Normic continued fractions in totally and tamely ramified extensions of local fields

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Abstract

The goal of this paper is to introduce a new way of constructing continued fractions in a Galois, totally and tamely ramified extension of local fields. We take a set of elements of a special form using the norm of that extension and we show that the set such defined is dense in the field by the means of continued fractions.

1 Introduction

A ring A is a discrete valuation ring (DVR) if it has a unique maximal ideal $\mathbb{m}_A$, it is a principal ideal domain, but not a field. The residue field of A is the quotient field $\overline{\mathbb{k}}_A = A/\mathbb{m}_A$. Recall that a complete discrete valuation ring is a DVR that is complete with respect to the topology in which $\{\mathbb{m}_A^n\}_{n\geq 1}$ forms a basis of open neighborhoods of 0; that is, every series $\sum_{j=0}^{\infty} a_j\pi^j$ converges to an element of A, where π is a generator (often called *uniformizer*) of the (principal) maximal ideal $\mathbb{m}_A$.

Throughout this paper, $\mathbb{k}$ denotes a local field with a discrete valuation $v_{\mathbb{k}}$, which is a field of fractions of a complete discrete valuation ring $A_{\mathbb{k}}$ [7, §2, P.3], with finite residue class fields. Its maximal ideal is $\pi_{\mathbb{k}}$, its finite residue field is $\overline{\mathbb{k}} = A_{\mathbb{k}}/\pi_{\mathbb{k}}$, and $U_{\mathbb{k}} = A_{\mathbb{k}} - \pi_{\mathbb{k}}$ is the multiplicative group

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of invertible elements of $A_{\Bbbk}$. The local fields are the p-adic fields, which are finite extensions of the field $\mathbb{Q}_p$ of p-adic numbers (characteristic char $= 0$), and the finite extensions of the power series field $\mathbb{F}_p((x))$ (case char $= p > 0$); these are also locally compact, but we do not need that here. We refer to [7, 3], for example, for more on this topic.

If $\mathbb{K}$ is a finite extension of $\Bbbk$ (here, we write this as $\Bbbk \hookrightarrow \mathbb{K}$), we denote by $A_{\mathbb{K}}$ the integral closure of $A_{\Bbbk}$ in $\mathbb{K}$. We define $v_{\mathbb{K}}$, $\Pi_{\mathbb{K}}$, $U_{\mathbb{K}}$, $\overline{\mathbb{K}}$ as before. We will always assume that $\Bbbk \hookrightarrow \mathbb{K}$ is Galois, totally and tamely ramified extension. The ramification index of $\mathbb{K}/\Bbbk$, which is the degree of this extension will be denoted by e. We also assume that $v_{\Bbbk}$ is the restriction to $\Bbbk$ of $v_{\mathbb{K}}$, so we will use the same notation v for both of them. Choose $\Pi \in \mathbb{K}$, $\pi \in \Bbbk$ prime elements, such that $\Pi^e = \pi$ (see [5, Theorem 5.11]). Denoting the norm of $\mathbb{K}/\Bbbk$ by $N_{\mathbb{K}/\Bbbk}$, it is known that

$$v(x) = \frac{1}{e} v\left(N_{\mathbb{K}/\Bbbk}(x)\right), \ \forall \ x \in \mathbb{K}.$$

and we may assume that $v(\Pi) = 1$ and $v(\pi) = e$.

For easy writing, we use the notation $[\alpha, \beta, \gamma, \ldots]$ to mean

$$\alpha + \cfrac{1}{\beta + \cfrac{1}{\gamma + \cfrac{1}{\ddots}}}.$$

We want to mention that there are several nonequivalent definitions of continued fractions in the the field $\mathbb{Q}_p$ of p-adic numbers (see [1, 2] and the references therein). There are similarities as well as differences between these definitions and the classical real continued fractions. Among other continued fractions approaches, we want to mention the expansion of $\alpha \in \mathbb{Q}_p$ in the form

$$\alpha = [b_0, b_1, \ldots],$$

where $b_j \in \mathbb{Z}\left[\frac{1}{p}\right] \cap (0, p)$ (see Ruban [6]), and $b_j \in \mathbb{Z}\left[\frac{1}{p}\right] \cap \left(-\frac{p}{2}, \frac{p}{2}\right)$ (see Browkin [1, 2] and the references therein).

The groups of norms in such extensions play a very important role in class field theory. The goal of this paper is to introduce a new way of constructing continued fractions in a Galois, totally and tamely ramified extension of local fields $\mathbb{K}/\Bbbk$. We take a set of elements of a special form using the norm of that extension and we show that the defined set is dense in the larger field $\mathbb{K}$ by the means of continued fractions. This will give a glance to the "topological distance" between the set of norms of $\mathbb{K}/\Bbbk$ and $\mathbb{K}$. The approximation will be exact, and we will give the degree of the approximation as exact as we can by our method. In the last section we solve an equation in two variables using our continued fraction expansion.

We take $A\cup\{0\}$ to be a complete system of representatives of $\overline{\mathbb{K}}=\overline{\Bbbk}$, such that $A\subset A_{\Bbbk}$, $A^p=A$ where p is the (prime) characteristic of the residue field $\overline{\Bbbk}$. A has the structure of a group that is isomorphic to $\overline{\Bbbk}-\{0\}=\overline{\mathbb{K}}-\{0\}$ [4, Theorem 4.10]. Put

$$\begin{aligned}\Re := \big\{[\Pi^{p_1}N_0c_0,\dots,\Pi^{p_s}N_sc_s] \mid p_i\in(1-e)\mathbb{Z},\ c_i\in A^{1-e}\\ \text{and } \exists x_i\in\mathbb{K},\ N_{\mathbb{K}/\Bbbk}(x_i)=N_i,\ i=1,\dots,s\big\}.\end{aligned}$$

We define the *choice map* $c:U_{\mathbb{K}}\to A^*=A-\{0\}$ by $c(u):=a$, where a is the unique element of A such that $u\equiv a \pmod{\Pi}$ [4, Theorem 4.10]. The map c has the following properties:

(i) c is surjective and $c_{|A}=1_A$.

(ii) $c(u_1u_2)=c(u_1)c(u_2)$.

(iii) $c(u^{-1})=c(u)^{-1}$.

2 The normic continued fractions approach

We shall need the following lemma.

Lemma 2.1. *We have*

$$v\left(1+\Pi x-N_{\mathbb{K}/\Bbbk}(1+\Pi x)\right)\geq 1+v(x),\ \textit{whenever } v(x)\geq 0.$$

Proof. We have

$$\begin{aligned}1+\Pi x-N_{\mathbb{K}/\Bbbk}(1+\Pi x) &= 1+\Pi x-(1+\Pi x)(1+\Pi^{(1)}x)\cdots(1+\Pi^{(e-1)}x)\\ &= \Pi x-Tr_{\mathbb{K}/\Bbbk}(\Pi x)-\sum\Pi^{(i)}\Pi^{(j)}x^{(i)}x^{(j)}-\cdots\end{aligned}$$

where $x^{(i)},\Pi^{(i)}$ are the conjugates of x,Π in the extension. Since $v(\Pi^{(i)})=v(\Pi)$ and $v(x^{(i)})=v(x)$ for all conjugates $\Pi^{(i)}$ of Π and $x^{(i)}$ of x, we get

$$v\left(1+\Pi x-N_{\mathbb{K}/\Bbbk}(1+\Pi x)\right)\geq\min\left(v(\Pi x),v\left(Tr_{\mathbb{K}/\Bbbk}(\Pi x)\right),\dots\right)=1+v(x)$$

when $v(x)\geq 0$. We have used here the fact that we deal with local fields, hence with Henselian fields (fields where Hensel's lemma holds, that is, a simple root in a residue field can be lifted in the field above). $\square$

Take an element $\alpha \in \mathbb{K} - \{0\}$, and define the (finite or infinite) sequences $\{\alpha_n\}_n$, $\{a_n\}_n$, $\{u_n\}_n$ as follows:

$$\alpha_0 := \alpha, a_0 := N_{\mathbb{K}/\Bbbk}(\alpha), u_0 := \alpha \Pi^{-v(\alpha)}$$

If α_n, a_n, u_n are defined, then

$$\alpha_{n+1} := \Big(\alpha_n - c(u_n)^{1-e}\Pi^{(1-e)v(\alpha_n)} N_{\mathbb{K}/\Bbbk}(\alpha_n)\Big)^{-1}, \tag{1}$$

(if the inverse exists, otherwise the sequence "*terminates*" at n)

$$a_{n+1} := N_{\mathbb{K}/\Bbbk}(\alpha_{n+1}), u_{n+1} := \alpha_{n+1}\Pi^{-v(\alpha_{n+1})},$$

where c is the choice map defined in Section 1. Putting

$$\alpha_n = \Pi^{v(\alpha_n)} u_n = \Pi^{v(\alpha_n)} c(u_n) u'_n$$

where u'_n is a unit in $U_{\mathbb{K}}$ which starts with 1 in the canonical expansion after powers of Π and coefficients in A, that is, $u'_n = 1 + \Pi x_n$ and $v(x_n) \geq 0$, we see that (1) can be rewritten in the following form:

$$\alpha_{n+1} = (c(u_n))^{-1}\Pi^{-v(\alpha_n)} \left(u'_n - N_{\mathbb{K}/\Bbbk}(u'_n)\right)^{-1}. \tag{2}$$

Thus, the sequence terminates if $u'_n - N_{\mathbb{K}/\Bbbk}(u'_n) = 0$ (we will deal with this condition in Theorem 3.5).

Our intuition tells us that $\alpha \neq 0$ can be expanded as

$$c(u_0)^{1-e} a_0 \Pi^{(1-e)v(\alpha_0)} + \cfrac{1}{c(u_1)^{1-e} a_1 \Pi^{(1-e)v(\alpha_1)} + \cfrac{1}{c(u_2)^{1-e} a_2 \Pi^{(1-e)v(\alpha_2)} + \cfrac{1}{\ddots}}}$$

and proving this and other basic properties will be our goal in the main section of this paper.

3 The results

We start with a lemma on the valuation of α_n.

Lemma 3.1. *With the notations of the previous section, let*

$$t_n := v\left({u_n}' - N_{\mathbb{K}/\Bbbk}({u_n}')\right).$$

We assume that $N_{\mathbb{K}/\Bbbk}({u_n}') \neq {u_n}'$, hence $t_n < \infty$. Then

$$v(\alpha_n \alpha_{n+1}) = -t_n < 0, \ \textit{for all } n \in \mathbb{N}. \tag{3}$$

Furthermore,

$$v(\alpha_{n+1}) = -t_n + t_{n-1} + \cdots + (-1)^n t_0 + (-1)^n v(\alpha_0),\ \textit{for all } n \in \mathbb{N}.$$

Proof. We first observe that α_{n+1} exists since $N_{\mathbb{K}/\mathbb{k}}({u_n}') \neq {u_n}'$. The first claim is immediate from Lemma 2.1 and equation (2). The last claim follows by induction. □

We will define now the approximation of elements of $\mathbb{K}$ with elements of $\Re$. Take

$$p_{-1} := 1,\ q_{-1} := 0,\ p_0 := a_0 c(u_0)^{1-e}\Pi^{(1-e)v(\alpha_0)},\ q_0 := 1, \tag{4}$$

and

$$\begin{aligned} p_{n+1} &:= a_{n+1}c(u_{n+1})^{1-e}\Pi^{(1-e)v(\alpha_{n+1})}p_n + p_{n-1},\\ q_{n+1} &:= a_{n+1}c(u_{n+1})^{1-e}\Pi^{(1-e)v(\alpha_{n+1})}q_n + q_{n-1}, \end{aligned} \tag{5}$$

assuming that α_{n+1} defined by (1) exists. We will call $\left\{\frac{p_n}{q_n}\right\}_{n\in\mathbb{N}\cup\{-1\}}$ the *convergents* of α and we observe that they belong to the set $\Re$.

Lemma 3.2. *We have*

$$q_{n+1}p_n - p_{n+1}q_n = (-1)^n.$$

Proof. Follows from the definitions (4) and (5) of p_n and q_n. □

Theorem 3.3. *Let $\alpha_0 \in \mathbb{K}^*$. We have $v(q_0) = 0$ and*

$$\begin{aligned} v(p_n) &= v(\alpha_0\alpha_1\cdots\alpha_n)\\ v(q_n) &= v(\alpha_1\alpha_2\ldots\alpha_n) \leq -\left[\frac{n+1}{2}\right] - \varepsilon\ v(\alpha_0),\ \textit{for all } n > 0, \end{aligned} \tag{6}$$

where $\varepsilon = 0,\ 1$, if n is even, respectively, odd.

Proof. The first assertion follows from (4) and the second claim will be proved by induction. Obviously, from (4) and (5) we get $v(p_0) = v(\alpha_0)$ and $v(p_1) = v(\alpha_0\alpha_1)$. Now we show that

$$v(p_{n+1}) = v(\alpha_0\alpha_1\cdots\alpha_{n+1}),$$

using the induction assumption. So,

$$v(p_{n+1}) = v(a_{n+1}c(u_{n+1})^{(1-e)}\Pi^{(1-e)v(\alpha_{n+1})}p_n + p_{n-1})$$

$$
\begin{aligned}
&= \min\{v(\alpha_0 \cdots \alpha_{n+1}), v(\alpha_0 \cdots \alpha_{n-1})\} \\
&= v(\alpha_0 \alpha_1 \cdots \alpha_{n+1}),
\end{aligned}
$$

since $v(\alpha_n \alpha_{n+1}) = -t_n < 0$, according to the Lemma 3.1.

The second claim of (5) will also be proved by induction. From (5), for $n = 1$ we have

$$v(q_1) = (1-e)v(\alpha_1) + v(a_1) + v(q_0) = v(\alpha_1) + v(q_0) = v(\alpha_1).$$

Suppose that the assertion is true for $q_1, \dots, q_n,$ for $n \geq 2$. Then,

$$
\begin{aligned}
v(q_{n+1}) &= v\left(c(u_{n+1})^{1-e} a_{n+1} \Pi^{(1-e)v(\alpha_{n+1})} q_n + q_{n-1}\right) \\
&= v(\alpha_{n+1}) + v(q_n) = v(\alpha_1 \alpha_2 \cdots \alpha_{n+1}),
\end{aligned}
$$

since

$$
\begin{aligned}
v(a_{n+1} c(u_{n+1})^{1-e} \Pi^{(1-e)v(\alpha_{n+1})} q_n) &= (1-e)v(\alpha_{n+1}) + v(a_{n+1}) + v(q_n) \\
&= v(\alpha_{n+1}) + v(q_n) = v(\alpha_1 \alpha_2 \cdots \alpha_{n+1}) \\
&< v(\alpha_1 \alpha_2 \cdots \alpha_{n-1}) = v(q_{n-1}),
\end{aligned}
$$

using (3).

We now show the inequality (5) satisfied by $v(q_n)$. From Lemma 3.1 and the previous result of this theorem we have

$$
\begin{aligned}
v(q_{2m}) &= v(\alpha_1 \alpha_2) + \cdots + v(\alpha_{2m-1} \alpha_{2m}) \\
&= -t_1 - t_2 - \cdots t_{2m-1} \leq -m
\end{aligned}
$$

and

$$
\begin{aligned}
v(q_{2m+1}) &= v(\alpha_0 \alpha_1) + \cdots + v(\alpha_{2m} \alpha_{2m+1}) - v(\alpha_0) \\
&= -t_0 - t_1 - \cdots - t_{2m} - v(\alpha_0) \leq -(m+1) - v(\alpha_0).
\end{aligned}
$$

The theorem is shown. □

Now we will study the behavior of the sequence $\left\{\frac{p_n}{q_n}\right\}_{n \in \mathbb{N} \cup \{-1\}}$. We shall prove now that our sequence is Cauchy and, consequently, it has a limit.

Theorem 3.4. *The sequence* $\left\{\frac{p_n}{q_n}\right\}_{n \in \mathbb{N} \cup \{-1\}}$ *is convergent and its limit is* α.

Proof. First observe that

$$v\left(\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n}\right) = v\left(\frac{(-1)^{n+1}}{q_n q_{n+1}}\right) = -v(q_n q_{n+1})$$
$$= v(\alpha_0) + t_0 + t_1 + \cdots + t_n \geq n + 1 + v(\alpha_0)$$

and

$$v\left(\frac{p_s}{q_s} - \frac{p_r}{q_r}\right) \geq \min\left(v\left(\frac{p_s}{q_s} - \frac{p_{s-1}}{q_{s-1}}\right), \ldots, v\left(\frac{p_{r+1}}{q_{r+1}} - \frac{p_r}{q_r}\right)\right)$$
$$= v(\alpha_0) + t_0 + t_1 + \cdots + t_r \to \infty \text{ as } \ s, r \to \infty$$

assuming, without loss of generality, that $s \geq r$.

Next, take

$$v\left(\alpha - \frac{p_n}{q_n}\right) = v\left(\frac{(-1)^n}{q_n(\alpha_{n+1}q_n + q_{n-1})}\right)$$
$$= -v(q_n) - v(\alpha_{n+1}q_n + q_{n-1})$$

since

$$\alpha = \frac{\alpha_{n+1}p_n + p_{n-1}}{\alpha_{n+1}q_n + q_{n-1}},$$

which follows from our definition (1) of α_n. Now set $w_{n+1} := \alpha_{n+1}q_n + q_{n-1}$ and estimate

$$\begin{aligned} w_{n+1} &= \alpha_{n+1}(q_n + \alpha_{n+1}^{-1}q_{n-1}) \\ &= \alpha_{n+1}\left(q_n + \left(\alpha_n - a_n c(u_n)^{1-e}\Pi^{(1-e)v(\alpha_n)}\right) q_{n-1}\right) \\ &= \alpha_{n+1}\left(q_n + \alpha_n q_{n-1} - a_n c(u_n)^{1-e}\Pi^{(1-e)v(\alpha_n)} q_{n-1}\right) \\ &= \alpha_{n+1}(\alpha_n q_{n-1} + q_{n-2}) = \alpha_{n+1}w_n = \alpha_1 \cdots \alpha_{n+1}. \end{aligned} \tag{7}$$

Hence

$$v\left(\alpha - \frac{p_n}{q_n}\right) = -v(q_n) - v(\alpha_1 \cdots \alpha_{n+1}) = -v(q_n q_{n+1}) \to \infty,$$

as $n \to \infty$, so α is the limit of our sequence. □

It is known that in the classical case, finite continued fractions with integer terms represent rational numbers. We investigate the same problem next for our continued fraction expansion.

Theorem 3.5. *The sequence $\{\alpha_n\}_n$ is finite if and only if there exists n such that*

$$\alpha_n = a\xi_{e-1}\Pi^{v(\alpha_n)}, \tag{8}$$

where $a \in A$ and ξ_{e-1} is an $(e-1)$–root of unity in $\Bbbk$.

Proof. Our sequence terminates if and only if there exists n such that

$$u_n - c(u_n)^{1-e}N_{\mathbb{K}/\Bbbk}(u_n) = 0.$$

This is the same as saying that

$$1 + \Pi x_n = N_{\mathbb{K}/\Bbbk}\,(1 + \Pi x_n) \in \Bbbk$$

where $u_n = c(u_n)\,(1 + \Pi x_n)$, for an element $x_n \in \mathbb{K}$ with $v(x_n) \geq 0$. So there exists an element $x'_n \in \Bbbk$ such that

$$x_n = x'_n\Pi^{e-1} \ \text{ and } \ v(x'_n) \geq 0.$$

We also must have the condition

$$N_{\mathbb{K}/\Bbbk}(1 + \pi x'_n) = 1 + \pi x'_n$$

fulfilled, which is equivalent to (knowing that $(1 + \pi x'_n) \in \Bbbk$)

$$N_{\mathbb{K}/\Bbbk}(1 + \pi x'_n) = (1 + \pi x'_n)^e = 1 + \pi x'_n. \tag{9}$$

Obviously, $1 + \pi x'_n$ can never be zero, so the only case we could have (9) is when

$$(1 + \pi x'_n)^{e-1} = 1,$$

hence u_n must be of the form

$$u_n = c(u_n)\xi_{e-1} \ \text{ and } \alpha_n = c(u_n)\xi_{e-1}\Pi^{v(\alpha_n)} \tag{10}$$

where $\xi_{e-1} = 1 + \Pi x_n \in \Bbbk$ is an $(e-1)$-root of unity. □

Remark 3.6. *In the p-adic field $\mathbb{Q}_p$, the condition (8) could be re-written as $\mathrm{Log_p}(\alpha_n) = 0$, in terms of the analytic continuation of the usual logarithm, called the Iwasawa logarithm $\mathrm{Log_p}$, (for example, if $x \in \mathbb{Z}_p^*$, then $\mathrm{Log_p(x)} = \frac{1}{\mathrm{p}-1}\mathrm{Log_p}(\mathrm{x}^{\mathrm{p}-1}) = \frac{1}{1-\mathrm{p}}\sum_{\mathrm{k}\geq 1}\frac{(1-\mathrm{x}^{\mathrm{p}-1})^{\mathrm{k}}}{\mathrm{k}}$), but this gives no other indication on the set of elements of the form (8).*

4 An application

We will use our continued fraction process to solve an equation, namely

$$ax + by + d = 0 \tag{11}$$

where

$$\gcd(a, b) = 1 \text{ and } a, b, d \in A_{\mathbb{K}}$$

are such that

$$\frac{a}{b} - c\left(\frac{a}{b}\Pi^{v(\frac{b}{a})}\right)^{(1-e)} \ \Pi^{(1-e)v(\frac{a}{b})} \ N_{\mathbb{K}/\mathbb{k}}\left(\frac{a}{b}\right) = \xi_{e-1}$$

is an $(e-1)$-root of unity in a Galois, totally and tamely ramified extension $\mathbb{k} \hookrightarrow \mathbb{K}$ of degree e and

$$v(d) \geq v\left(\frac{b}{a}\right).$$

We are looking for solutions in $A_{\mathbb{K}}$. Suppose that we found a solution of (11), say (x_0, y_0). Thus

$$ax_0 + by_0 + d = 0. \tag{12}$$

Subtracting (12) from (11) we get

$$a(x - x_0) + b(y - y_0) = 0$$

or

$$y - y_0 = \frac{a}{b}(x_0 - x).$$

Since $\gcd(a, b) = 1$ we must have $b \big| (x - x_0)$ in $A_{\mathbb{K}}$, so

$$\begin{aligned} x =& x_0 - bt \\ y =& y_0 + at \end{aligned} \tag{13}$$

for some $t \in A_{\mathbb{K}}$. So we have showed that if (x, y) is solution of (11), then it must satisfies (13) for some $t \in A_{\mathbb{K}}$. Conversely, we take (x_1, y_1) of the form (13) and we show that it is a solution of (11). We have

$$ax_1 + by_1 + d = ax_0 + by_0 + d + abt_1 - abt_1 = ax_0 + by_0 + d = 0.$$

We must find now a particular solution of (11). This can be done using our continued fraction expansion for $\alpha_0 = a/b$. We will use the notations of Section 2. Since

$$\alpha_1 = \left(\frac{a}{b} - c\left(\frac{a}{b}\Pi^{v(\frac{b}{a})}\right)^{(1-e)} \ \Pi^{(1-e)v(\frac{a}{b})} N_{\mathbb{K}/\mathbb{k}} \ \left(\frac{a}{b}\right)\right)^{-1}$$

is an $(e-1)$-root of unity this implies that α_2 does not exist. Hence

$$\frac{p_1}{q_1} = \frac{a}{b}$$

and

$$\frac{p_1}{q_1} - \frac{p_0}{q_0} = \frac{1}{q_1 q_0}$$

or

$$\frac{a}{b} - \frac{p_0}{q_0} = \frac{1}{b q_0}.$$

Furthermore, $aq_0 - bp_0 = 1$ or $aq_0 - bp_0 - 1 = 0$. Multiplying the previous relation by $-d$ we get

$$-adq_0 + bdp_0 + d = 0$$

and taking

$$\begin{aligned} x_0 &= -\,dq_0 = -d \\ y_0 &= dp_0 = d\ a_0\ c\left(\frac{a}{b}\ \Pi^{v(\frac{a}{b})}\right)^{1-e}\ \Pi^{(1-e)v(\frac{a}{b})} \end{aligned} \tag{14}$$

we have produced a particular solution of (11) and consequently, we have found all the solution of our equation in algebraic integers of the extension $\mathbb{k} \hookrightarrow \mathbb{K}$. However we must make sure that our particular solution is in $A_{\mathbb{K}}$, so we have to check that both $v(x_0)$ and $v(y_0)$ are positive. We have no trouble with x_0 since $q_0 = 1$ and $d \in A_{\mathbb{K}}$. For y_0 we get

$$v(y_0) = v(d) + v(p_0) = v(d) + v\left(a_0 c(u_0)^{1-e}\Pi^{(1-e)v(\frac{a}{b})}\right) = v(d) + v\left(\frac{a}{b}\right) \geq 0$$

and we have solved the problem.

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