly  $e_X$  if  $w(\mathbf{a}, \mathbf{b}) \in X_1$ , respectively,  $d_X$  if  $w(\mathbf{a}, \mathbf{b}) \in X_2$ .

Note that our definition (see also [10] for an alternative concept, which we mention in the last section) is a natural extension of the classical definition: Let  $q = 2$ , and  $X = \{1\}$ . A classical strongly regular graph is then equivalent to an  $(X; X)$ -strongly regular graph.

We first show that (part of) Proposition 1 can be adapted to this notion, as well, in some cases, and we deal below with one such instance.

**Proposition 8.** Let  $G = (V, E, w)$  be a weighted  $(X; X)$ -strongly regular graph with parameters  $(v; r_0, r_1, \dots, r_{q-1}; e_X, d_X)$ , where  $X \subseteq \mathbb{Z}_q, v = |V|$ . Then,

$$r_X(r_X - e_X - 1) = d_X(v - r_X - 1),$$

where  $r_X = \sum_{i \in X} r_i$ .

**Proof.** Without loss of generality we assume that the weights are additive, that is, they belong to  $\mathbb{Z}_q$ . Fix a vertex  $\mathbf{u} \in V$  and let  $A$  be the set of vertices adjacent to  $\mathbf{u}$  with connecting edges of weight in  $X$ , and  $B = V \setminus \{\mathbf{u}\}$ . Observe that  $|A| = \sum_{i \in X} r_i = r_X$  and  $|B| = v - r_X - 1$ . We somewhat follow the combinatorial method of the classical case, and we shall count the number of vertices between  $A$  and  $B$  in two different ways. For any vertex  $\mathbf{a} \in A$ , there are exactly  $e_X$  vertices in  $A$  adjacent to both  $\mathbf{u}, \mathbf{a}$  of edge weights in  $X$ , and so, exactly  $r_X - e_X - 1$  neighbors in  $B$  whose connecting edges have weight in  $X$ . Therefore, the number of edges of weight in  $X$  between  $A$  and  $B$  is  $r_X(r_X - e_X - 1)$ .

On the other hand, any vertex  $\mathbf{b} \in B$  is adjacent to  $d_X$  vertices in  $A$  of connecting edge with weight in  $X$  (since  $\mathbf{u}, \mathbf{b}$  must share  $d_X$  common vertices of connecting edges of weight in  $X$ ) and so, the total number of edges of weight in  $X$  between  $A$  and  $B$  is  $d_X(v - r_X - 1)$ . The proposition follows.  $\square$

Let  $G = (V, E, w)$  (where  $w : E \rightarrow \mathbb{Z}_q$ ) be a weighted graph, where  $w(E) \subseteq \mathbb{Z}_q$  (or  $w(E) \subseteq \mathbb{U}_q$ ). We define the complement of  $G$ , denoted by  $\bar{G}$  the graph of vertex set  $V$  with an edge between two vertices  $\mathbf{a}, \mathbf{b}$  having weight  $q - 1 - f(\mathbf{a} \oplus \mathbf{b})$  (or, multiplicatively,  $\zeta^{q-1-f(\mathbf{a} \oplus \mathbf{b})}$ ). This is a natural definition, since, if  $G$  is the Cayley graph associated to  $f = a_0 + 2a_1, a_0, a_i \in \mathcal{B}_n$ , then we observe that  $\bar{G}$  is the Cayley graph associated to  $\bar{f} = \bar{a}_0 + 2\bar{a}_1 + \dots + 2^{k-1}\bar{a}_{k-1}$ , where  $\bar{a}_i$  is the binary complement of  $a_i$  (that follows from  $2^k - 1 - f = (1 - a_0) + 2(1 - a_1) + \dots + 2^{k-1}(1 - a_{k-1}) = \bar{a}_0 + 2\bar{a}_1 + \dots + 2^{k-1}\bar{a}_{k-1}$ ).

**Lemma 9.** Let  $G = (V, E, w)$  ( $w : E \rightarrow \mathbb{Z}_q$ ) be a weighted regular graph with parameters  $(v; r_0, r_1, \dots, r_{q-1})$ . Then the complement  $\bar{G}$  is a weighted regular graph with parameters  $(v; \bar{r}_0, \dots, \bar{r}_{q-1})$ , where  $\bar{r}_{q-1-j} = r_j$ .

**Proof.** Let  $\mathbf{a}$  be an arbitrary vertex. Recall that we denote by  $N_j(\mathbf{a})$  the set of all neighbors of a vertex  $\mathbf{a}$  of corresponding edge weight  $j$ . Since  $G$  is weighted regular, then  $|N_j(\mathbf{a})| = r_j$ . In the graph  $\bar{G}$ , the weight  $j$  will transform into  $q - 1 - j$ , therefore  $\bar{r}_{q-1-j} = r_j$  and the lemma is shown.  $\square$

Let  $A \subset B$  and  $x \in B$ . As it is customary, we will denote by  $x + A$  the set  $\{x + a : a \in A\}$ .

**Theorem 10.** Let  $G = (V, E, w)$  ( $V \subseteq \mathbb{F}_n^q, w : E \rightarrow \mathbb{Z}_q$ ) be an  $(X; Y)$ -strongly regular, for some  $X, Y \subseteq \mathbb{Z}_q$  with  $|X| = 2^{k-1}, q = 2^k$ , with parameters  $(v; r_0, r_1, \dots, r_{q-1}; e_X, d_X)$  such that  $q - 1 - X = X$  or  $\bar{X}$ , and  $q - 1 - Y = Y$ . Then, the complement  $\bar{G}$  is a  $(q - 1 - X; Y)$ -strongly regular graph with parameters  $(v; \bar{r}_0, \dots, \bar{r}_{q-1}; \bar{e}_{q-1-X}, \bar{d}_{q-1-X})$ , where  $\bar{r}_{q-1-j} = r_j, \bar{e}_{q-1-X} = e_X$  and  $\bar{d}_{q-1-X} = d_X$ , if  $q - 1 - X = X$ , respectively,  $\bar{r}_{q-1-j} = r_j, \bar{e}_{q-1-X} = d_X$  and  $\bar{d}_{q-1-X} = e_X$ , if  $q - 1 - X = \bar{X}$ .

**Proof.** The first claim follows from Lemma 9. We consider the two cases  $q - 1 - X = X$ , or  $\bar{X}$ , separately. As before, for any two vertices  $\mathbf{a}, \mathbf{b}$  we denote by  $N_Y(\mathbf{a}, \mathbf{b})$  the set of all vertices  $\mathbf{c}$  adjacent to both  $\mathbf{a}, \mathbf{b}$  such that  $w(\mathbf{a}, \mathbf{c}) \in Y, w(\mathbf{b}, \mathbf{c}) \in Y$ .

Case 1. Let  $q - 1 - X = X$ . For any two vertices  $\mathbf{a}, \mathbf{b}$  with  $w(\mathbf{a}, \mathbf{b}) \in X$ , then  $|N_Y(\mathbf{a}, \mathbf{b})| = e_X$ , since the weight of the edge between  $\mathbf{a}, \mathbf{b}$  remains in  $X$ . Similarly, for two vertices  $\mathbf{a}, \mathbf{b}$  with  $w(\mathbf{a}, \mathbf{b}) \in \bar{X}$ , then  $|N_Y(\mathbf{a}, \mathbf{b})| = d_X$ .

Case 2. Let  $q - 1 - X = \bar{X}$ . For any two vertices  $\mathbf{a}, \mathbf{b}$  with  $w(\mathbf{a}, \mathbf{b}) \in X$ , then the weight of the edge between  $\mathbf{a}, \mathbf{b}$  in  $\bar{G}$  is now in  $\bar{X}$ , and we know that in that case  $N_Y(\mathbf{a}, \mathbf{b}) = d_X$ . Similarly, for two vertices  $\mathbf{a}, \mathbf{b}$  with  $w(\mathbf{a}, \mathbf{b}) \in \bar{X}$ , then  $|N_Y(\mathbf{a}, \mathbf{b})| = e_X$ .  $\square$

In the next theorem, we shall show a strong regularity theorem (a Bernasconi–Codonotti correspondence) for gbents  $f \in \mathcal{GB}_n^4$  when  $n$  is even. For two vertices  $\mathbf{a}, \mathbf{b}$  of the associated Cayley graph, for  $i, j \in \{0, 1, 2, 3\}$ , let  $N_{\{i,j\}}(\mathbf{a}, \mathbf{b})$  be the set of all “neighbor” vertices  $\mathbf{w}$  to both  $\mathbf{a}, \mathbf{b}$  such that the edges have additive weights  $f(\mathbf{w} \oplus \mathbf{a}) \in \{i, j\}, f(\mathbf{w} \oplus \mathbf{b}) \in \{i, j\}$ .

**Theorem 11.** Let  $f \in \mathcal{GB}_n^4$ ,  $n$  even. Then  $f$  is gbent if and only if the associated generalized Cayley graph is  $(X; \bar{X})$ -strongly regular with  $e_X = d_X$ , for both  $X = \{0, 1\}$ , and  $X = \{0, 3\}$ , that is, if and only if the following two conditions are satisfied:

- (i) For any two pairs of vertices  $\{\mathbf{a}, \mathbf{b}\}, \{\mathbf{c}, \mathbf{d}\}$ , then  $|N_{\{2,3\}}(\mathbf{a}, \mathbf{b})| = |N_{\{2,3\}}(\mathbf{c}, \mathbf{d})|$ .
- (ii) For any two pairs of vertices  $\{\mathbf{a}, \mathbf{b}\}, \{\mathbf{c}, \mathbf{d}\}$ , then  $|N_{\{1,2\}}(\mathbf{a}, \mathbf{b})| = |N_{\{1,2\}}(\mathbf{c}, \mathbf{d})|$ .

**Proof.** We know that  $f = a_0 + 2a_1$ , where  $a_0, a_1 \in \mathcal{B}_n$ , is gbent if and only if  $a_1, a_1 \oplus a_0$  are both bent (see [20,21]). Let  $\mathbf{u} \in \mathbb{V}_n$ . We have that:

1.  $f(\mathbf{u}) = 0 \Leftrightarrow a_0(\mathbf{u}) = 0, (a_1 \oplus a_0)(\mathbf{u}) = 0$
2.  $f(\mathbf{u}) = 1 \Leftrightarrow a_0(\mathbf{u}) = 1, (a_1 \oplus a_0)(\mathbf{u}) = 1$
3.  $f(\mathbf{u}) = 2 \Leftrightarrow a_0(\mathbf{u}) = 0, (a_1 \oplus a_0)(\mathbf{u}) = 1$
4.  $f(\mathbf{u}) = 3 \Leftrightarrow a_0(\mathbf{u}) = 1, (a_1 \oplus a_0)(\mathbf{u}) = 1$

If  $f$  is gbent, then  $a_1, a_1 \oplus a_0$  are both bent. Then, by [1], their respective graphs are srg with respective parameters  $e = d, e' = d'$ . We consider the following cases:

- (a) Let any  $\mathbf{a}, \mathbf{b}, \mathbf{c}$  such that  $f(\mathbf{a} \oplus \mathbf{c}) \in \{1, 2\}$  and  $f(\mathbf{b} \oplus \mathbf{c}) \in \{1, 2\}$ , then  $(a_1 \oplus a_0)(\mathbf{a} \oplus \mathbf{c}) = 1 = (a_1 \oplus a_0)(\mathbf{b} \oplus \mathbf{c})$ . Since the graph corresponding to  $a_1 \oplus a_0$  is srg with  $e' = d'$ , then  $|\{\mathbf{c} : (a_1 \oplus a_0)(\mathbf{a} \oplus \mathbf{c}) = 1 = (a_1 \oplus a_0)(\mathbf{b} \oplus \mathbf{c})\}| = e'$ . Therefore,  $|N_{\{1,2\}}(\mathbf{a}, \mathbf{b})| = e'$ .
- (b) Let any  $\mathbf{a}, \mathbf{b}, \mathbf{c}$  such that  $f(\mathbf{a} \oplus \mathbf{c}) \in \{2, 3\}$  and  $f(\mathbf{b} \oplus \mathbf{c}) \in \{2, 3\}$ , then  $a_1(\mathbf{a} \oplus \mathbf{c}) = 1 = a_1(\mathbf{b} \oplus \mathbf{c})$ . Since the graph corresponding to  $a_1$  is srg with  $e = d$ , then  $|\{\mathbf{c} : a_1(\mathbf{a} \oplus \mathbf{c}) = 1 = a_1(\mathbf{b} \oplus \mathbf{c})\}| = e$ . Therefore,  $|N_{\{2,3\}}(\mathbf{a}, \mathbf{b})| = e$ .

Conversely, let the generalized Cayley graph be such that, for any two pairs of vertices  $\{\mathbf{a}, \mathbf{b}\}, \{\mathbf{c}, \mathbf{d}\}$ , then  $|N_{\{2,3\}}(\mathbf{a}, \mathbf{b})| = |N_{\{2,3\}}(\mathbf{c}, \mathbf{d})|$ , and  $|N_{\{1,2\}}(\mathbf{a}, \mathbf{b})| = |N_{\{1,2\}}(\mathbf{c}, \mathbf{d})|$ . As seen in the first part of the proof,  $|N_{\{2,3\}}(\mathbf{a}, \mathbf{b})| = |\{\mathbf{c} : a_1(\mathbf{a} \oplus \mathbf{c}) = 1 = a_1(\mathbf{b} \oplus \mathbf{c})\}|$ . This number is a constant, regardless of the value of  $a_1(\mathbf{a} \oplus \mathbf{b})$ . This implies that the Cayley graph corresponding to  $a_1$  is srg with  $e = d$ , where  $e = |N_{\{2,3\}}(\mathbf{a}, \mathbf{b})|$ .

Similarly,  $|N_{\{1,2\}}(\mathbf{a}, \mathbf{b})| = |\{\mathbf{c} : (a_1 \oplus a_0)(\mathbf{a} \oplus \mathbf{c}) = 1 = (a_1 \oplus a_0)(\mathbf{b} \oplus \mathbf{c})\}|$ . This number is a constant, regardless of the value of  $(a_1 \oplus a_0)(\mathbf{a} \oplus \mathbf{b})$ . This implies that the Cayley graph corresponding to  $a_1 \oplus a_0$  is srg with  $e' = d'$ , where  $e' = |N_{\{1,2\}}(\mathbf{a}, \mathbf{b})|$ . Since both  $a_1$  and  $a_1 \oplus a_0$  are therefore bent, we conclude that  $f$  is gbent.  $\square$

It is not hard to show that in some instances a “uniform” strong regularity will hold.

**Corollary 12.** Let  $S$  be a bent set (see [3]), that is, every element of  $S$  is a bent function and the sum of any two such is also a bent function. Let  $a_0, a_1 \in S$ . Then, the generalized edge-weighted Cayley graph of  $f = a_0 + 2a_1$  is  $(X; \bar{X})$ -strongly regular for any  $X$  with  $|X| = 2$ .

**Remark 13.** One certainly could inquire whether a similar result holds for a gbent function for  $n$  odd, but we leave that question for a subsequent project of an interested reader. We suspect that the answer depends on a characterization of Boolean semibents in terms of their Cayley graphs and we point to [19], where that was achieved for general plateaued functions.

We can follow a similar approach as in Theorem 11 to find a necessary condition on the Cayley graph of a generalized bent in  $\mathcal{GB}_n^q$ ,  $q = 2^k$ , for any  $k > 2$ . As in the previous result, for  $X \subseteq \mathbb{Z}_q$  and two vertices  $\mathbf{u}, \mathbf{v}$ , let  $N_X(\mathbf{u}, \mathbf{v})$  be the set of vertices  $\mathbf{w}$  such that  $f(\mathbf{u} \oplus \mathbf{w}) \in X$  and  $f(\mathbf{v} \oplus \mathbf{w}) \in X$ . As usual,  $\bar{\mathbf{c}}$  is the complement of the vector  $\mathbf{c}$ , and for two vectors  $\mathbf{a} = (a_1, \dots, a_t), \mathbf{b} = (b_1, \dots, b_t)$ , the notation  $\mathbf{a} \leq \mathbf{b}$  means that  $a_i \leq b_i$ , for all  $1 \leq i \leq t$ . Recall that the canonical injection  $\iota : \mathbb{V}_s \rightarrow \mathbb{Z}_{2^s}, \iota(\mathbf{c}) = \mathbf{c} \cdot (1, 2, \dots, 2^{s-1}) = \sum_{j=0}^{s-1} c_j 2^j$ , where  $\mathbf{c} = (c_0, c_1, \dots, c_{s-1})$ . For  $\mathbf{c} \in \mathbb{V}_{k-1}$  and  $i = 0, 1$ , we let  $X_c^i = \{\iota(\bar{\mathbf{c}}) + \iota(\mathbf{d}) : \bar{\mathbf{c}} \leq (\mathbf{c}, 1), wt(\bar{\mathbf{c}}) \equiv i \pmod{2}, \mathbf{d} \leq \bar{\mathbf{c}}\}$ .

**Theorem 14.** Let  $n$  be even, and  $f = a_0 + 2a_1 + \dots + 2^{k-1}a_{k-1}$ ,  $k \geq 2$ ,  $a_i \in \mathcal{B}_n$ , be a generalized bent Boolean function. Then the associated edge-weighted Cayley graph is  $(X_c^0; X_c^1)$ -strongly regular with  $e_{X_c^0} = d_{X_c^0}$ , for all  $\mathbf{c} \in \mathbb{V}_{k-1}$ ; that is, for all  $\mathbf{c} \in \mathbb{V}_{k-1}$ , and for any two pairs of vertices  $(\mathbf{u}, \mathbf{v}), (\mathbf{x}, \mathbf{y})$ ,

$$|N_{X_c^1}(\mathbf{u}, \mathbf{v})| = |N_{X_c^1}(\mathbf{x}, \mathbf{y})|.$$

**Proof.** The weighted regularity of  $f$  follows from Proposition 7. We know [13] (see also [15, Theorem 8] and [22]) that  $f = a_0 + 2a_1 + \dots + 2^{k-1}a_{k-1}$  is gbent if and only if  $a_{k-1} \oplus \langle a_0, \dots, a_{k-2} \rangle$  forms an affine space of bent (semibent) functions when  $n$  is even (respectively,  $n$  is odd), with certain additional properties (properly described in [13]).

Take an arbitrary bent function from the affine space  $a_{k-1} \oplus \langle a_0, \dots, a_{k-2} \rangle$  of bent functions, which can be written as  $f_c(\mathbf{x}) = c_0 a_0(\mathbf{x}) \oplus c_1 a_1(\mathbf{x}) \oplus \dots \oplus c_{k-2} a_{k-2}(\mathbf{x}) \oplus a_{k-1}(\mathbf{x})$ , for some  $\mathbf{c} \in \mathbb{V}_{k-1}$ . For  $\mathbf{u} \in \mathbb{V}_n$ , the condition that  $f_c(\mathbf{u}) = 1$  means that an odd number of functions  $a_j$ , occurring (that is, the corresponding coefficient is nonzero) in  $f_c$  will output 1 at  $\mathbf{u}$ . The  $a_j$ 's corresponding to entries that are 0 in  $\mathbf{c}$  can be taken either 0 or 1 (hence the condition in the definition of  $X_c^i$  that  $\mathbf{d} \leq \bar{\mathbf{c}}$ ). We see that the set of values of  $f$  when  $f_c(\mathbf{u}) = 1$  is exactly  $X_c^1$ , and the set of values for  $f$  when  $f_c(\mathbf{u}) = 0$  is  $X_c^0$ , which were previously defined.

Since  $f_c$  is bent, then any two vertices,  $\mathbf{u}, \mathbf{v}$ , will have the same number of adjacent  $\mathbf{w}$  with  $f_c(\mathbf{u} \oplus \mathbf{w}) = f_c(\mathbf{v} \oplus \mathbf{w}) = 1$ , regardless of the value of  $f_c(\mathbf{u} \oplus \mathbf{v})$ . This implies that  $|N_{\chi_1}(\mathbf{u}, \mathbf{v})|$  is constant for all  $\mathbf{u}, \mathbf{v}$ .  $\square$

## 5. Further comments

In [10] it was defined yet another strong regularity concept. Let  $\Gamma$  be an edge-weighted graph (with no loops) with vertices  $V$ , edges  $E$ , and weight set  $W$  (in [10],  $W$  was taken to be  $\mathbb{Z}_q^*$ , although it could be arbitrary). As before, for each  $\mathbf{u} \in V$  and  $a \in W \cup \{0\}$ , the weighted  $a$ -neighborhood of  $u$ ,  $N_a(\mathbf{u})$ , is defined as follows:

- $N_a(\mathbf{u})$  = the set of all neighbors  $\mathbf{v}$  of  $\mathbf{u}$  in  $\Gamma$  for which the edge  $(\mathbf{u}, \mathbf{v}) \in E$  has weight  $a$  (for each  $a \in W$ ).
  - $N^0(\mathbf{u})$  = the set of all nonadjacent  $\mathbf{v}$  of  $\mathbf{u}$  in  $\Gamma$  (i.e., the set of  $\mathbf{v}$  such that  $(\mathbf{u}, \mathbf{v}) \notin E$ ), that is,  $N^0(\mathbf{u}) = V \setminus \cup_{a \in W} N_a(\mathbf{u})$ .
- In particular,  $\mathbf{u} \in N^0(\mathbf{u})$ .

In [10], the following definition of weighted strongly regular graph is given. Let  $\Gamma$  be a connected edge-weighted graph which is regular as a simple (unweighted) graph. Let  $W$  be the set of edge-weights of  $\Gamma$ . The graph  $\Gamma$  is called an *edge-weighted local strongly regular* (to distinguish it from our definition we inserted the adjective “local”) with parameters  $v, k = (k_a)_{a \in W}, \lambda = (\lambda_a)_{a \in W^3}$ , and  $\mu = (\mu_a)_{a \in W^2}$ , denoted  $SRG_W(v, k, \lambda, \mu)$ , if  $\Gamma$  has  $v$  vertices, and there are constants  $k_a, \lambda_{a_1, a_2, a_3}$ , and  $\mu_{a_1, a_2}$ , for  $a, a_1, a_2, a_3 \in W$ , such that

$$|N_a(\mathbf{u})| = k_a \text{ for all vertices } \mathbf{u},$$

and for vertices  $\mathbf{u}_1 \neq \mathbf{u}_2$  we have

$$|N_{a_1}(\mathbf{u}_1) \cap N_{a_2}(\mathbf{u}_2)| = \begin{cases} \lambda_{a_1, a_2, a_3} & \text{if } \exists a_3 \in W \text{ with } \mathbf{u}_1 \in N_{a_3}(\mathbf{u}_2); \\ \mu_{a_1, a_2} & \text{if } \mathbf{u}_1 \notin N_{a_3}(\mathbf{u}_2) \text{ for all } a_3. \end{cases}$$

As was observed in [10] for functions  $f : \mathbb{F}_p^n \rightarrow \mathbb{F}_p$ , where several questions were posed, it is not clear what the connection between this concept and generalized (or  $p$ -ary) bentness is. Our strong regularity definition does allow us to show such a connection and in the case  $k = 2$ , we have a complete Bernasconi–Codonotti correspondence [1,2].

## Acknowledgments

The authors express their deep appreciation to the editor, as well as to the anonymous referee, whose thorough reading and constructive comments have improved the paper, significantly.

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