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uniform distribution theory

ON THE FIRST DIGITS OF THE FIBONACCI NUMBERS AND THEIR EULER FUNCTION

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ABSTRACT. Here, we show that given any two finite strings of base b digits, say s_1 and s_2, there are infinitely many Fibonacci numbers F_n such that the base b representation of F_n starts with s_1 and the base b representation of $\phi(F_n)$ starts with s_2.

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1. Introduction

Let $b \geq 2$ be and integer. Let $s_1 = \overline{c_1 \cdots c_k}_{(b)}$ be a positive integer s_1 written in base b. Washington [4] proved that there exist infinitely many Fibonacci numbers F_n whose base b representation starts with s_1. In fact, the first digits of the Fibonacci sequence obey Benford's law in that the proportion of the positive integers n such that F_n starts with s_1 is precisely $\log((s_1+1)/s_1)/\log b$. Here, we take this one step further. Let $\phi(m)$ be the Euler function of the positive integer m. We put $s_2 = \overline{d_1 \ldots d_\ell}_{(b)}$ for some other positive integer written in base b and prove the following theorem.

Theorem. *Given positive integers $s_1 = \overline{c_1 \cdots c_k}_{(b)}$ and $s_2 = \overline{d_1 \ldots d_\ell}_{(b)}$ written in base b, there exist infinitely many positive integers n such that the base b representation of F_n starts with the digits of s_1 and the base b representation of $\phi(F_n)$ starts with the digits of s_2.*

We use the fact that with $(\alpha, \beta) = ((1+\sqrt{5})/2, (1-\sqrt{5})/2)$ the Binet formula

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \qquad \text{holds for all} \qquad n \geq 0.$$

For a positive real number x we write $\log x$ for the natural logarithm of x, and $\lfloor x \rfloor$, respectively, $\{x\}$, for the integer part, respectively, fractional part of x.

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2. The proof

By replacing s_1 with $s_1 b^m$ for some positive integer m, if needed, whose effect is adding m zeros at the end of the base b representation of s_1, we may assume that $s_1 > s_2$. By replacing s_1, s_2 by $s_1 b^m$, respectively, $s_2 b^m$ for an arbitrary positive integer m, we may assume that the length of the base b representation of s_1, that is k, is as large as we wish. In Section 4 of [2], it is shown that $\phi(F_n)/F_n$ is dense in $[0,1]$. So, we take $\varepsilon \in (0, 1/(15b^{2k}))$ and choose a positive integer a such that

$$\frac{\phi(F_a)}{F_a} \in \left(\frac{s_2}{s_1} + \varepsilon, \frac{s_2}{s_1} + 2\varepsilon\right). \tag{1}$$

Now we take any prime $p > F_a$ and look at F_{ap}. Since $p > F_a$, it follows that

$$F_{ap} = F_a \left(\frac{F_{ap}}{F_a}\right),$$

and the two factors F_a and F_{ap}/F_a on the right above are coprime (indeed, the only common prime factor of these two numbers could be p, which is not the case since $p > F_a$). Any prime factor q of F_{ap}/F_a is a *primitive prime factor* of F_{dp} for some divisor d of a. Recall that a prime number q is said to be a primitive prime factor of F_n if q divides F_n, but does not divide any F_m for $1 \le m < n$. One of the properties of primitive prime factors q of F_n when $n > 5$ is that $q \equiv \pm 1 \pmod{n}$. In particular, every prime factor q of F_{ap}/F_a is congruent to $\pm 1 \pmod{p}$.

Let $q_1, \ldots, q_t$ be all the prime factors of F_{ap}/F_a. Then

$$(2p-1)^t \le q_1 \cdots q_t \le \frac{F_{ap}}{F_a} \le F_{ap} \le \alpha^{ap}.$$

Thus, $t = O(p/\log p)$. Then

$$\begin{aligned}
\frac{\phi(F_{ap})}{F_{ap}} &= \left(\frac{\phi(F_a)}{F_a}\right) \prod_{i=1}^{t} \left(1 - \frac{1}{q_i}\right) \\
&= \frac{\phi(F_a)}{F_a} \exp\left(-\sum_{i=1}^{t} \frac{1}{q_i} + O\left(\sum_{q \ge q_1} \frac{1}{q^2}\right)\right) \\
&= \frac{\phi(F_a)}{F_a} \exp\left(O\left(\frac{t}{q_1}\right)\right) \\
&= \frac{\phi(F_a)}{F_a} \exp\left(O\left(\frac{1}{\log p}\right)\right)
\end{aligned}$$

$$= \quad \frac{\phi(F_a)}{F_a}\left(1+O\left(\frac{1}{\log p}\right)\right).$$

It implies that, if $p > \exp(\kappa\varepsilon^{-1})$, where $\kappa > 0$ is some absolute constant, then

$$\frac{\phi(F_{ap})}{F_{ap}} \in \left(\frac{s_2}{s_1}+0.5\varepsilon, \frac{s_2}{s_1}+1.5\varepsilon\right). \tag{2}$$

We now follow Washington's argument [4] to prove that there exist infinitely many primes p such that the base b representation of F_{ap} starts with s_1. For this, it is enough to show that

$$F_{ap} = s_1 b^N + \zeta_{ap} \qquad \text{for some integer} \qquad 0 \le \zeta_{ap} \le b^N - 1. \tag{3}$$

Note that since $q_1 \ge 2p-1$, it follows that if p is sufficiently large (say, $p > b^k$), then F_{ap} cannot equal $s_1 b^N$, and in particular, if in the above formula (3) we have $\zeta_{ap} \ge 0$, then in fact $\zeta_{ap} \ge 1$. The above formula (3) yields

$$\alpha^{ap} = \sqrt{5}s_1 b^N + \sqrt{5}\zeta_{ap} + \beta^{ap} = \sqrt{5}s_1 b^N\left(1+x_{ap}\right).$$

Since $\zeta_{ap} \ge 1$, it follows that $\sqrt{5}\zeta_{ap} + \beta^{ap} > \sqrt{5}-1 > 1$, and

$$0 < x_{ap} = \frac{\sqrt{5}\zeta_{ap}+\beta^{ap}}{\sqrt{5}s_1 b^N}.$$

So, if $x_{ap} \in (0, 1/b^k)$, and $p > b^k$ is sufficiently large, it then follows that $\zeta_{ap} < b^N$, which is what we want. Thus,

$$ap\log\alpha = \log(\sqrt{5}s_1) + N\log b + \log(1+x_{ap}),$$

or

$$ap\frac{\log\alpha}{\log b} - N - \left\lfloor\frac{\log(\sqrt{5}s_1)}{\log b}\right\rfloor = \left\{\frac{\log(\sqrt{5}s_1)}{\log b}\right\} + \frac{\log(1+x_{ap})}{\log b}. \tag{4}$$

Observe that $\log(\sqrt{5}s_1)/\log b$ is never an integer. Assume that k is sufficiently large such that

$$\frac{1}{b^k\log b} < 1 - \left\{\frac{\log(\sqrt{5}s_1)}{\log b}\right\}.$$

Then putting

$$\delta = \frac{\log(1+1/b^k)}{\log b},$$

we see that a relation like (4) with $x_{ap} \in (0, 1/b^k)$ holds provided that

$$\left\{p\left(\frac{a\log\alpha}{\log b}\right)\right\} \in \left(\left\{\frac{\log(\sqrt{5}s_1)}{\log b}\right\}, \left\{\frac{\log(\sqrt{5}s_1)}{\log b}\right\}+\delta\right). \tag{5}$$

The number $\gamma = a\log\alpha/\log b$ is irrational. By a result of Vinogradov [3], the sequence of fractional parts $\{p\gamma\}_{p \text{ prime}}$ is uniformly distributed. In particular, containment (5) holds for a positive proportion of primes p, and therefore certainly for infinitely many of them. So, indeed relation (3) holds. Relation (2) now shows that

$$\phi(F_{ap}) = s_2 b^N + \theta,$$

where

$$\theta \in \left(\zeta_{ap}\left(\frac{s_2}{s_1} + 0.5\varepsilon\right) + 0.5\varepsilon b^N, \zeta_{ap}\left(\frac{s_2}{s_1} + 1.5\varepsilon\right) + 1.5\varepsilon s_1 b^N\right).$$

Since $\varepsilon < 1/(15b^{2k})$, the above upper bound is

$$\begin{aligned}\zeta_{ap}\left(\frac{s_2}{s_1} + 1.5\varepsilon\right) + 1.5\varepsilon s_1 b^N &< (b^N - 1)\left(\frac{b^k - 1}{b^k} + \frac{0.1}{b^k}\right) + \frac{0.1(b^k - 1)}{b^{2k}} b^N \\ &< b^N - 1,\end{aligned}$$

where the last inequality above is implied by

$$\frac{1}{9} < \frac{b^N - 1}{b^N},$$

which holds true for all $b \geq 2$ and $N \geq 1$. This completes the proof of the theorem.

3. Comments

It was shown in [1] that with $\sigma(m)$ being the sum of divisors of the positive integer m, the ratio $\sigma(F_n)/F_n$ is dense in $[1, \infty)$. The present method now shows that there are infinitely many positive integers n such that the base b representation of F_n starts with the digits of s_1 and the base b representation of $\sigma(F_n)$ starts with the digits of s_2. Also, one may replace the Fibonacci sequence F_n in the above statements with some other sequence u_n for which it has been proved that $\phi(u_n)/u_n$ and $u_n/\sigma(u_n)$, respectively, are dense in $[0, 1]$. For example, one can take $u_n = 2^n - 1$ (see [1]) and the main result of this paper still holds provided that b is not a power of 2. We give no further details.

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