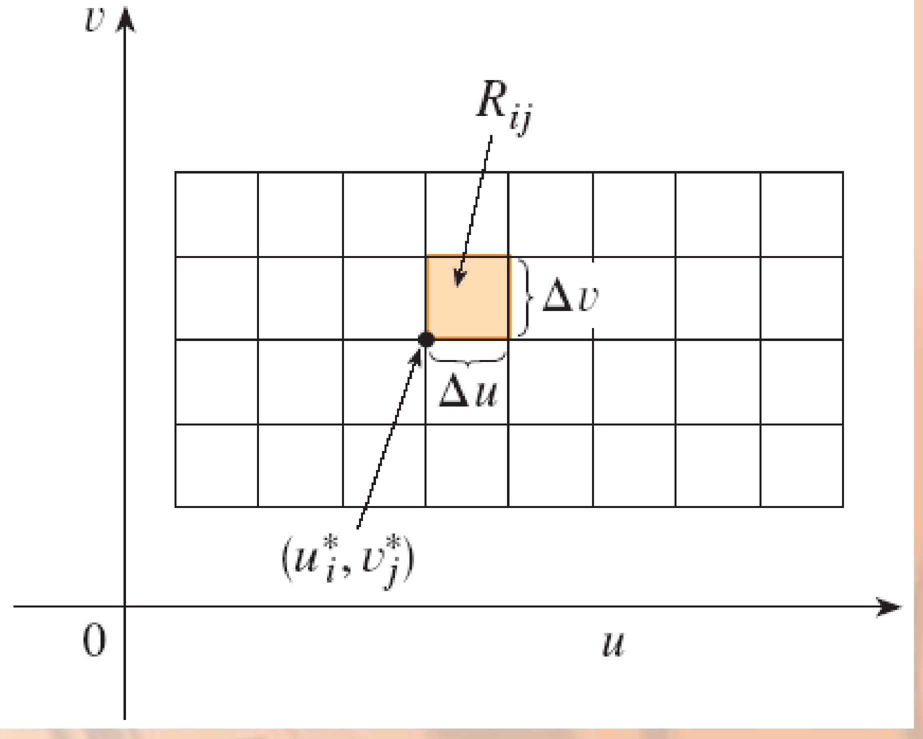
SURFACE AREAS

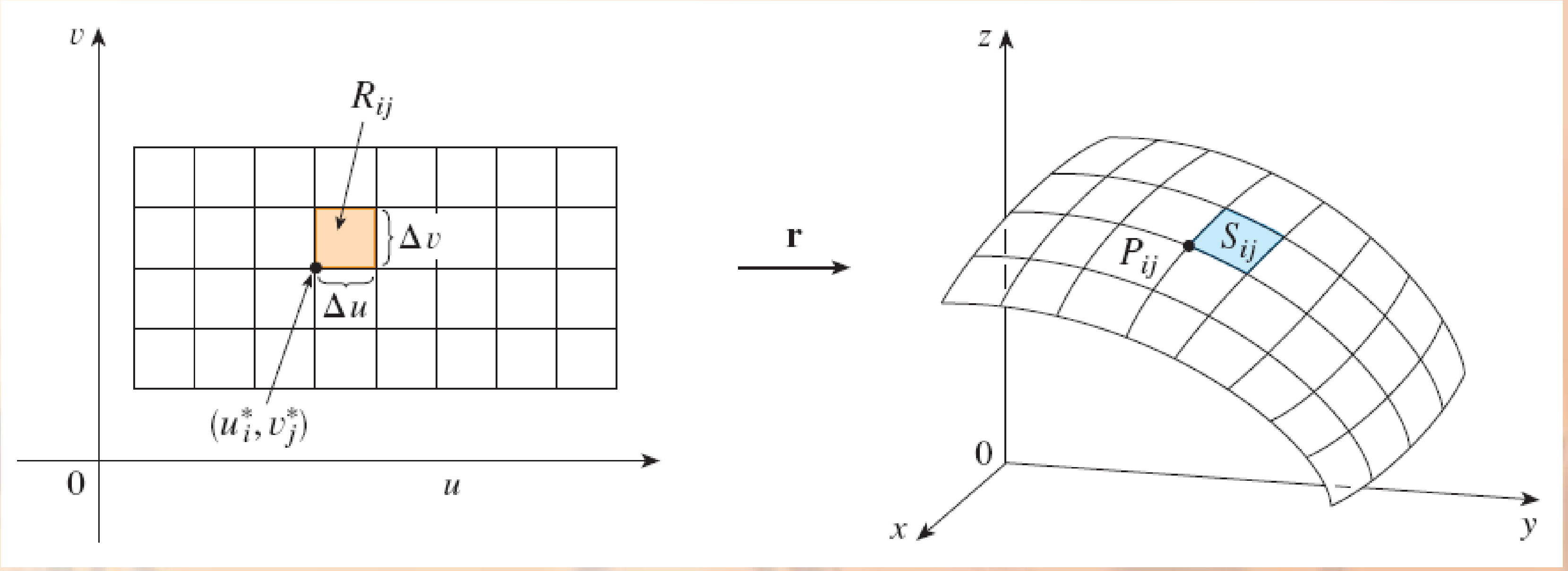
Let's choose (u_i*, v_j*) to be the lower left corner of R_{ij}.

For simplicity, we start by considering a surface whose parameter domain D is a rectangle, and we divide it into subrectangles R_{ij}.



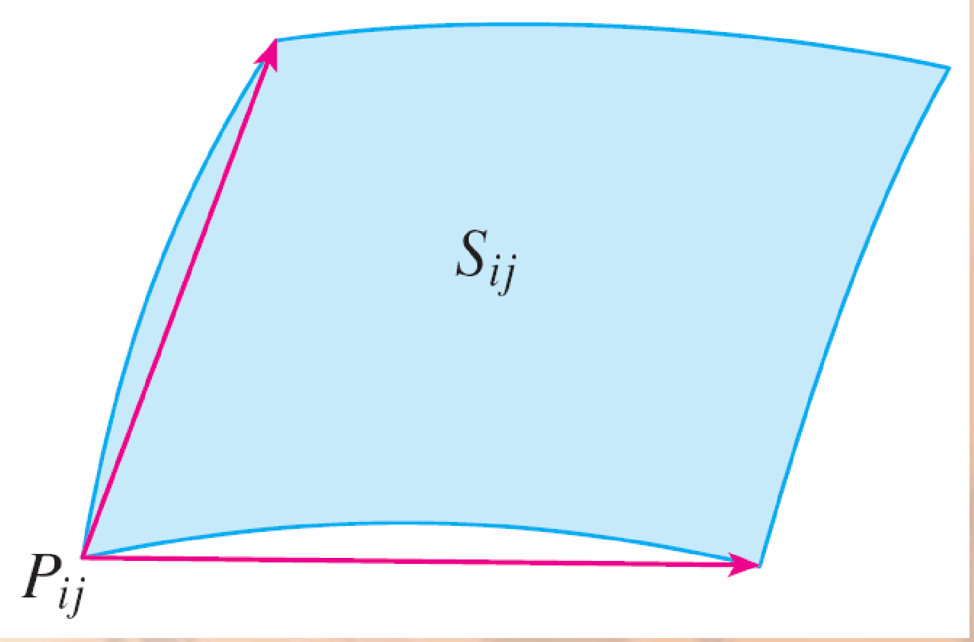
PATCH

The part S_{ij} of the surface S that corresponds to R_{ij} is called a patch and has the point P_{ij} with position vector $\mathbf{r}(u_i^*, v_j^*)$ as one of its corners.



SURFACE AREAS

- Let $\mathbf{r}_u^* = \mathbf{r}_u(u_i^*, v_j^*)$ and $\mathbf{r}_v^* = \mathbf{r}_v(u_i^*, v_j^*)$ be the tangent vectors at P_{ij}
- The figure shows how the two edges of the patch that meet at P_{ij} can be approximated by vectors.



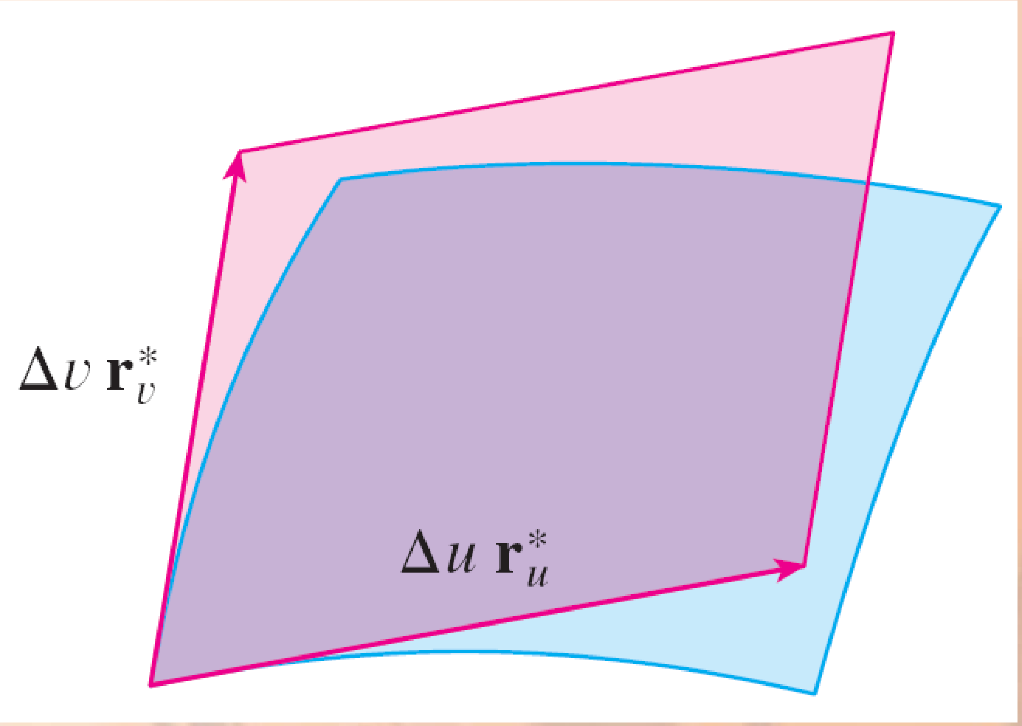
SURFACE AREAS

These vectors, in turn, can be approximated by the vectors $\Delta u\, \mathbf{r}_u^*$ and $\Delta v\, \mathbf{r}_v^*$ because partial derivatives can be approximated by difference quotients.

- So, we approximate S_{ij} by the parallelogram determined by the vectors $\Delta u\, \mathbf{r}_u^*$ and $\Delta v\, \mathbf{r}_v^*$.

This parallelogram is shown here.

- It lies in the tangent plane to S at P_{ij}.



SURFACE AREAS

- The area of this parallelogram is: $$|(\Delta u\, \mathbf{r}_u^*) \times (\Delta v\, \mathbf{r}_v^*)| = |\mathbf{r}_u^* \times \mathbf{r}_v^*|\, \Delta u\, \Delta v$$

So, an approximation to the area of *S* is: $$\sum_{i=1}^{m} \sum_{j=1}^{n} |\mathbf{r}_u^* \times \mathbf{r}_v^*|\, \Delta u\, \Delta v$$

- Our intuition tells us that this approximation gets better as we increase the number of subrectangles.
- Also, we recognize the double sum as a Riemann sum for the double integral $$\iint_D |r_u \times r_v|\, du\ dv$$

  - This motivates the following definition.

Suppose a smooth parametric surface *S* is:

- Given by **r**(*u*, *v*) = *x*(*u*, *v*) **i** + *y*(*u*, *v*) **j** + *z*(*u*, *v*) **k**
  (covered once (*u*, *v*) ∈ *D*
- Then, the surface area of *S* is

$$A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| \, dA$$

- where: $\mathbf{r}_u = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k}$ $\mathbf{r}_v = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}$

Find the surface area of a sphere of radius *a*.

- In Example 4, we found

$$x = a \sin \Phi \cos \theta, \quad y = a \sin \Phi \sin \theta, \quad z = a \cos \Phi$$

where the parameter domain is:

$$D = \{(\Phi, \theta) \mid 0 \le \Phi \le \pi, 0 \le \theta \le 2\pi)$$

We first compute the cross product of the tangent vectors:

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial x}{\partial \phi} & \dfrac{\partial y}{\partial \phi} & \dfrac{\partial z}{\partial \phi} \\ \dfrac{\partial x}{\partial \theta} & \dfrac{\partial y}{\partial \theta} & \dfrac{\partial z}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi \\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= a^2\sin^2\phi\cos\theta\,\mathbf{i} + a^2\sin^2\phi\sin\theta\,\mathbf{j} + a^2\sin\phi\cos\phi\,\mathbf{k}$$

Thus, $|r_\phi \times r_\theta|$

$$= \sqrt{a^4 \sin^4 \phi \cos^2 \theta + a^4 \sin^4 \phi \sin^2 \theta + a^4 \sin^2 \phi \cos^2 \phi}$$

$$= \sqrt{a^4 \sin^4 \phi + a^4 \sin^2 \phi \cos^2 \phi}$$

$$= a^2 \sqrt{\sin^2 \phi} = a^2 \sin \phi$$

since sin $\Phi \geq 0$ for $0 \leq \Phi \leq \pi$.

Hence, by Definition 6, the area of the sphere is:

$$A = \iint_D |r_\phi \times r_\theta| \, dA = \int_0^{2\pi} \int_0^{\pi} a^2 \sin \phi \, d\phi \, d\theta$$

$$= a^2 \int_0^{2\pi} d\theta \int_0^{\pi} \sin \phi \, d\phi$$

$$= a^2 (2\pi) 2 = 4\pi a^2$$

SURFACE AREA OF THE GRAPH OF A FUNCTION

Now, consider the special case of a surface S with equation $z = f(x, y)$, where (x, y) lies in D and f has continuous partial derivatives.

- Here, we take x and y as parameters.
- The parametric equations are:

$$x = x \qquad y = y \qquad z = f(x, y)$$

Thus, $\mathbf{r}_x = \mathbf{i} + \left(\frac{\partial f}{\partial x}\right)\mathbf{k}$ $\mathbf{r}_y = \mathbf{j} + \left(\frac{\partial f}{\partial y}\right)\mathbf{k}$

and $\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \frac{\partial f}{\partial x} \\ 0 & 1 & \frac{\partial f}{\partial y} \end{vmatrix} = -\frac{\partial f}{\partial x}\mathbf{i} - \frac{\partial f}{\partial y}\mathbf{j} + \mathbf{k}$

And, so, we have:

$$|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}$$

$$= \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}$$

Then, the surface area formula in Definition 6 becomes:

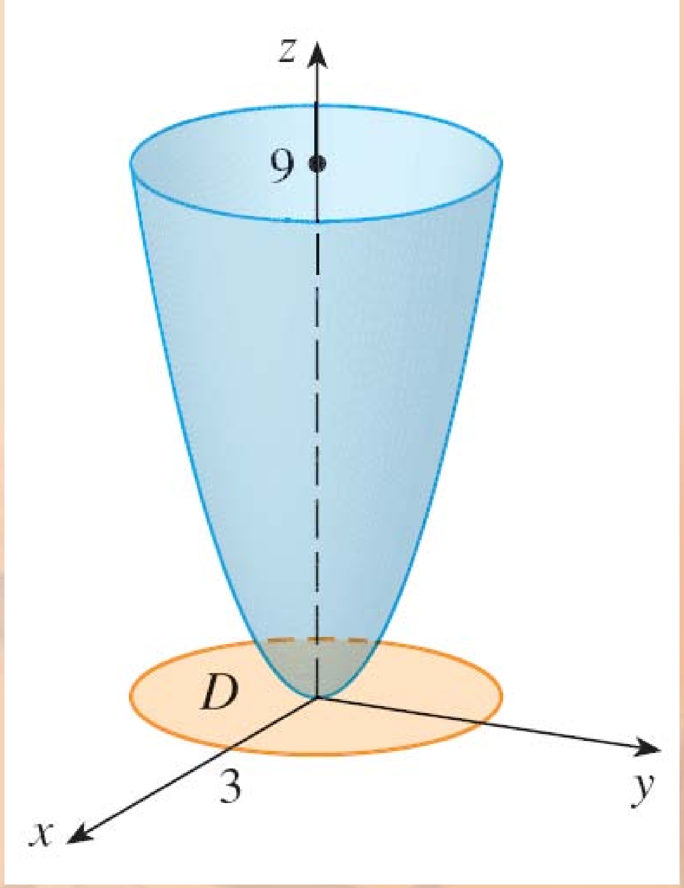
$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.

- The plane intersects the paraboloid in the circle

$$x^2 + y^2 = 9, \; z = 9$$

Therefore, the given surface lies above the disk *D* with center the origin and radius 3.



Using Formula 9, we have:

$$A = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

$$= \iint_D \sqrt{1 + (2x)^2 + (2y)^2}\, dA$$

$$= \iint_D \sqrt{1 + 4(x^2 + y^2)}\, dA$$

Converting to polar coordinates,

we obtain:

$$A = \int_0^{2\pi} \int_0^3 \sqrt{1+4r^2}\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^3 r\sqrt{1+4r^2}\, dr$$

$$= 2\pi\left(\tfrac{1}{8}\right)\tfrac{2}{3}(1+4r^2)^{3/2}\Big]_0^3$$

$$= \frac{\pi}{6}(37\sqrt{37}-1)$$