Then, the curl of **F** is the vector field on R^3 defined by:

$$\text{curl } \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

CURL

$$\nabla \times \mathbf{F}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

$$= \text{curl } \mathbf{F}$$

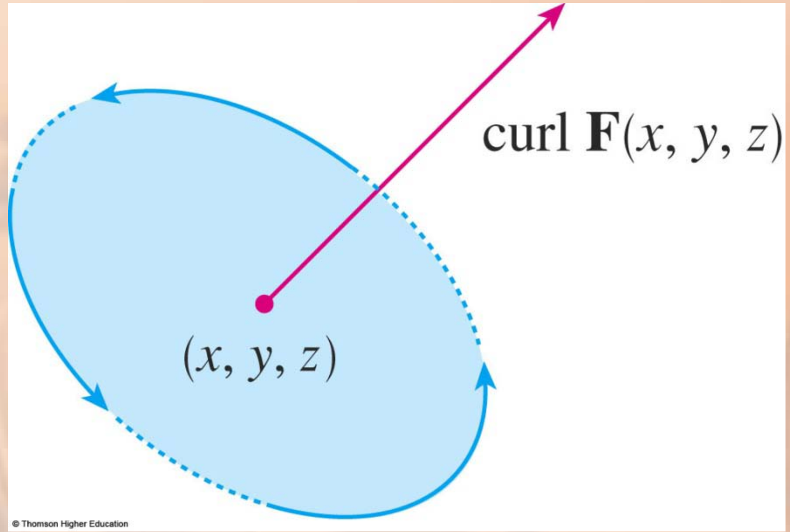
CURL

The reason for the name curl is that the curl vector is associated with rotations.

- One connection is explained in Exercise 37.
- Another occurs when **F** represents the velocity field in fluid flow (Example 3 in Section 16.1).

CURL

- Particles near (x, y, z) in the fluid tend to rotate about the axis that points in the direction of curl **F**(x, y, z).
- If curl **F** = **0** at a point P, the fluid is free from rotations at P.
- **F** is called irrotational at P: there is no whirlpool at P.
  - The length of this curl vector is a measure of how quickly the particles move around the axis.



curl F = 0 & curl F ≠ 0

If curl $\mathbf{F} = \mathbf{0}$, a tiny paddle wheel moves with the fluid but doesn't rotate about its axis.

If curl $\mathbf{F} \neq \mathbf{0}$, the paddle wheel rotates about its axis.

If **F** = P **i** + Q **j** + R **k** is a vector field on R^3 and $\partial P/\partial x$, $\partial Q/\partial y$, and $\partial R/\partial z$ exist, the divergence of **F** is the function of three variables defined by:

$$\text{div}\,\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Observe that:

- Curl **F** is a vector field.
- Div **F** is a scalar field.

DIVERGENCE

Equation 10

In terms of the gradient operator

$$\nabla = \left(\frac{\partial}{\partial x}\right)\mathbf{i} + \left(\frac{\partial}{\partial y}\right)\mathbf{j} + \left(\frac{\partial}{\partial z}\right)\mathbf{k}$$

the divergence of **F** can be written symbolically as the dot product of ∇ and **F**:

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F}$$

Theorem 11

If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is a vector field on R^3 and P, Q, and R have continuous second-order partial derivatives, then

$$\text{div curl } \mathbf{F} = 0$$

(Note the analogy with the scalar triple product: $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0$)

By the definitions of divergence and curl,

$$\begin{aligned}
&\text{div curl } \mathbf{F} \\
&= \nabla \cdot (\nabla \times \mathbf{F}) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \\
&= \frac{\partial^2 R}{\partial x\,\partial y} - \frac{\partial^2 Q}{\partial x\,\partial z} + \frac{\partial^2 P}{\partial y\,\partial z} - \frac{\partial^2 R}{\partial y\,\partial x} + \frac{\partial^2 Q}{\partial z\,\partial x} - \frac{\partial^2 P}{\partial z\,\partial y} = 0
\end{aligned}$$

- The terms cancel in pairs by Clairaut's Theorem.

Again, the reason for the name divergence can be understood in the context of fluid flow.

- If **F**(x, y, z) is the velocity of a fluid (or gas), div **F**(x, y, z) represents the net rate of change (with respect to time) of the mass of fluid (or gas) flowing from the point (x, y, z) per unit volume.

INCOMPRESSIBLE DIVERGENCE

In other words, div **F**(x, y, z) measures the tendency of the fluid to diverge from the point (x, y, z).

If div **F** = 0, **F** is said to be incompressible.

Another differential operator occurs when we compute the divergence of a gradient vector field ∇f.

- If *f* is a function of three variables, we have:

$$\operatorname{div}(\nabla f) = \nabla \cdot (\nabla f)$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

LAPLACE OPERATOR

•This expression occurs so often that we abbreviate it as $\nabla^2 f$.

•The operator $\nabla^2 = \nabla \cdot \nabla$ is called the Laplace operator due to its relation to Laplace's equation $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$

•We can also apply the Laplace operator

to a vector field $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$

in terms of its components:

$$\nabla^2\mathbf{F} = \nabla^2 P\,\mathbf{i} + \nabla^2 Q\,\mathbf{j} + \nabla^2 R\,\mathbf{k}$$