Perfect Squares as Concatenation of Consecutive Integers

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Abstract. We find an infinite family of positive integers a such that concatenating a and $a-1$ in base 10 (from left to right) results in a number that is a perfect square and estimates for such concatenations.

1. INTRODUCTION. In [**2**], Sastry noticed that $183184 = 428^2$ and asked if there are other examples of positive integers a such that concatenating a with $a+1$ (from left to right) in base 10 results in a perfect square. While this is discussed in [**1**], in this article we shall consider the concatenation of a with $a+1$ (from right to left) in base 10, which results in a perfect square.

We shall now briefly revisit the approach of [**1**] for dealing with Sastry's problem. Denote by n the number of digits of $a+1$; then the question reduces to finding other instances when

$$10^n a + (a+1) = x^2 \tag{1}$$

with positive integers $a,\ x$, with $10^{n-1} \le a < 10^n - 1$. The equation above can be rewritten as

$$a(10^n+1) = x^2 - 1 = (x-1)(x+1).$$

We see that the arithmetic of 10^n+1 plays a role. For example, if 10^n+1 is a prime, then the equation above implies that 10^n+1 divides one of $x-1$ or $x+1$. Thus, $x+1 \ge 10^n+1$, so $x^2 \ge 10^{2n}$ is a number with at least $2n+1$ digits; this contradicts the fact that it should have exactly $2n$ digits. Since 10^n+1 is never a perfect power of exponent larger than 1 of some other integer (a fact which can either be easily proved in an elementary way or follows by invoking known facts about Catalan's equation), it follows that if 10^n+1 is not prime, then it has at least two distinct prime factors. To find solutions, we write $10^n+1 = A_1A_2$ and $a = a_1a_2$ and try to solve

$$x+1 = A_1a_1 \quad \text{and} \quad x-1 = A_2a_2.$$

Eliminating x, we get

$$A_1a_1 - A_2a_2 = 2. \tag{2}$$

Since A_1, A_2 are odd (as divisors of 10^n+1), we deduce from (2) that they must be coprime and the argument from the case where we assumed that 10^n+1 is prime shows that neither A_1, A_2 can be 1. Given A_1, A_2, equation (2) has infinitely many solutions (a_1, a_2), but they all come from the minimal one, let us call it $(a_{1,0}, a_{2,0})$, via

$$a_1 = a_{1,0} + A_2m \qquad \text{and} \qquad a_2 = a_{2,0} + A_1m$$

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for some nonnegative integer m. Since we also want to have $a_1a_2 = a < 10^n - 1 < A_1A_2$, the only chance that we have is that $(a_1, a_2) = (a_{1,0}, a_{2,0})$. If $a_0 = a_{1,0}a_{2,0}$ has n digits, then we found a convenient solution to our problem. Sometimes, a_0 is "too short." For example, taking $m = 3,\ A_1 = 11,\ A_2 = 91$, the minimal solution of the equation

$$11a_1 - 91a_2 = 2$$

is $(a_{1,0}, a_{2,0}) = (25, 3)$ for which $a_0 = 75$ has only two digits. If we pretend that it has three digits, namely that it is 075, then indeed concatenating a_0 with $a_0 + 1$ results in the perfect square

$$75076 = 274^2.$$

But note that if (a_1, a_2) is a solution of (2), then $(a_1', a_2') = (A_2 - a_1, A_1 - a_2)$ is a solution of

$$A_1a_1' - A_2a_2' = -2,$$

which is the same equation as (2) with the pair (A_1, A_2) replaced by the pair (A_2, A_1). It is shown in [**1**] that given A_1, A_2, not both a_0 and a_0' can be short. For the example with $m = 3$, $A_1 = 11,\ A_2 = 91$, we have $(a_{1,0}', a_{2,0}') = (66, 8)$, so $a_0' = 66 \times 8 = 528$ has three digits and

$$528529 = 727^2.$$

All this is proved in [**1**]. As a byproduct, one finds that if one denotes by $N_+(n)$ the number of positive integers a with n digits satisfying Sastry's requirement, then $N_+(n) \neq 0$ if and only if $10^n + 1$ has at least two distinct prime factors. Furthermore, denoting by $\omega(n)$ the number of distinct prime factors of $10^n + 1$, it follows that

$$2^{\omega(n)-1} - 1 \leq N_+(n) \leq 2(2^{\omega(n)-1} - 1),$$

since at least one and at most two nontrivial factorizations of $10^n + 1$ as A_1A_2 will render solutions to Sastry's question. Giving m the values $3, 6, 9, \ldots$, one notices that

$$66 \times 8, \qquad 6666 \times 68, \qquad 666666 \times 668, \qquad \ldots$$

are all valid examples of numbers a. Once one sees these examples, one extrapolates that perhaps for all m, the number

$$a = \underbrace{66\ldots6}_{2m \text{ times}} \times \underbrace{66\ldots6}_{m-1 \text{ times}}8 \tag{3}$$

is a valid example with $3m$ digits, which is easy to verify as for such values of a, the number shown in (3) is a polynomial of degree 6 in 10^m which turns out to be the square of a polynomial of degree 3 in 10^m.

Professor I. Shparlinski noticed that the problem above was easy because $x^2 - 1$ factors as $(x - 1)(x + 1)$, so he asked, what if we concatenate a with $a + 1$ in the reverse order and ask for *that* to be a square [**3**]. We decided to answer this question in this article. Writing a for the first (leftmost number) with n digits, the analog to equation (1) is now

$$10^n a + (a - 1) = x^2,$$

so

$$a(10^n + 1) = x^2 + 1. \tag{4}$$

We see that n is necessarily even, since if n is odd, then 11 divides the left-hand side of equation (4), which implies that $x^2 \equiv -1 \pmod{11}$, and this is impossible. This argument not only shows that n is even, but it also shows that all prime factors of both a and $10^n + 1$ are congruent to 1 modulo 4. Now factor $x^2 + 1 = (x + i)(x - i)$ in $\mathbb{Z}[i]$. Then $x + i$ is a divisor in $\mathbb{Z}[i]$ of $a(10^n + 1)$. It then follows that there are integers a_1, a_2, A_1, A_2 such that

$$x + i = (a_2 + a_1 i)(A_1 - A_2 i), \tag{5}$$

where $a_2 + a_1 i$ is the greatest common divisor (in $\mathbb{Z}[i]$) of a and $x + i$ and $A_1 - A_2 i$ is the greatest common divisor (in $\mathbb{Z}[i]$) of $x + i$ and $10^n + 1$. By multiplying $A_1 - A_2 i$ by one of the units of $\mathbb{Z}[i]$ (namely, one of $\pm 1, \pm i$), we may assume that A_1 and A_2 are positive. Hence,

$$x^2 + 1 = (a_1^2 + a_2^2)(A_1^2 + A_2^2), \quad \text{where } a_1^2 + a_2^2 = a \text{ and } A_1^2 + A_2^2 = 10^n + 1.$$

In (5) we identify the imaginary parts of the two sides of the equation, getting

$$a_1 A_1 - a_2 A_2 = 1, \tag{6}$$

an equation very similar to (2). Since A_1, A_2 are positive, a_1, a_2 have the same sign, and up to replacement in the left-hand side of (5), that is, replacing $x + i$ by $(-x - i)$, we may assume them to be positive. Let $(a_{1,0}, a_{2,0})$ be the minimal solution of (6). Then

$$(a_1, a_2) = (a_{1,0} + A_2 m, a_{2,0} + A_1 m) \quad \text{for some} \quad m \geq 0.$$

If $m \geq 1$,

$$a = a_1^2 + a_2^2 > (A_1^2 + A_2^2)m^2 \geq A_1^2 + A_2^2 > 10^n$$

is “too long.” Hence, the only chance is that $(a_1, a_2) = (a_{1,0}, a_{2,0})$. We treat the cases of $10^n + 1$ being prime or composite separately. If $10^n + 1$ is prime, then by a result of Fermat, it has only one representation as a sum of two squares, so it must be $(10^{n/2})^2 + 1^2$. Hence, $\{A_1, A_2\} = \{10^{n/2}, 1\}$, and we get, up to interchanging a_1 with a_2 if needed, that

$$a_1 - a_2 10^{n/2} = \pm 1.$$

Thus, $a_1 = 10^{n/2} \pm 1$ and $a_2 = 1$. The possibility $a_1 = 10^{n/2} + 1$ gives $a = a_1^2 + a_2^2 > 10^n$, which is false, while the possibility $a_1 = 10^{n/2} - 1$ gives

$$a = a_1^2 + a_2^2 = 10^n - 2 \cdot 10^{n/2} + 2,$$

which works, since

$$\begin{aligned} 10^n a + (a - 1) &= (10^n - 2 \cdot 10^{n/2} + 2)10^n + (10^n - 2 \cdot 10^{n/2} + 1) \\ &= (10^n - 10^{n/2} + 1)^2. \end{aligned}$$

Now, let (A_1, A_2) be any representation of $10^n + 1 = A_1^2 + A_2^2$ such that both $A_1 > 1$ and $A_2 > 1$. Such a representation exists if and only if $10^n + 1$ is not a prime. Then one takes $a_0 = a_{1,0}^2 + a_{2,0}^2$, where $(a_{1,0}, a_{2,0})$ is the minimal solution of (6). As in the case of Sastry's problem, it could be that a_0 is "too short." That is, it could be that $a_0 < 10^{n-1}$. Then, we note that if (a_1, a_2) satisfies equation (6), the pair $(a_1', a_2') = (A_2 - a_1, A_1 - a_2)$ satisfies

$$a_1' A_1 - a_2' A_2 = -1,$$

which is the same equation as (6) with (A_1, A_2) replaced by (A_2, A_1). If $a_0' = (a_{1,0}')^2 + (a_{2,0}')^2$ is also "short," namely, if $a_0' < 10^{n-1}$, then we get a contradiction by the triangle inequality, namely,

$$10^{n/2} < \sqrt{A_1^2 + A_2^2} < \sqrt{a_{1,0}^2 + a_{2,0}^2} + \sqrt{(a_{1,0}')^2 + (a_{2,0}')^2} \le 2\sqrt{10^{n-1}},$$

which is absurd.

Recalling that the number of representations as a sum of two squares of $10^n + 1$ equals $2^{\omega(n)}$, and letting $N_-(n)$ denote the number of positive integers a with n digits satisfying Shparlinski's requirement that concatenating a with $a - 1$ (from left to right) one obtains a square, we conclude that we have proved the following theorem.

Theorem 1. *Let $n \ge 1$ be a positive integer. Then $N_-(n) = 0$ unless n is even. Furthermore, the inequality*

$$2^{\omega(n)-1} \le N_-(n) \le 2(2^{\omega(n)-1} - 1) + 1$$

holds for all even n.

So, although the expression $x^2 + 1$ does not factor (at least not in any obvious way), one may apply the same argument and get the same conclusions for $N_-(n)$ as one did for $N_+(n)$.

How about finding parametric families of solutions? That works, too. Taking $n = 6k$ and giving k values $1, 2, 3$, one obtains the examples

$$146^2 + 719^2, \qquad 13466^2 + 673199^2, \qquad 1334666^2 + 667331999^2, \ldots$$

from which one deduces that perhaps the pair (a_1, a_2) given by

$$a_1 = 1\underbrace{33\ldots3}_{k-1 \text{ times}}4\underbrace{66\ldots6}_{k \text{ times}}$$
$$a_2 = \underbrace{66\ldots6}_{k-1 \text{ times}}7\underbrace{33\ldots3}_{k-1 \text{ times}}1\underbrace{99\ldots9}_{k \text{ times}}$$

has the property that

$$a = a_1^2 + a_2^2$$

is a valid solution (with n digits) to Shparlinski's question. One checks easily that

$$a_1 = \frac{4 \cdot 10^{2k} + 4 \cdot 10^k - 2}{3},$$

$$a_2 = \frac{2 \cdot 10^{3k} + 2 \cdot 10^{2k} - 4 \cdot 10^k - 3}{3},$$

and indeed

$$a = a_1^2 + a_2^2 = \left(\frac{2 \cdot 10^{6k} + 2 \cdot 10^{5k} + 10^{3k} + 2 \cdot 10^k + 2}{3} \right)^2$$

is a perfect square. We also found a parametric family for $n = 10k$, and even a "short parametric family" for such n, where by short we mean that a has $8k$ digits, instead of $10k$, so it has to be "beefed up" by $2k$ zeros to the left in order to create an example. The "short parametric family" is given by

$$a_1 = 7 \underbrace{99\ldots9}_{2(k-1) \text{ times}} 84 \underbrace{00\ldots0}_{k-1 \text{ times}} = \frac{4 \cdot 10^{3k} - 8 \cdot 10^k}{5}$$

$$a_2 = 3 \underbrace{99\ldots9}_{2(k-1) \text{ times}} 88 \underbrace{00\ldots0}_{k-1 \text{ times}} 1 = \frac{2 \cdot 10^{4k} - 6 \cdot 10^{2k} + 5}{5},$$

which, of course, can be changed into the "right" one by the previously mentioned trick. We leave the details to the interested reader.

We can certainly provide (see table below) all the examples to Shparlinski's question up to, say, 10 decimal digits:

$$82\ 81 = 91^2$$
$$8242\ 8241 = 9079^2$$
$$9802\ 9801 = 9901^2$$
$$538277\ 538276 = 733674^2$$
$$998002\ 998001 = 999001^2$$
$$77837026\ 77837025 = 88225295^2$$
$$99980002\ 99980001 = 99990001^2$$
$$7922547265\ 7922547264 = 8900869208^2$$
$$8643251345\ 8643251344 = 9296908812^2$$
$$9223797610\ 9223797609 = 9604060397^2$$
$$9999800002\ 9999800001 = 9999900001^2.$$

We conclude this discussion with the following open problem for the reader.

Problem. Is it true that for every integer d, there are infinitely many positive integers a such that a and $a + d$ have the same number of digits and concatenating a with $a + d$ (from left to right) one gets a perfect square?

In this article, we treated the cases $d = \pm 1$. The case $d = 0$ can be related to $10^n + 1$ not being square-free. Indeed in this case, the analog to equation (1) is

$$a(10^n + 1) = x^2,$$

and if $10^n + 1$ is squarefree, it would imply that $10^n + 1$ divides x, which in turn would give that $a(10^n + 1) \geq (10^n + 1)^2$, so $a > 10^n$, a contradiction. For example, for $n = 11$, $10^{11} + 1$ is a multiple of 11^2, and taking

$$a = \left(\frac{10^{11}+1}{11^2}\right) y^2$$

for some integer y such that a has exactly 11 digits, one gets the examples

$$\begin{aligned}
&13223140496\ 13223140496 = 36363636364^2, && y = 4;\\
&20661157025\ 20661157025 = 45454545455^2, && y = 5;\\
&29752066116\ 29752066116 = 54545454546^2, && y = 6;\\
&40495867769\ 40495867769 = 63636363637^2, && y = 7;\\
&52892561984\ 52892561984 = 72727272728^2, && y = 8;\\
&66942148761\ 66942148761 = 81818181819^2, && y = 9;\\
&82644628100\ 82644628100 = 90909090910^2, && y = 10.
\end{aligned}$$

More generally, if $m^2 > 1$ is any square factor of $10^n + 1$, then taking

$$a = \left(\frac{10^n+1}{m^2}\right) y^2$$

with an integer y in the interval $[m/\sqrt{10}, m-1]$ gives a valid answer to our problem for $d = 0$. How about for other values of d? One can quickly check and see that for all d with $|d| \leq 10$, one can find examples of perfect squares by concatenating a with $a + d$ from left to right, except for $d = -3, 7$. With a little modular arithmetic work, one can give an argument for why those values of d will never generate perfect squares. Certainly, one can ask similar questions about concatenating a sequence of consecutive integers (all with the same number of digits) in some order giving rise to a perfect square. We invite the reader to investigate such questions.

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