16.7

Surface Integrals

In this section, we will learn about:

Integration of different types of surfaces.

Suppose a surface S has a vector equation

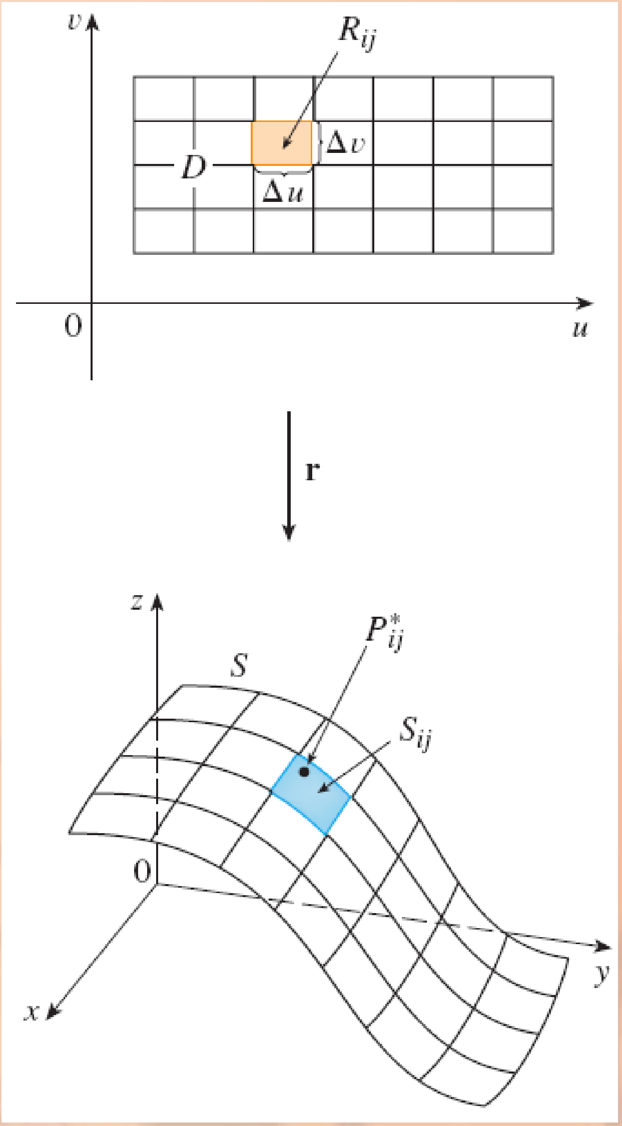
$$\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k}$$

$$(u, v) \in D$$

PARAMETRIC SURFACES

- We first assume that the parameter domain D is a rectangle and we divide it into subrectangles R_{ij} with dimensions Δu and Δv.
- Then, the surface S is divided into corresponding patches S_{ij}.
- We evaluate f at a point P_{ij}^* in each patch, multiply by the area ΔS_{ij} of the patch, and form the Riemann sum

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij}^{*})\,\Delta S_{ij}$$



SURFACE INTEGRAL **Equation 1**

Then, we take the limit as the number of patches increases and define the surface integral of *f* over the surface *S* as:

$$\iint_S f(x,y,z)\,dS = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij}^*)\,\Delta S_{ij}$$

- Analogues to: The definition of a line integral (Definition 2 in Section 16.2);The definition of a double integral (Definition 5 in Section 15.1)
- To evaluate the surface integral in Equation 1, we approximate the patch area ΔS_{ij} by the area of an approximating parallelogram in the tangent plane.

In our discussion of surface area in Section 16.6, we made the approximation

$$\Delta S_{ij} \approx |\mathbf{r}_u \times \mathbf{r}_v| \, \Delta u \, \Delta v$$

where:

$$\mathbf{r}_u = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k} \qquad \mathbf{r}_v = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}$$

are the tangent vectors at a corner of S_{ij}.

If the components are continuous and $\mathbf{r}_u$ and $\mathbf{r}_v$ are nonzero and nonparallel in the interior of *D*, it can be shown from Definition 1—even when *D* is not a rectangle—that:

$$\iint_S f(x, y, z)\, dS = \iint_D f(\mathbf{r}(u, v))\, |\mathbf{r}_u \times \mathbf{r}_v|\, dA$$

This should be compared with the formula for a line integral:

$$\int_C f(x, y, z)\,ds = \int_a^b f(\mathbf{r}(t))\,|\mathbf{r}'(t)|\,dt$$

Observe also that:

$$\iint_S 1\,dS = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\,dA = A(S)$$

Example 1

Compute the surface integral $\iint_S x^2 \, dS$, where S is the unit sphere

$x^2 + y^2 + z^2 = 1$.

As in Example 4 in Section 16.6,

we use the parametric representation

$$x = \sin \Phi \cos \theta,\ y = \sin \Phi \sin \theta,\ z = \cos \Phi$$

$$0 \le \Phi \le \pi,\ 0 \le \theta \le 2\pi$$

- That is,
  $\mathbf{r}(\Phi, \theta) = \sin \Phi \cos \theta\, \mathbf{i} + \sin \Phi \sin \theta\, \mathbf{j} + \cos \Phi\, \mathbf{k}$
- we can compute: $|\mathbf{r}_\Phi \times \mathbf{r}_\theta| = \sin \Phi$

Therefore, by Formula 2,

$$\iint_S x^2 \, dS$$

$$= \iint_D (\sin\phi\cos\theta)^2 \, |\mathbf{r}_\phi \times \mathbf{r}_\theta| \, dA$$

$$= \int_0^{2\pi} \int_0^{\pi} (\sin^2\phi \cos^2\theta \sin\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \cos^2\theta \, d\theta \int_0^{\pi} \sin^3\phi \, d\phi$$

$$= \int_0^{2\pi} \tfrac{1}{2}(1+\cos 2\theta) \, d\theta \int_0^{\pi} (\sin\phi - \sin\phi\cos^2\phi) \, d\phi$$

$$= \tfrac{1}{2}\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_0^{2\pi} \left[-\cos\phi + \tfrac{1}{3}\cos^3\phi\right]_0^{\pi}$$

$$= \frac{4\pi}{3}$$

APPLICATIONS

For example, suppose a thin sheet (say, of aluminum foil) has:

- The shape of a surface *S*.
- The density (mass per unit area) at the point (x, y, z) as $\rho(x, y, z)$.

CENTER OF MASS

Then, the total mass of the sheet is: $m = \iint_S \rho(x, y, z)\, dS$

The center of mass is: $\left(\bar{x}, \bar{y}, \bar{z}\right)$

$$\bar{x} = \frac{1}{m} \iint_S x\, \rho(x, y, z)\, dS$$

where $$\bar{y} = \frac{1}{m} \iint_S y\, \rho(x, y, z)\, dS$$

$$\bar{z} = \frac{1}{m} \iint_S z\, \rho(x, y, z)\, dS$$

Any surface S with equation $z = g(x, y)$ can be regarded as a parametric surface with parametric equations

$$x = x \qquad y = y \qquad z = g(x, y)$$

- So, we have:

$$\mathbf{r}_x = \mathbf{i} + \left(\frac{\partial g}{\partial x}\right)\mathbf{k} \qquad \mathbf{r}_y = \mathbf{j} + \left(\frac{\partial g}{\partial y}\right)\mathbf{k}$$

Equation 3

- Thus, $\mathbf{r}_x \times \mathbf{r}_y = -\dfrac{\partial g}{\partial x}\mathbf{i} - \dfrac{\partial g}{\partial x}\mathbf{j} + \mathbf{k}$

and $|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2 + 1}$

- Formula 2 becomes:

$$\iint_S f(x, y, z)\, dS$$

$$= \iint_D f(x, y, g(x, y)) \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA$$

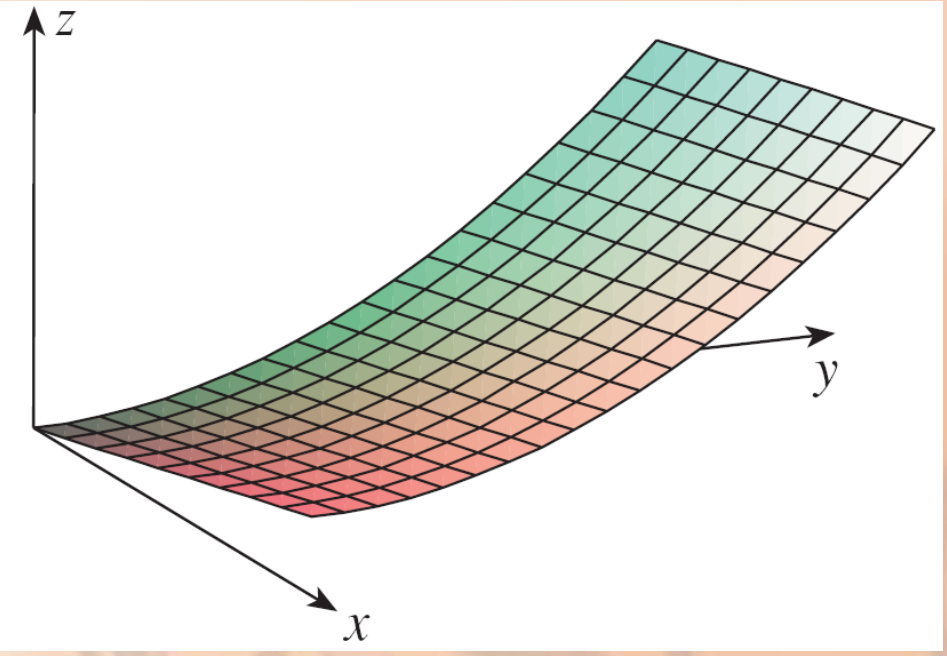
Example 2

Evaluate $\iint_S y\,dS$ where S is the surface $z = x + y^2$, $0 \le x \le 1$, $0 \le y \le 2$

- $\dfrac{\partial z}{\partial x} = 1$

  and

  $\dfrac{\partial z}{\partial y} = 2y$



So, Formula 4 gives:

$$\iint_S y\,dS = \iint_D y\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$$

$$= \int_0^1 \int_0^2 y\sqrt{1+1+4y^2}\,dy\,dx$$

$$= \int_0^1 dx\,\sqrt{2}\int_0^2 y\sqrt{1+2y^2}\,dy$$

$$= \sqrt{2}\left(\tfrac{1}{4}\right)\tfrac{2}{3}\left(1+2y^2\right)^{3/2}\Big]_0^2 = \frac{13\sqrt{2}}{3}$$

If S is a piecewise-smooth surface—a finite union of smooth surfaces $S_1, S_2, \ldots, S_n$ that intersect only along their boundaries—then the surface integral of f over S is defined by:

$$\iint_S f(x,y,z)\,dS$$

$$= \iint_{S_1} f(x,y,z)\,dS + \cdots\cdots + \iint_{S_n} f(x,y,z)\,dS$$

Evaluate $\iint_S z\, dS$, where S is the surface whose:

- Sides S_1 are given by the cylinder $x^2 + y^2 = 1$.
- Bottom S_2 is the disk $x^2 + y^2 \leq 1$ in the plane $z = 0$.
- Top S_3 is the part of the plane $z = 1 + x$ that lies above S_2.

For S_1, we use θ and z as parameters (Example 5 in Section 16.6) and write its parametric equations as:

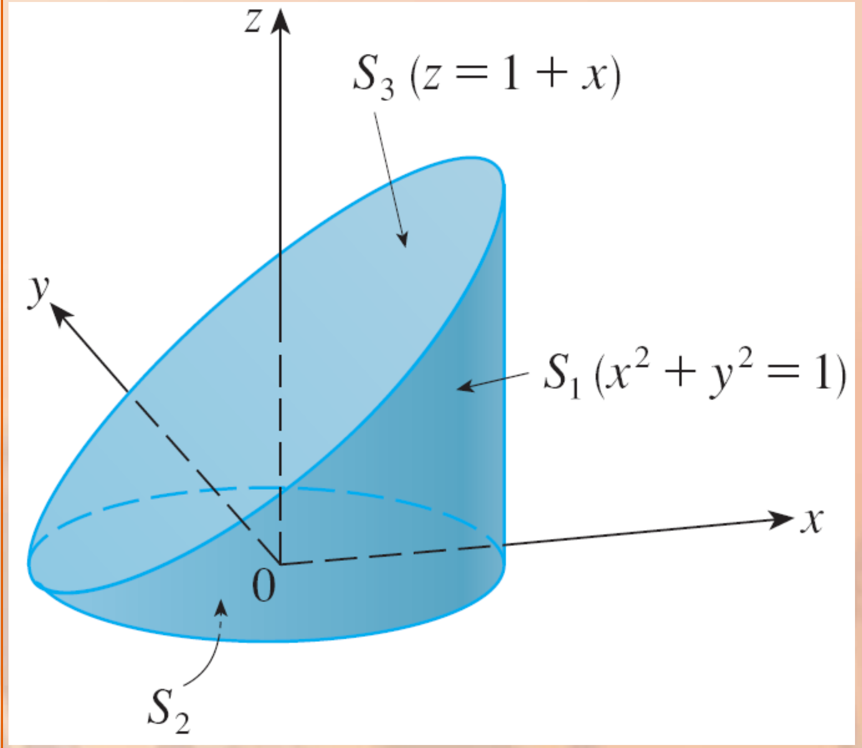
$$x = \cos\theta$$

$$y = \sin\theta$$

$$z = z$$

where:

- $0 \leq \theta \leq 2\pi$
- $0 \leq z \leq 1 + x = 1 + \cos\theta$



Therefore,

$$\mathbf{r}_\theta \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$$

and

$$|\mathbf{r}_\theta \times \mathbf{r}_z| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

Thus, the surface integral over S_1 is:

$$\iint_{S_1} z\,dS = \iint_D z\,|\mathbf{r}_\theta \times \mathbf{r}_z|\,dA$$

$$= \int_0^{2\pi} \int_0^{1+\cos\theta} z\,dz\,d\theta$$

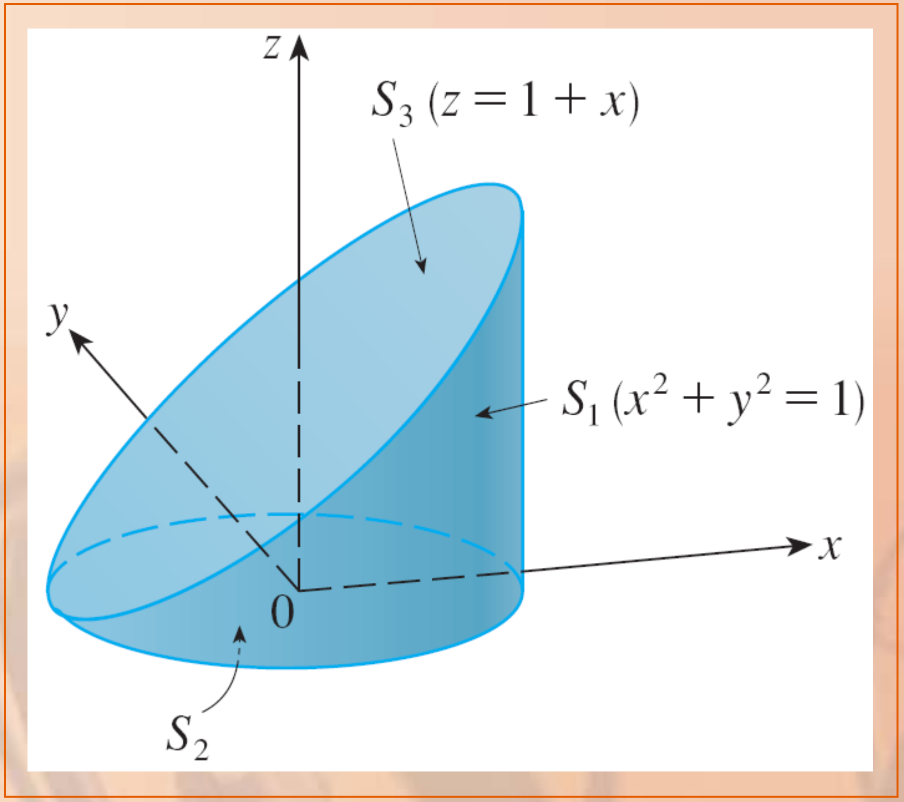
$$= \int_0^{2\pi} \tfrac{1}{2}(1+\cos\theta)^2\,d\theta$$

$$= \tfrac{1}{2}\int_0^{2\pi} \left[1 + 2\cos\theta + \tfrac{1}{2}(1+\cos 2\theta)\right] d\theta$$

$$= \tfrac{1}{2}\left[\tfrac{3}{2}\theta + 2\sin\theta + \tfrac{1}{4}\sin 2\theta\right]_0^{2\pi} = \frac{3\pi}{2}$$

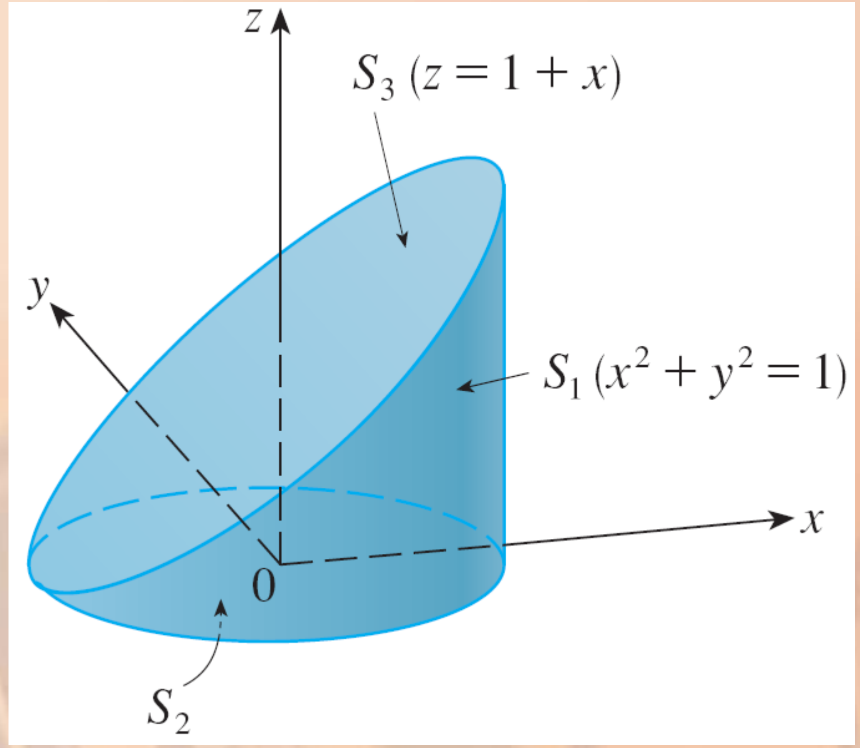
Since S_2 lies in the plane $z = 0$, we have:

$$\iint_{S_2} z\, dS = \iint_{S_2} 0\, dS = 0$$



S_3 lies above the unit disk D and is part of the plane $z = 1 + x$.

- So, taking $g(x, y) = 1 + x$ in Formula 4 and converting to polar coordinates, we have the following result.



Example 3

$$\iint_{S_3} z\,dS = \iint_D (1+x)\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$$

$$= \int_0^{2\pi}\int_0^1 (1+r\cos\theta)\sqrt{1+1+0}\,r\,dr\,d\theta$$

$$= \sqrt{2}\int_0^{2\pi}\int_0^1 (r+r^2\cos\theta)\,dr\,d\theta$$

$$= \sqrt{2}\int_0^{2\pi}\left(\tfrac{1}{2}+\tfrac{1}{3}\cos\theta\right)d\theta$$

$$= \sqrt{2}\left[\frac{\theta}{2}+\frac{\sin\theta}{3}\right]_0^{2\pi} = \sqrt{2}\,\pi$$

Therefore,

$$\iint_S z\,dS = \iint_{S_1} z\,dS + \iint_{S_2} z\,dS + \iint_{S_3} z\,dS$$

$$= \frac{3\pi}{2} + 0 + \sqrt{2}\,\pi$$

$$= \left(\tfrac{3}{2} + \sqrt{2}\right)\pi$$

SURFACE INTEGRALS OF VECTOR FIELDS

Suppose that S is an oriented surface with unit normal vector **n**.

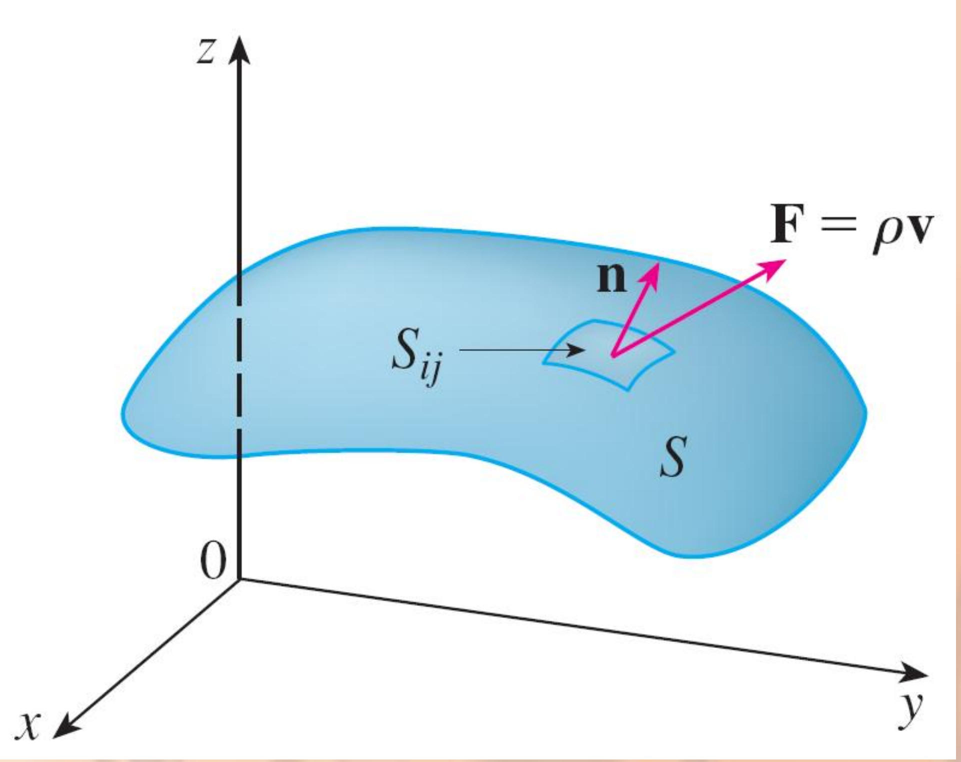
Then, imagine a fluid with density $\rho(x, y, z)$ and velocity field $\mathbf{v}(x, y, z)$ flowing through S.

- Think of S as an imaginary surface that doesn't impede the fluid flow—like a fishing net across a stream.

Then, the rate of flow (mass per unit time) per unit area is $\rho\mathbf{v}$.

SURFACE INTEGRALS OF VECTOR FIELDS

If we divide S into small patches S_{ij} , then S_{ij} is nearly planar.



SURFACE INTEGRALS OF VECTOR FIELDS

So, we can approximate the mass of fluid crossing S_{ij} in the direction of the normal **n** per unit time by the quantity

$$(\rho \mathbf{v} \cdot \mathbf{n}) A(S_{ij})$$

where ρ, **v**, and **n** are evaluated at some point on S_{ij}.

- Recall that the component of the vector $\rho\mathbf{v}$ in the direction of the unit vector **n** is $\rho\mathbf{v} \cdot \mathbf{n}$.

Summing these quantities and taking the limit, we get, according to Definition 1, the surface integral of the function $\rho\mathbf{v} \cdot \mathbf{n}$ over S:

$$\iint_S \rho\mathbf{v} \cdot \mathbf{n}\, dS$$

$$= \iint_S \rho(x, y, z)\ \mathbf{v}(x, y, z) \cdot \mathbf{n}(x, y, z)\, dS$$

- This is interpreted physically as the rate of flow through S.

If we write **F** = *ρ***v**, then **F** is also a vector field on $\mathbb{R}^3$. Then, the integral in Equation 7 becomes:

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS$$

A surface integral of this form occurs frequently in physics—even when **F** is not *ρ***v**.

It is called the surface integral (or flux integral) of **F** over *S*.

Definition 8

If **F** is a continuous vector field defined on an oriented surface *S* with unit normal vector **n**, then the surface integral of **F** over *S* is:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS$$

- This integral is also called the flux of **F** across *S*.

FLUX INTEGRAL

If S is given by a vector function $\mathbf{r}(u, v)$, then $\mathbf{n}$ is given by Equation 6.

- Then, from Definition 8 and Equation 2, we have (D is the parameters' domain):

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} dS$$

$$= \iint_D \left[\mathbf{F}(\mathbf{r}(u,v)) \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} \right] |\mathbf{r}_u \times \mathbf{r}_v| \, dA$$

- So,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) \, dA$$

- Find the flux of the vector field

$$\mathbf{F}(x, y, z) = z\,\mathbf{i} + y\,\mathbf{j} + x\,\mathbf{k}$$

across the unit sphere : $x^2 + y^2 + z^2 = 1$

- Using the parametric representation:

$\mathbf{r}(\Phi, \theta) = \sin \Phi \cos \theta\,\mathbf{i} + \sin \Phi \sin \theta\,\mathbf{j} + \cos \Phi\,\mathbf{k}$

$0 \leq \Phi \leq \pi \qquad 0 \leq \theta \leq 2\pi$

$\mathbf{F}(\mathbf{r}(\Phi, \theta)) = \cos \Phi\,\mathbf{i} + \sin \Phi \sin \theta\,\mathbf{j} + \sin \Phi \cos \theta\,\mathbf{k}$

From Example 10 in Section 16.6,

$\mathbf{r}_\Phi \times \mathbf{r}_\theta = \sin^2 \Phi \cos \theta\, \mathbf{i} + \sin^2 \Phi \sin \theta\, \mathbf{j} + \sin \Phi \cos \Phi\, \mathbf{k}$

Therefore, $\mathbf{F}(\mathbf{r}(\Phi, \theta)) \cdot (\mathbf{r}_\Phi \times \mathbf{r}_\theta) = \cos \Phi \sin^2 \Phi \cos \theta + \sin^3 \Phi \sin^2 \theta + \sin^2 \Phi \cos \Phi \cos \theta$

Then, by Formula 9, the flux is:

$$\iint_S \mathbf{F} \cdot d\mathbf{S}$$

$$= \iint_D \mathbf{F} \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta)\, dA$$

$$= \int_0^{2\pi} \int_0^{\pi} (2\sin^2\phi \cos\phi \cos\theta + \sin^3\phi \sin^2\theta)\, d\phi\, d\theta$$

$$= 2\int_0^{\pi} \sin^2\phi \cos\phi \, d\phi \int_0^{2\pi} \cos\theta \, d\theta + \int_0^{\pi} \sin^3\phi \, d\phi \int_0^{2\pi} \sin^2\theta \, d\theta$$

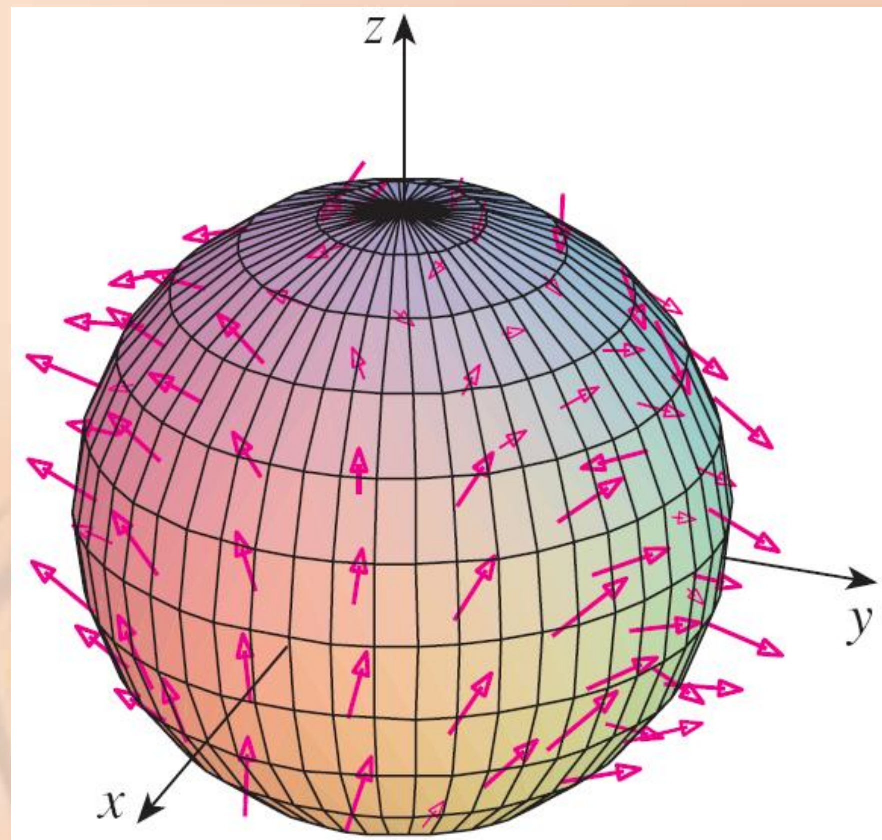
$$= 0 + \int_0^{\pi} \sin^3\phi \, d\phi \int_0^{2\pi} \sin^2\theta \, d\theta$$

$$= \frac{4\pi}{3}$$

- This is by the same calculation as in Example 1.

FLUX INTEGRALS

The figure shows the vector field **F** in Example 4 at points on the unit sphere.



If, for instance, the vector field in Example 4 is a velocity field describing the flow of a fluid with density 1, then the answer, $4\pi/3$, represents:

- The rate of flow through the unit sphere in units of mass per unit time.

In the case of a surface *S* given by a graph $z = g(x, y)$, we can think of *x* and *y* as parameters and use Equation 3 to write:

$$\mathbf{F}\cdot(\mathbf{r}_x \times \mathbf{r}_y) = (P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k})\cdot\left(-\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial y}\mathbf{j} + \mathbf{k}\right)$$

Formula 10

Thus, Formula 9 becomes:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R \right) dA$$

- This formula assumes the upward orientation of *S*.
- For a downward orientation, we multiply by –1.

Example 5

Evaluate

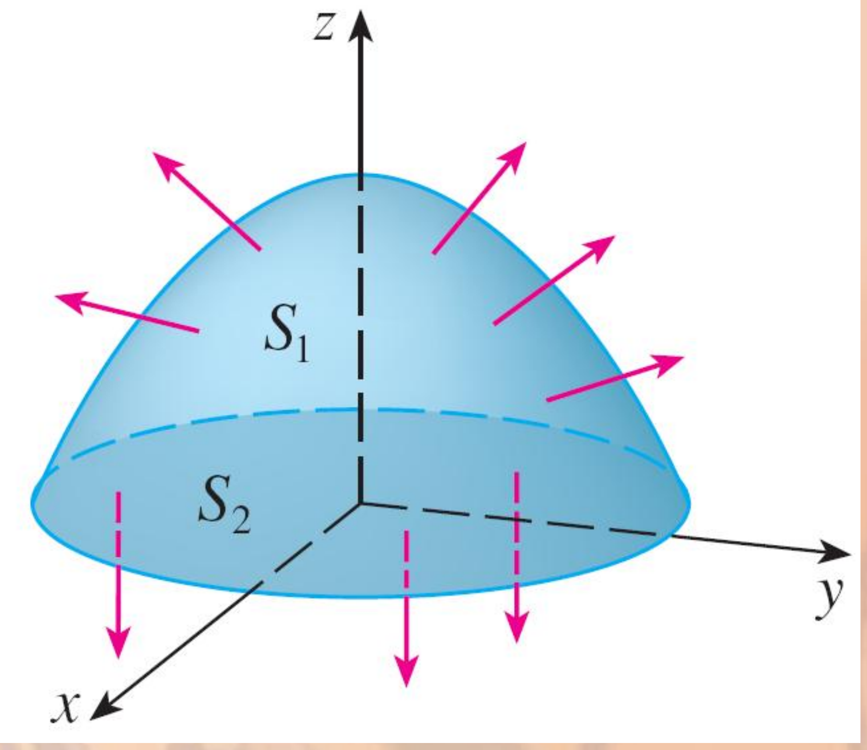
$$\iint_S \mathbf{F} \cdot d\mathbf{S}$$

where:

- $\mathbf{F}(x, y, z) = y\,\mathbf{i} + x\,\mathbf{j} + z\,\mathbf{k}$
- S is the boundary of the solid region E enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$.

S consists of:

- A parabolic top surface S_1.
- A circular bottom surface S_2.



Since S is a closed surface, we use the convention of positive (outward) orientation.

- This means that S_1 is oriented upward.
- So, we can use Equation 10 with D being the projection of S_1 on the xy-plane, namely, the disk $x^2 + y^2 \leq 1$.

On S_1,

$$P(x, y, z) = y$$

$$Q(x, y, z) = x$$

$$R(x, y, z) = z = 1 - x^2 - y^2$$

Also,

$$\frac{\partial g}{\partial x} = -2x \qquad \frac{\partial g}{\partial y} = -2y$$

So, we have:

$$\iint_{S_1} \mathbf{F} \cdot d\mathbf{S}$$

$$= \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R \right) dA$$

$$= \iint_D [-y(-2x) - x(-2y) + 1 - x^2 - y^2]\, dA$$

$$= \iint_D (1 + 4xy - x^2 - y^2)\, dA$$

Example 5

$$= \int_0^{2\pi} \int_0^1 (1 + 4r^2 \cos\theta \sin\theta - r^2)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 (r - r^3 + 4r^3 \cos\theta \sin\theta)\, dr\, d\theta$$

$$= \int_0^{2\pi} (\tfrac{1}{4} \cos\theta \sin\theta)\, d\theta$$

$$= \tfrac{1}{4}(2\pi) + 0$$

$$= \frac{\pi}{2}$$

The disk S_2 is oriented downward.

So, its unit normal vector is **n** = –**k**

and we have:

$$\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_2} \mathbf{F} \cdot (-\mathbf{k})\, dS = \iint_D (-z)\, dA$$

$$= \iint_D 0\, dA = 0$$

since $z = 0$ on S_2.

Finally, we compute, by definition, $\iint_S \mathbf{F} \cdot d\mathbf{S}$ as the sum of the surface integrals of **F** over the pieces S_1 and S_2:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_2} \mathbf{F} \cdot d\mathbf{S}$$

$$= \frac{\pi}{2} + 0 = \frac{\pi}{2}$$

APPLICATIONS

Although we motivated the surface integral of a vector field using the example of fluid flow, this concept also arises in other physical situations.

ELECTRIC FLUX

For instance, if **E** is an electric field (Example 5 in Section 16.1), the surface integral

$$\iint_S \mathbf{E} \cdot d\mathbf{S}$$

is called the electric flux of **E** through the surface *S*.

GAUSS'S LAW

Equation 11

One of the important laws of electrostatics is Gauss's Law, which says that the net charge enclosed by a closed surface *S* is:

$$Q = \varepsilon_0 \iint_S \mathbf{E} \cdot d\mathbf{S}$$

where ε_0 is a constant (called the permittivity of free space) that depends on the units used.

- In the SI system, $\varepsilon_0 \approx 8.8542 \times 10^{-12}\ C^2/N \cdot m^2$

Thus, if the vector field **F** in Example 4 represents an electric field, we can conclude that the charge enclosed by *S* is:

$$Q = 4\pi\varepsilon_0/3$$

Another application occurs in the study of heat flow.

- Suppose the temperature at a point (x, y, z) in a body is u(x, y, z).

HEAT FLOW

- Then, the heat flow is defined as the vector field $\mathbf{F} = -K \nabla u$

where K is an experimentally determined constant called the conductivity of the substance.

- Then, the rate of heat flow across the surface S in the body is given by the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S} = -K \iint_S \nabla u \cdot d\mathbf{S}$