16

VECTOR CALCULUS

In this chapter, we study the calculus of vector fields.

- These are functions that assign vectors to points in space.

We define:

- Line integrals—which can be used to find the work done by a force field in moving an object along a curve.
- Surface integrals—which can be used to find the rate of fluid flow across a surface.

The connections between these new types of integrals and the single, double, and triple integrals we have already met are given by the higher-dimensional versions of the Fundamental Theorem of Calculus:

- Green’s Theorem
- Stokes’ Theorem
- Divergence Theorem

16.1
Vector Fields

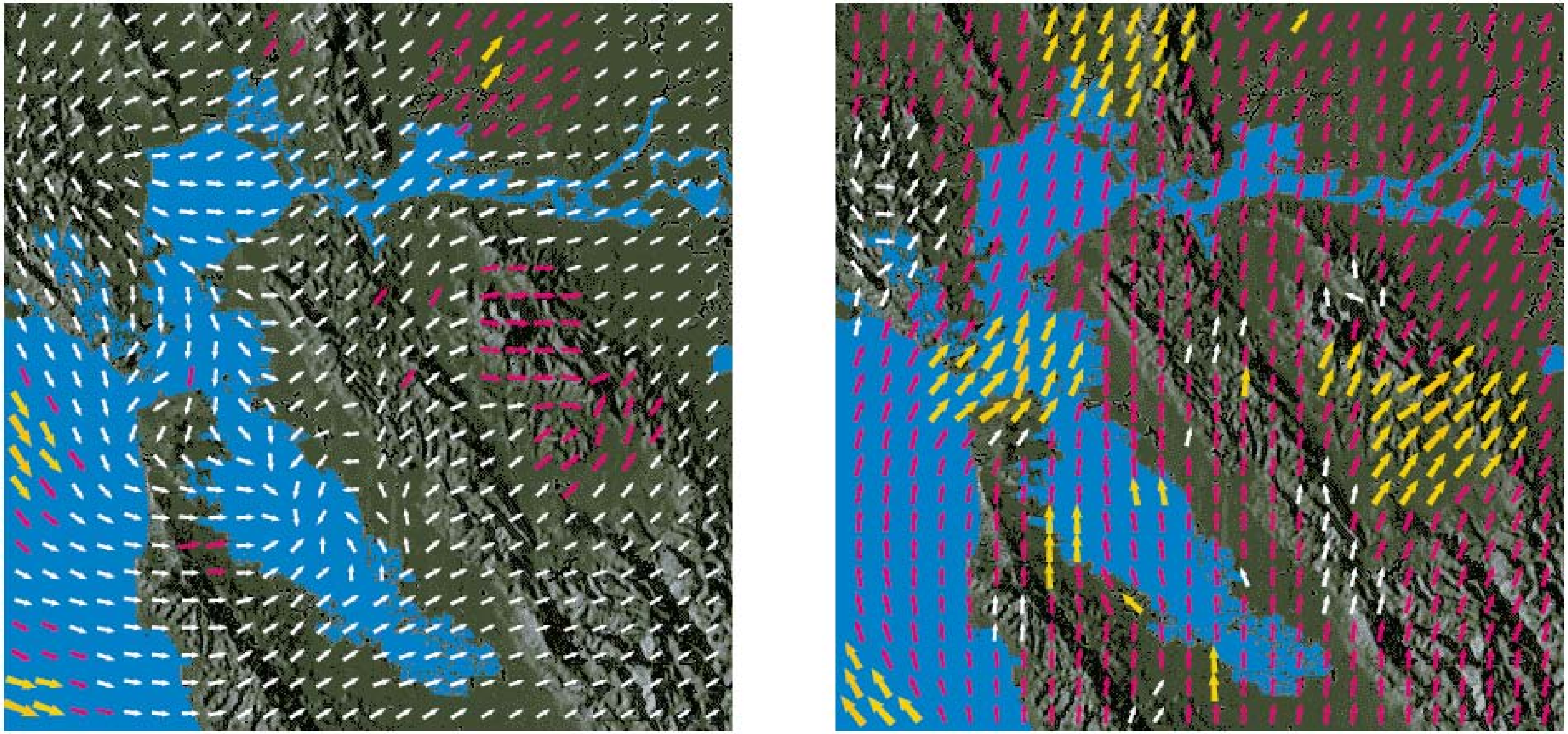
In this section, we will learn about:

Various types of vector fields.

VECTOR FIELDS

The vectors displayed are air velocity vectors.

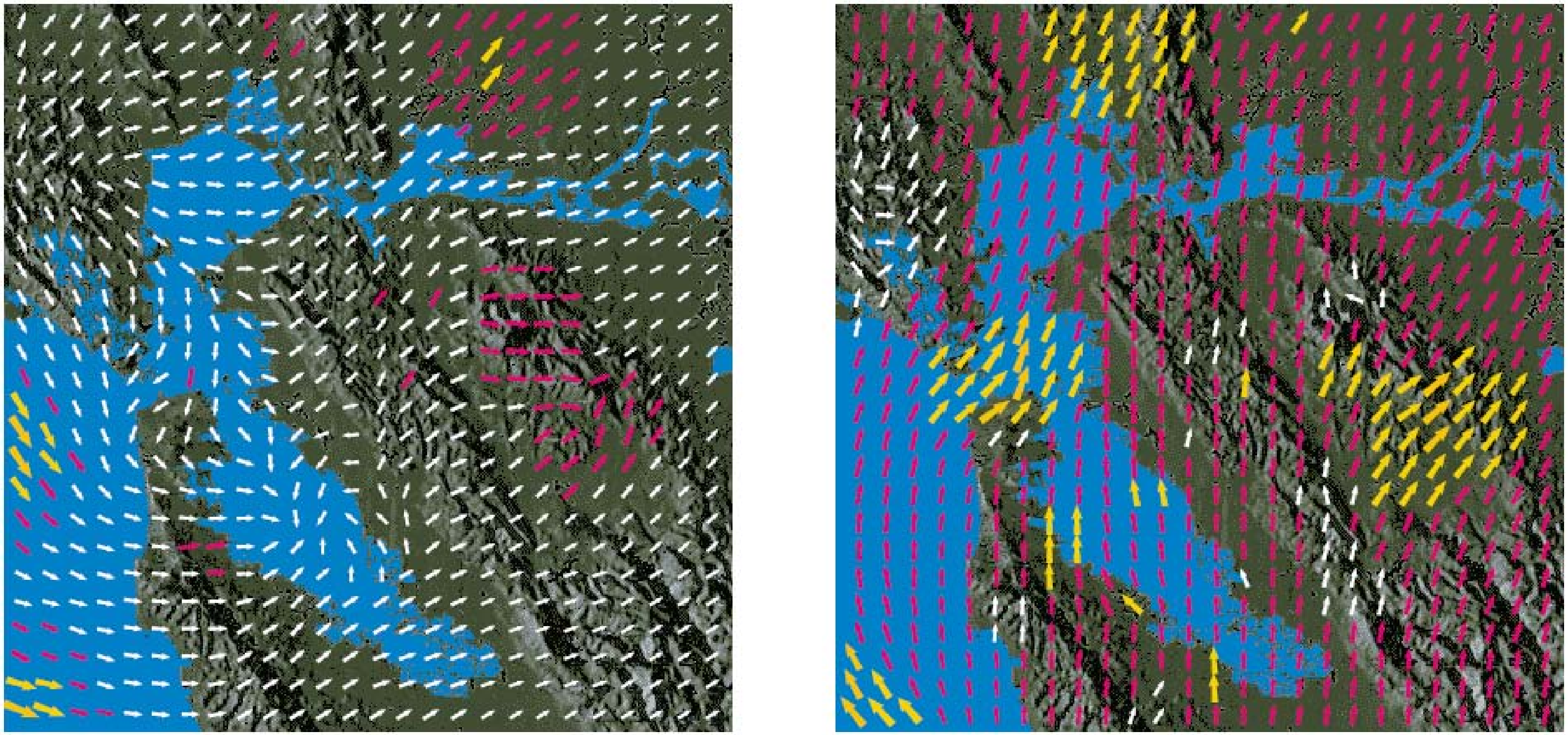
- They indicate the wind speed and direction at points 10 m above the surface elevation in the San Francisco Bay area.



(a) 12:00 AM, February 20, 2007 (b) 2:00 PM, February 21, 2007

VECTOR FIELDS

Notice that the wind patterns on consecutive days are quite different.



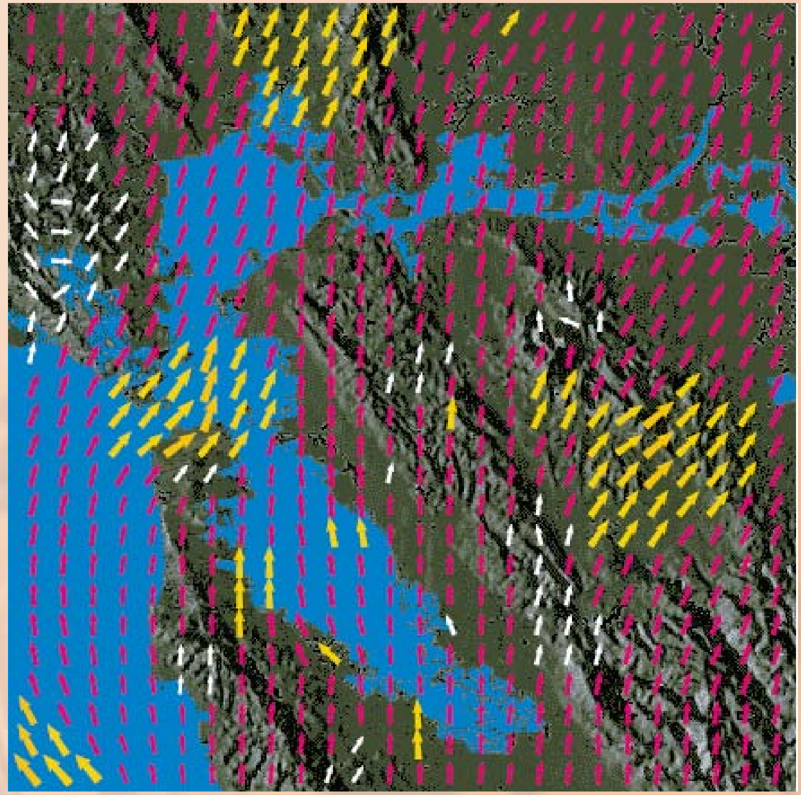
(a) 12:00 AM, February 20, 2007

(b) 2:00 PM, February 21, 2007

VELOCITY VECTOR FIELD

Associated with every point in the air, we can imagine a wind velocity vector.

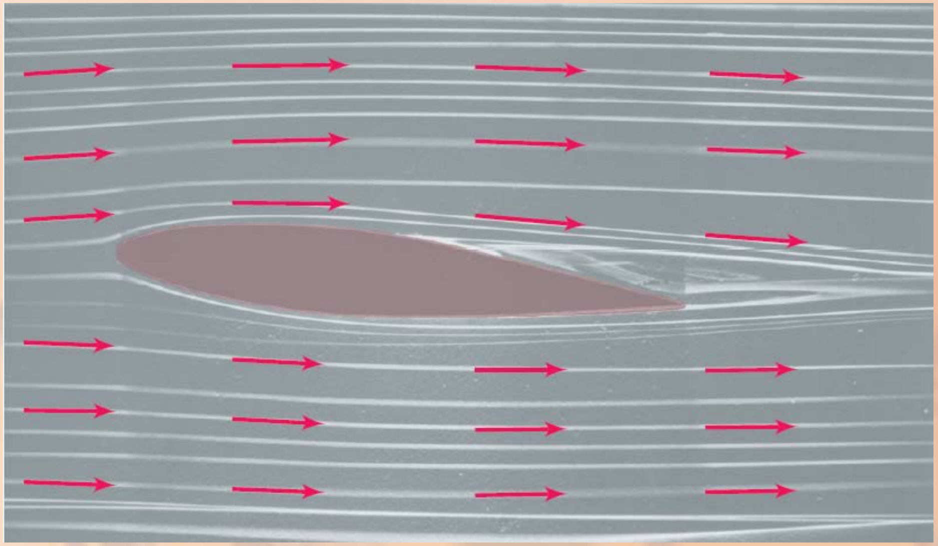
This is an example of a velocity vector field.



VELOCITY VECTOR FIELDS

Other examples of velocity vector fields are:

- Ocean currents
- Flow past an airfoil



FORCE FIELD

Another type of vector field, called a force field, associates a force vector with each point in a region.

- An example is the gravitational force field that we will look at in Example 4.

VECTOR FIELD

In general, a vector field is a function whose:

- Domain is a set of points in R^2 (or R^3).
- Range is a set of vectors in V_2 (or V_3).

VECTOR FIELD ON R^2

Let D be a set in R^2 (a plane region).

A vector field on R^2 is a function **F** that assigns to each point (x, y) in D a two-dimensional (2-D) vector $\mathbf{F}(x, y)$.

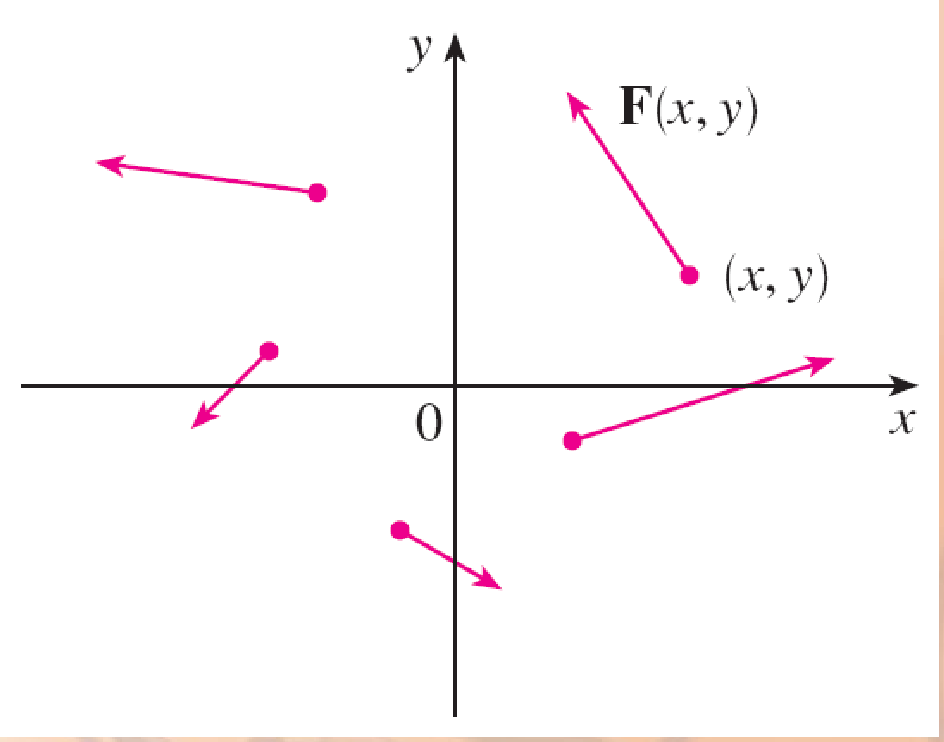
VECTOR FIELDS ON R^2

The best way to picture a vector field is to draw the arrow representing the vector $\mathbf{F}(x, y)$ starting at the point (x, y).

- Of course, it's impossible to do this for all points (x, y)

VECTOR FIELDS ON R^2

Still, we can gain a reasonable impression of **F** by doing it for a few representative points in *D*, as shown.



VECTOR FIELDS ON R^2

Since $\mathbf{F}(x, y)$ is a 2-D vector, we can write it in terms of its component functions P and Q as:

$$\mathbf{F}(x, y) = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j}$$

$$= \langle P(x, y), Q(x, y)\rangle$$

or, for short,

$$\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$$

SCALAR FIELDS

Notice that *P* and *Q* are scalar functions of two variables.

- They are sometimes called scalar fields to distinguish them from vector fields.

VECTOR FIELD ON R^3

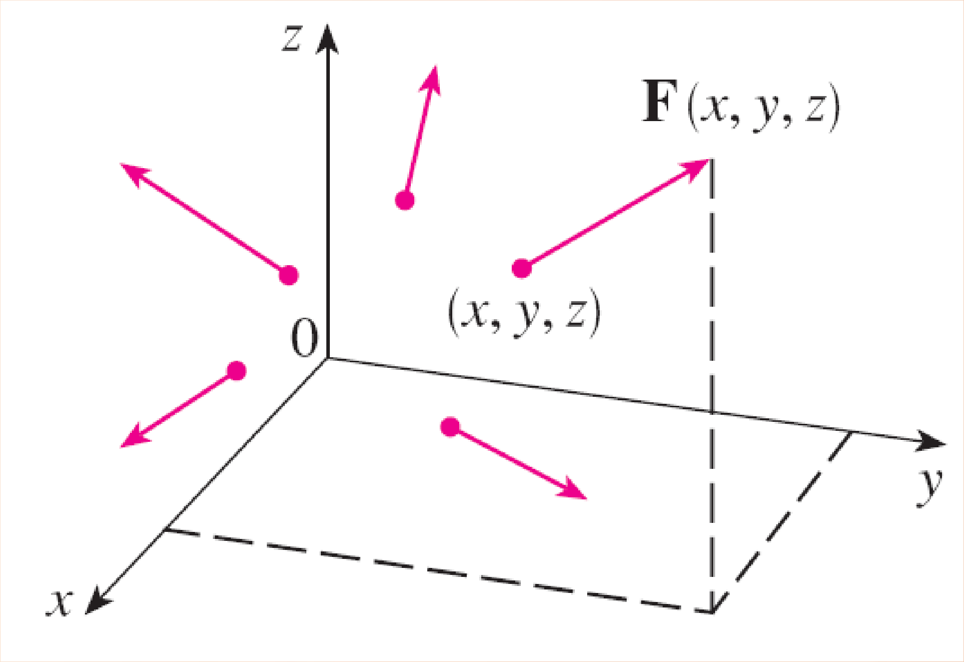
Definition 2

Let *E* be a subset of R^3.

A vector field on R^3 is a function **F** that assigns to each point (x, y, z) in *E* a three-dimensional (3-D) vector $\mathbf{F}(x, y, z)$.

VECTOR FIELDS ON R^3

A vector field **F** on R^3 is shown.



- We can express it in terms of its component functions P, Q, and R as:

$$\mathbf{F}(x, y, z) = P(x, y, z)\,\mathbf{i} + Q(x, y, z)\,\mathbf{j} + R(x, y, z)\,\mathbf{k}$$

CONTINUOUS VECTOR FIELDS ON R^3

As with the vector functions in Section 13.1, we can define continuity of vector fields.

We can show that **F** is continuous if and only if its component functions *P*, *Q*, and *R* are continuous.

VECTOR FIELDS ON R^3

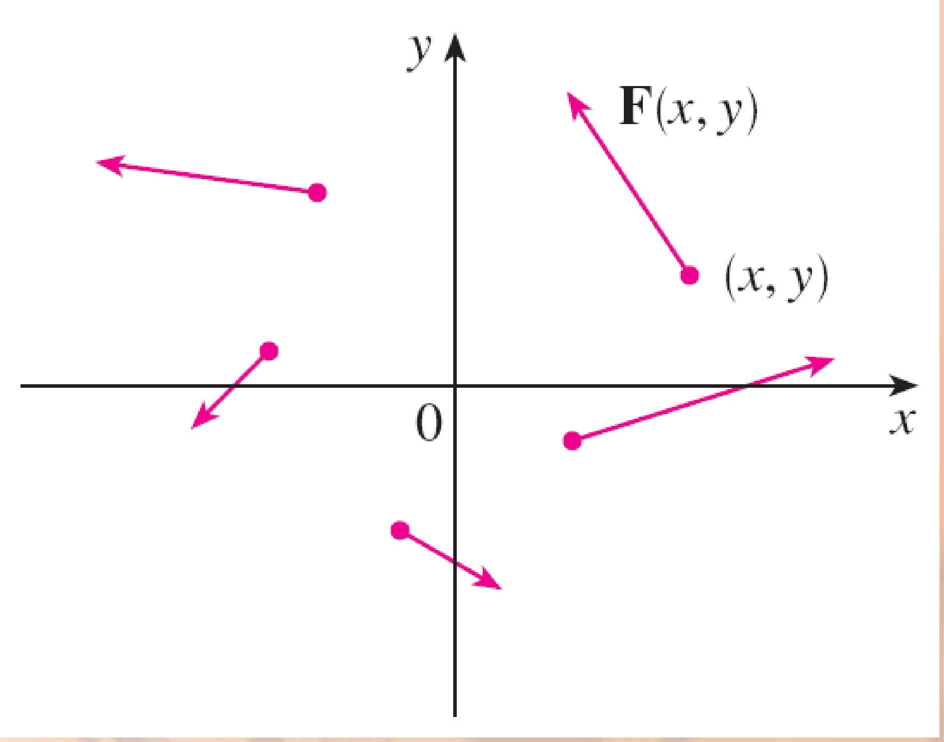
We sometimes identify a point (x, y, z) with its position vector $\mathbf{x} = \langle x, y, z \rangle$ and write $\mathbf{F}(\mathbf{x})$ instead of $\mathbf{F}(x, y, z)$.

- Then, **F** becomes a function that assigns a vector $\mathbf{F}(\mathbf{x})$ to a vector $\mathbf{x}$.

A vector field on R^2 is defined by:

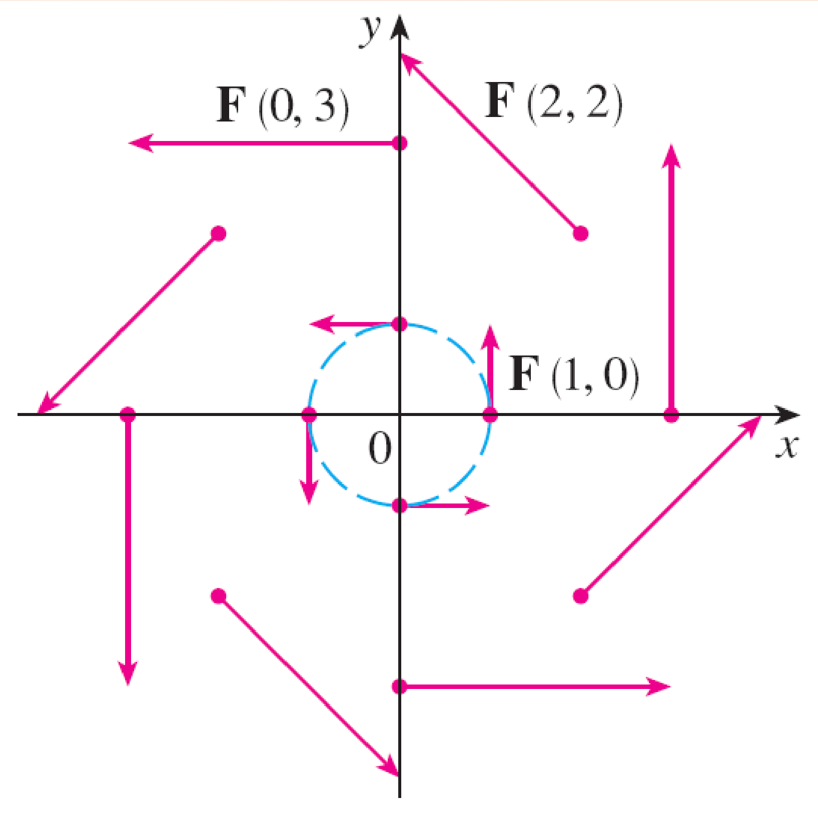
$$\mathbf{F}(x, y) = -y\,\mathbf{i} + x\,\mathbf{j}$$

Describe **F** by sketching some of the vectors **F**(x, y) as shown.



Since **F**(1, 0) = **j**, we draw the vector **j** = ⟨0, 1⟩ starting at the point (1, 0).

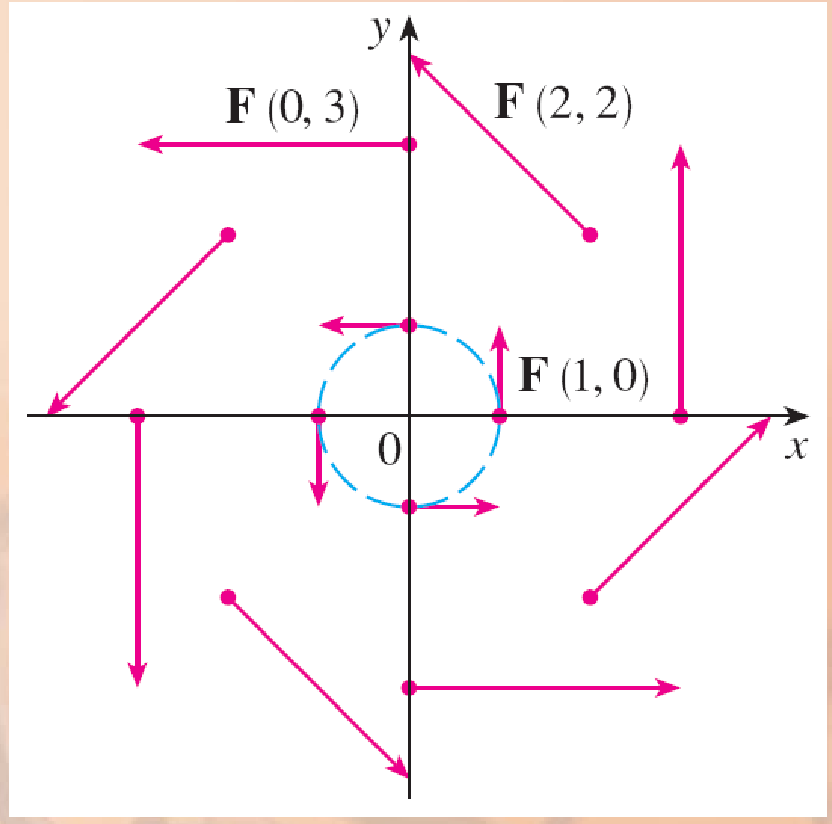
Since **F**(1, 0) = –**i**, we draw the vector ⟨–1, 0⟩ with starting point (0, 1).



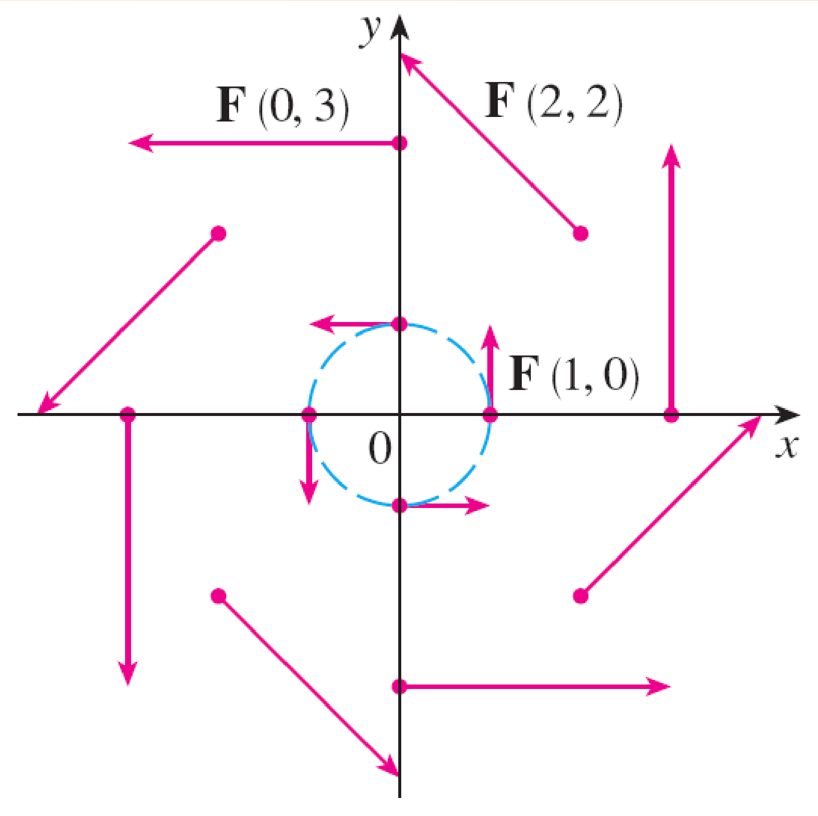
Continuing in this way, we calculate several other representative values of $\mathbf{F}(x, y)$ in this table.

(x, y)	$\mathbf{F}(x, y)$	(x, y)	$\mathbf{F}(x, y)$
$(1, 0)$	$\langle 0, 1 \rangle$	$(-1, 0)$	$\langle 0, -1 \rangle$
$(2, 2)$	$\langle -2, 2 \rangle$	$(-2, -2)$	$\langle 2, -2 \rangle$
$(3, 0)$	$\langle 0, 3 \rangle$	$(-3, 0)$	$\langle 0, -3 \rangle$
$(0, 1)$	$\langle -1, 0 \rangle$	$(0, -1)$	$\langle 1, 0 \rangle$
$(-2, 2)$	$\langle -2, -2 \rangle$	$(2, -2)$	$\langle 2, 2 \rangle$
$(0, 3)$	$\langle -3, 0 \rangle$	$(0, -3)$	$\langle 3, 0 \rangle$

We draw the corresponding vectors to represent the vector field shown.



It appears that each arrow is tangent to a circle with center the origin.



To confirm this, we take the dot product of the position vector $\mathbf{x} = x\,\mathbf{i} + y\,\mathbf{j}$ with the vector $\mathbf{F}(x) = \mathbf{F}(x, y)$:

$$\begin{aligned}\mathbf{x} \cdot \mathbf{F}(\mathbf{x}) &= (x\,\mathbf{i} + y\,\mathbf{j}) \cdot (-y\,\mathbf{i} + x\,\mathbf{j}) \\ &= -xy + yx \\ &= 0\end{aligned}$$

This shows that **F**(*x*, *y*) is perpendicular to the position vector $\langle x, y \rangle$ and is therefore tangent to a circle with center the origin and radius $|\mathbf{x}| = \sqrt{x^2 + y^2}$.

Notice also that:

$$|\mathbf{F}(x, y)| = \sqrt{(-y)^2 + x^2}$$
$$= \sqrt{x^2 + y^2} = |\mathbf{x}|$$

- So, the magnitude of the vector **F**(*x*, *y*) is equal to the radius of the circle.

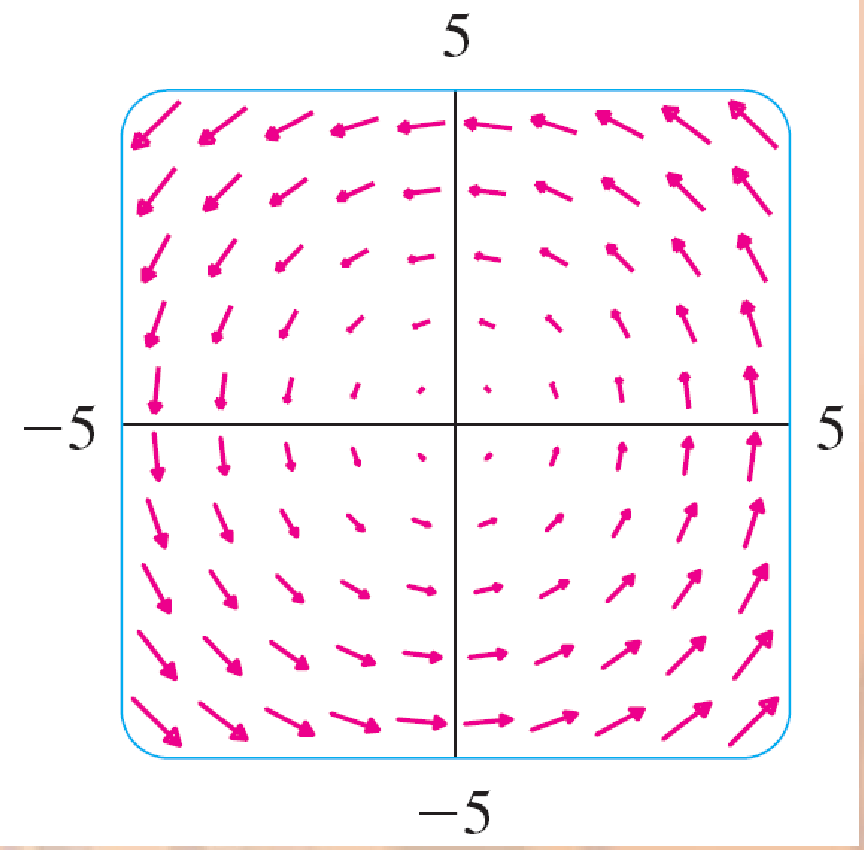
VECTOR FIELDS

Some computer algebra systems (CAS) are capable of plotting vector fields in two or three dimensions.

- They give a better impression of the vector field than is possible by hand because the computer can plot a large number of representative vectors.

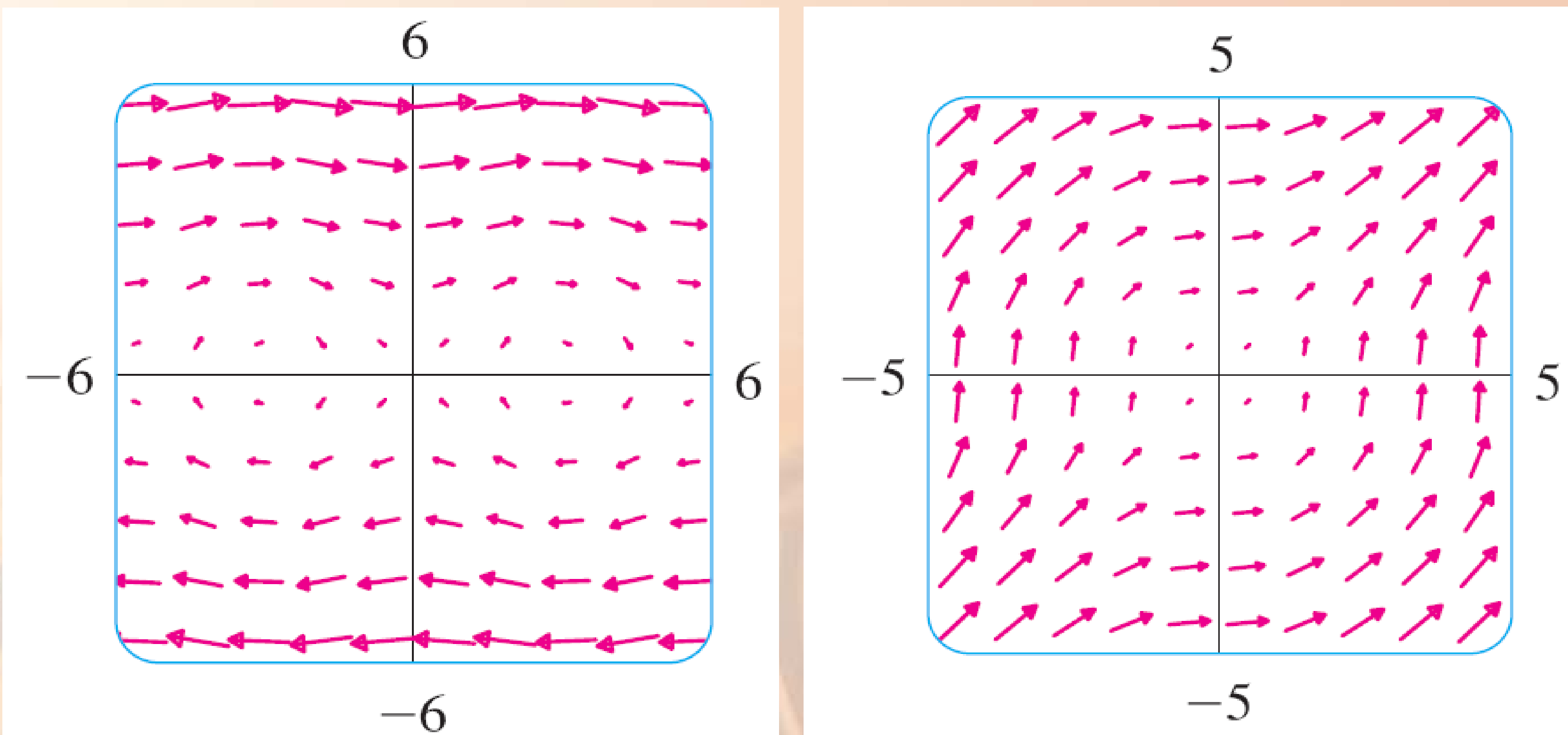
The figure shows a computer plot of the vector field in Example 1.

- Notice that the computer scales the lengths of the vectors so they are not too long and yet are proportional to their true lengths.



VECTOR FIELDS

These figures show two other vector fields.

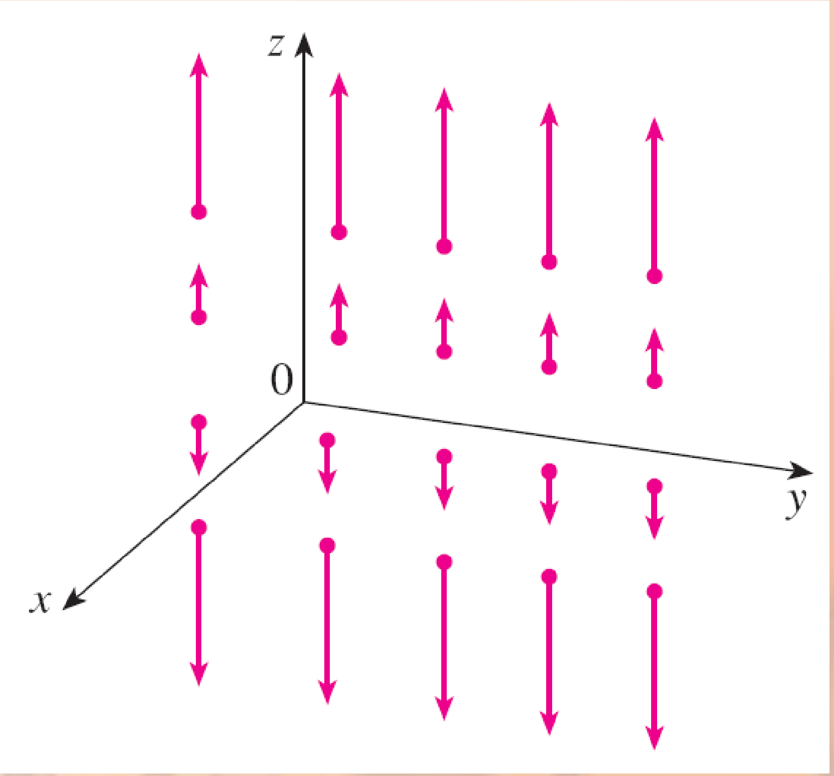


Sketch the vector field on R^3 given by:

$$\mathbf{F}(x, y, z) = z\,\mathbf{k}$$

The sketch is shown.

- Notice that all vectors are vertical and point upward above the *xy*-plane or downward below it.
- The magnitude increases with the distance from the *xy*-plane.



We were able to draw the vector field in Example 2 by hand because of its particularly simple formula.

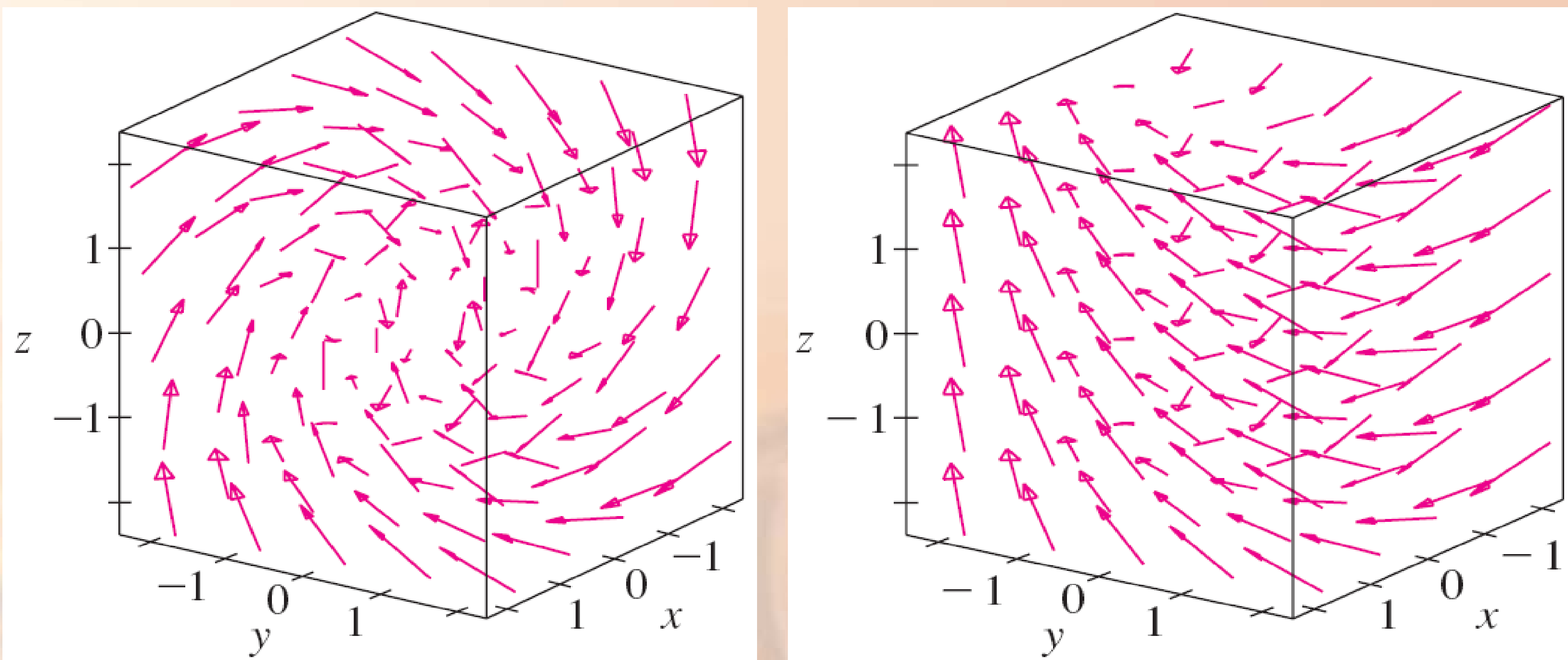
Most 3-D vector fields, however, are virtually impossible to sketch by hand.

- So, we need to resort to a CAS.
- Examples are shown in the following figures.

VECTOR FIELDS BY CAS

These vector fields have similar formulas.

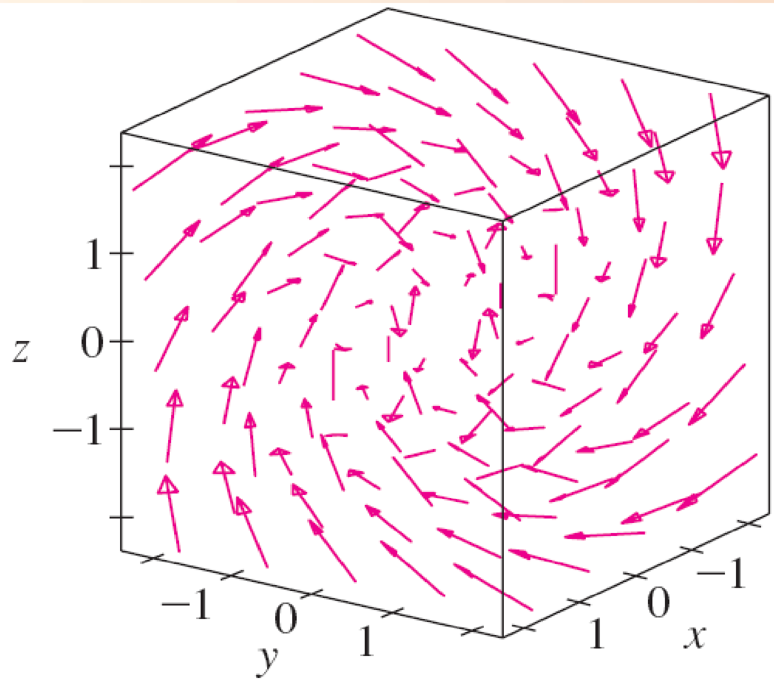
Still, all the vectors in the second figure point in the general direction of the negative *y*-axis.



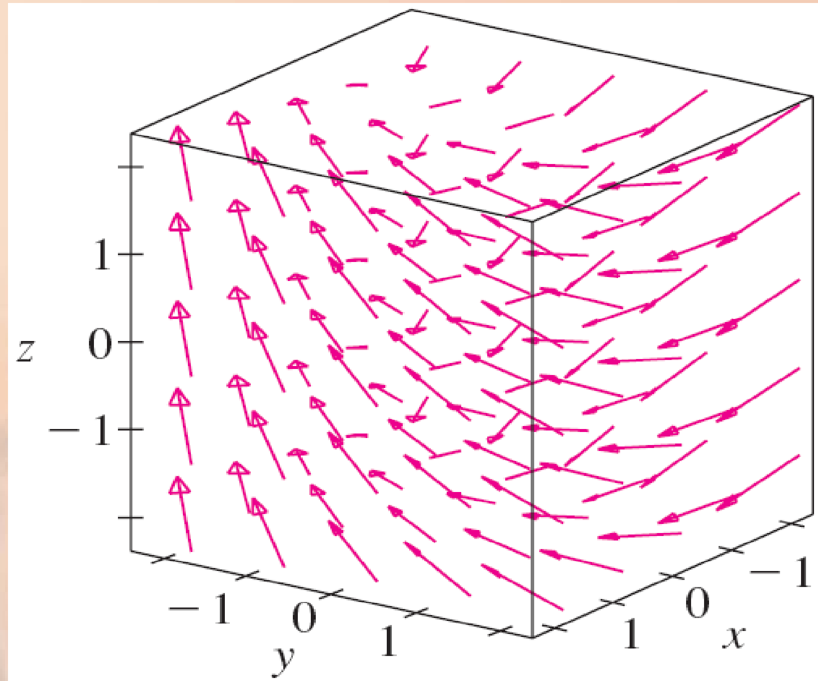
$\mathbf{F}(x, y, z) = y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$

$\mathbf{F}(x, y, z) = y\,\mathbf{i} - 2\,\mathbf{j} + x\,\mathbf{k}$

This is because their *y*-components are all –2.



$\mathbf{F}(x, y, z) = y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$

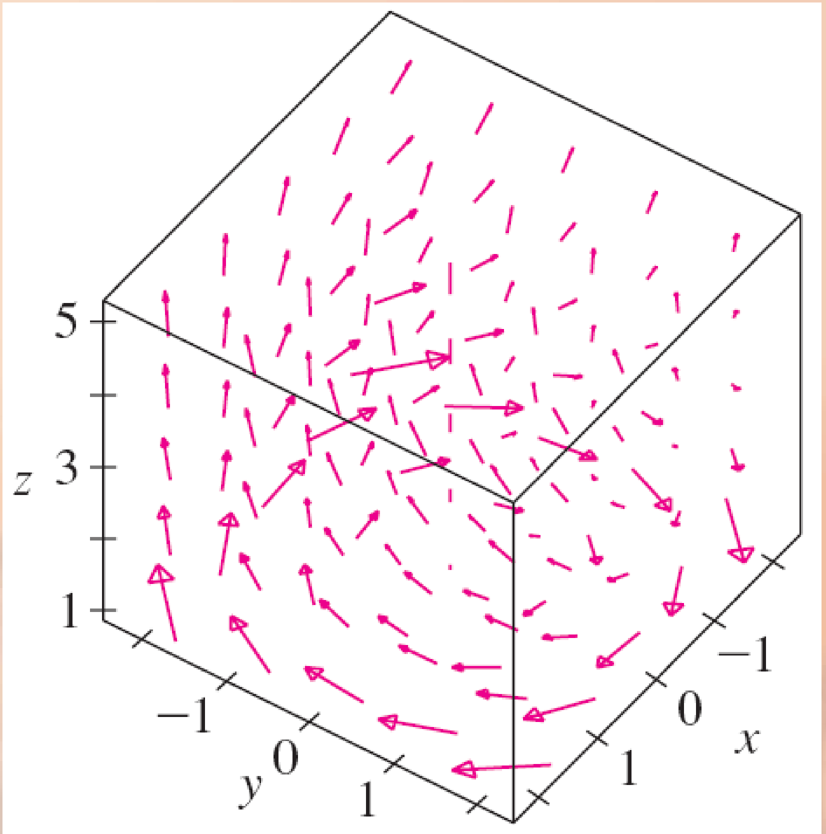


$\mathbf{F}(x, y, z) = y\,\mathbf{i} - 2\,\mathbf{j} + x\,\mathbf{k}$

VECTOR FIELDS BY CAS

If the vector field in this figure represents a velocity field, then a particle would:

- Be swept upward.
- Spiral around the *z*-axis in the clockwise direction as viewed from above.



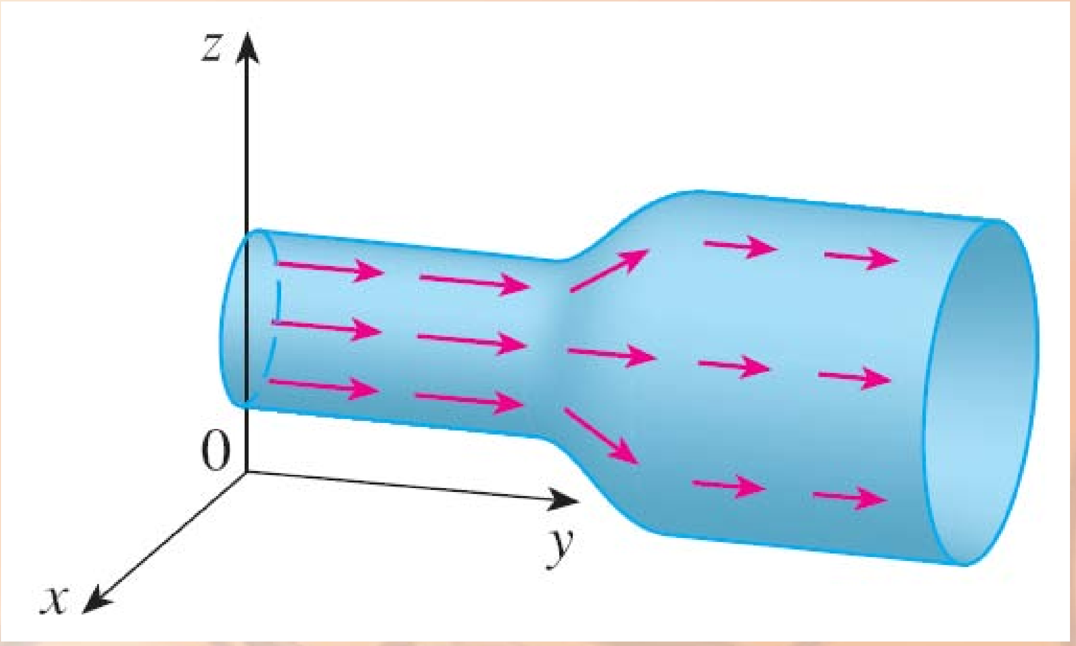
Example 3

Imagine a fluid flowing steadily along a pipe and let **V**(x, y, z) be the velocity vector at a point (x, y, z).

- Then, **V** assigns a vector to each point (x, y, z) in a certain domain E (the interior of the pipe).
- So, **V** is a vector field on R^3 called a velocity field.

A possible velocity field is illustrated here.

- The speed at any given point is indicated by the length of the arrow.



Example 3

Velocity fields also occur in other areas of physics.

- For instance, the vector field in Example 1 could be used as the velocity field describing the counterclockwise rotation of a wheel.

Newton's Law of Gravitation states that the magnitude of the gravitational force between two objects with masses *m* and *M* is

$$|\mathbf{F}| = \frac{mMG}{r^2}$$

- *r* is the distance between the objects.
- *G* is the gravitational constant ($G \sim 6.673 * 10^{-11}\ m^3 * kg^{-1}$)

Example 4

- Let's assume that the object with mass M is located at the origin in R^3.
- For instance, M could be the mass of the earth and the origin would be at its center.
- Let the position vector of the object with mass m be $\mathbf{x} = \langle x, y, z \rangle$.
- Then, $\mathbf{r} = |\mathbf{x}|$, so $\mathbf{r}^2 = |\mathbf{x}|^2$.

The gravitational force exerted on this second object acts toward the origin.

The unit vector in this direction is: $-\frac{\mathbf{x}}{|\mathbf{x}|}$

Thus, the gravitational force acting on the object at $\mathbf{x} = \langle x, y, z \rangle$ is:

$$\mathbf{F}(\mathbf{x}) = -\frac{mMG}{|\mathbf{x}|^3}\mathbf{x}$$

Physicists often use the notation **r** instead of **x** for the position vector.

- So, you may see Formula 3 written in the form

$$\mathbf{F} = -(mMG/r^3)\mathbf{r}$$

The function given by Equation 3 is an example of a vector field because it associates a vector [the force **F**(**x**)] with every point **x** in space.

- It is called the gravitational field.

Formula 3 is a compact way of writing the gravitational field.

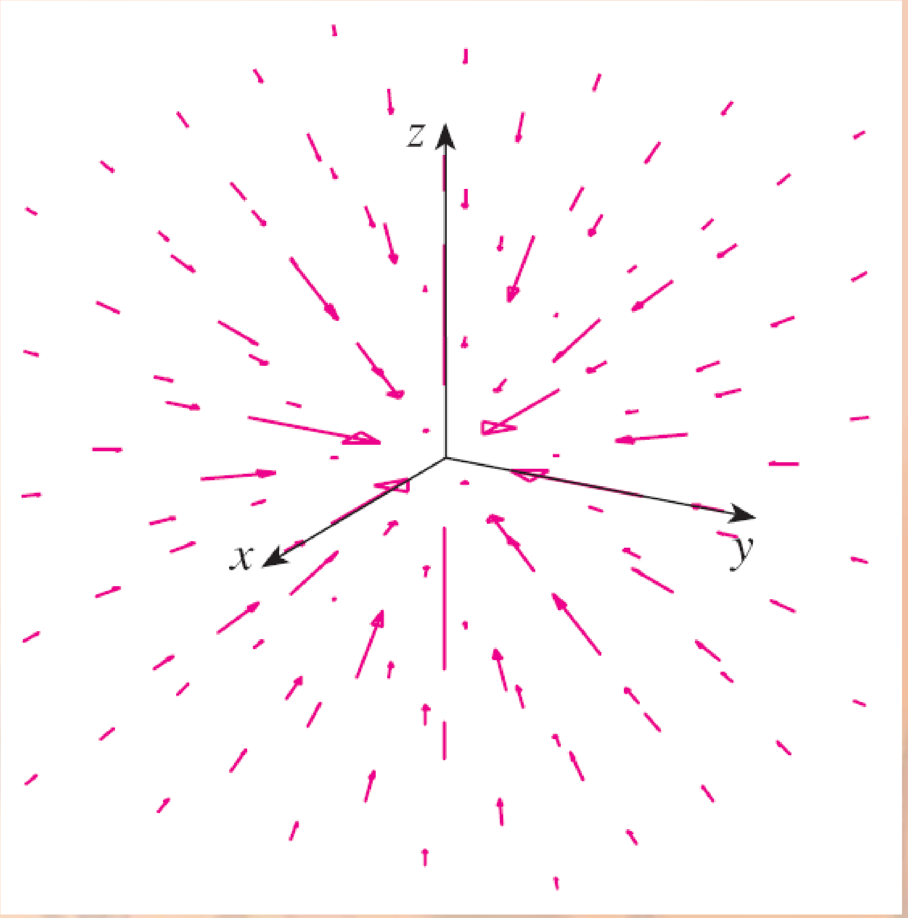
However, we can also write it in terms of its component functions.

We do this by using the facts that

x = x **i** + y **j** + z **k** and |**x**| = $\sqrt{x^2 + y^2 + z^2}$:

$$\mathbf{F}(x, y, z) = \frac{-mMGx}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{i} + \frac{-mMGy}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{j} + \frac{-mMGz}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{k}$$

The gravitational field **F** is pictured here.



GRADIENT VECTOR FIELD ON R^2

If f is a scalar function of two variables, recall from Section 14.6 that its gradient ∇f (or grad f) is defined by:

$$\nabla f(x, y) = f_x(x, y)\,\mathbf{i} + f_y(x, y)\,\mathbf{j}$$

- Thus, ∇f is really a vector field on R^2 and is called a gradient vector field.

GRADIENT VECTOR FIELD ON R^3

Likewise, if *f* is a scalar function of three variables, its gradient is a vector field on R^3 given by:

$$\nabla f(x, y, z)$$

$$= f_x(x, y, z)\,\mathbf{i} + f_y(x, y, z)\,\mathbf{j} + f_z(x, y, z)\,\mathbf{k}$$

Example 6

Find the gradient vector field of

$$f(x, y) = x^2y - y^3$$

Plot the gradient vector field together with a contour map of *f*.

- How are they related?

The gradient vector field is given by:

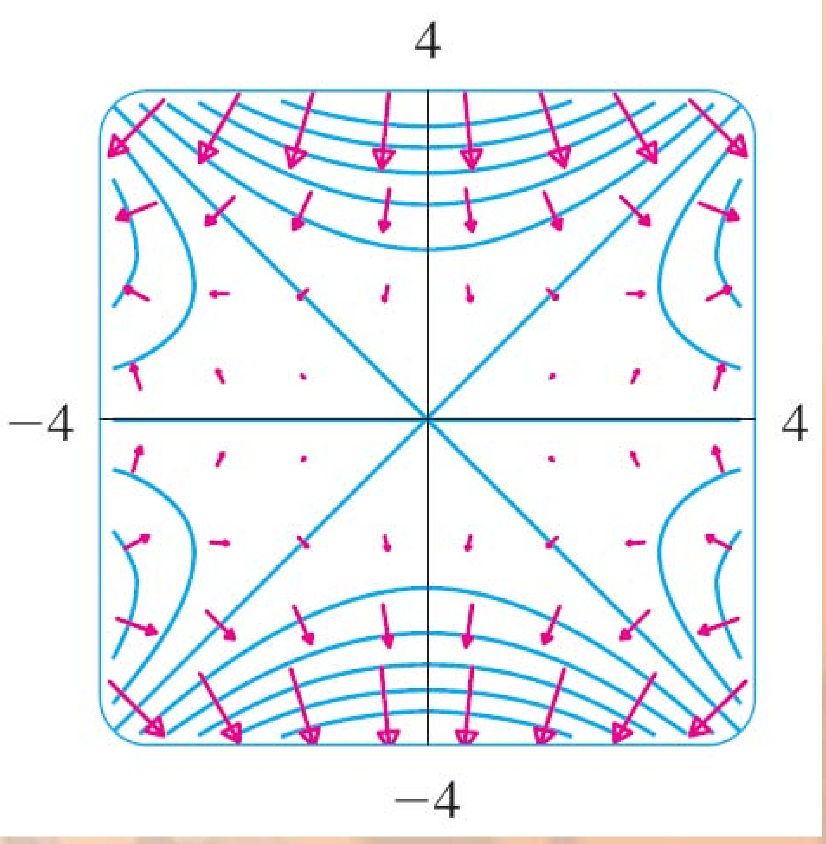
$$\nabla f(x, y) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

$$= 2xy\,\mathbf{i} + (x^2 - 3y^2)\,\mathbf{j}$$

GRADIENT VECTOR FIELDS ON R^2 Example 6

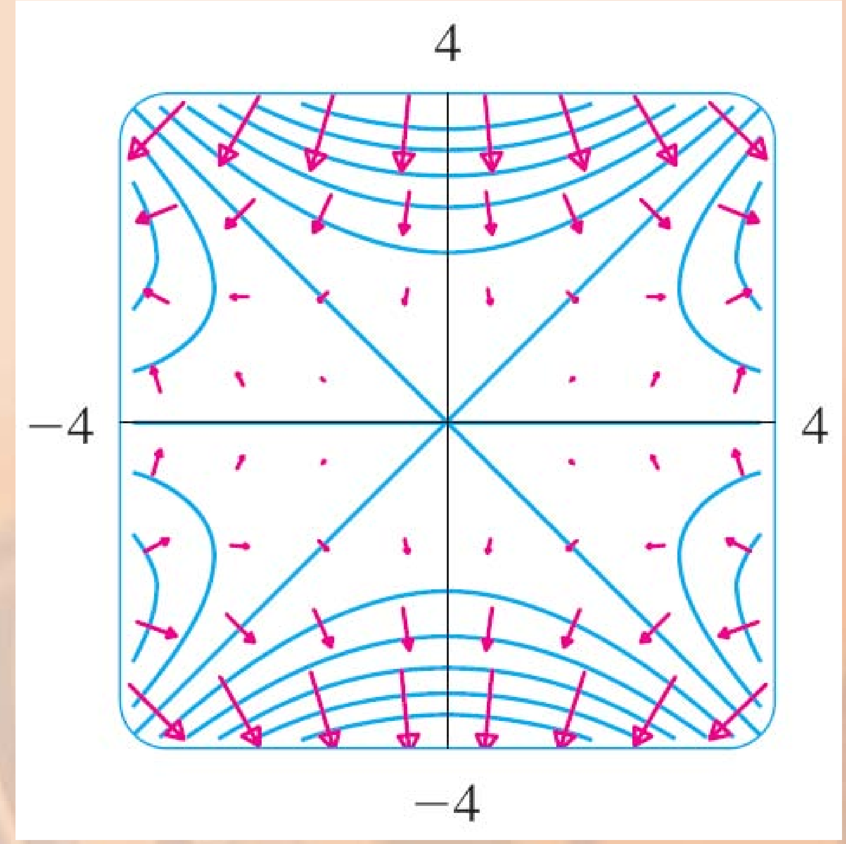
The figure shows a contour map of *f* with the gradient vector field.

- Notice that the gradient vectors are perpendicular to the level curves—as we would expect from Section 14.6

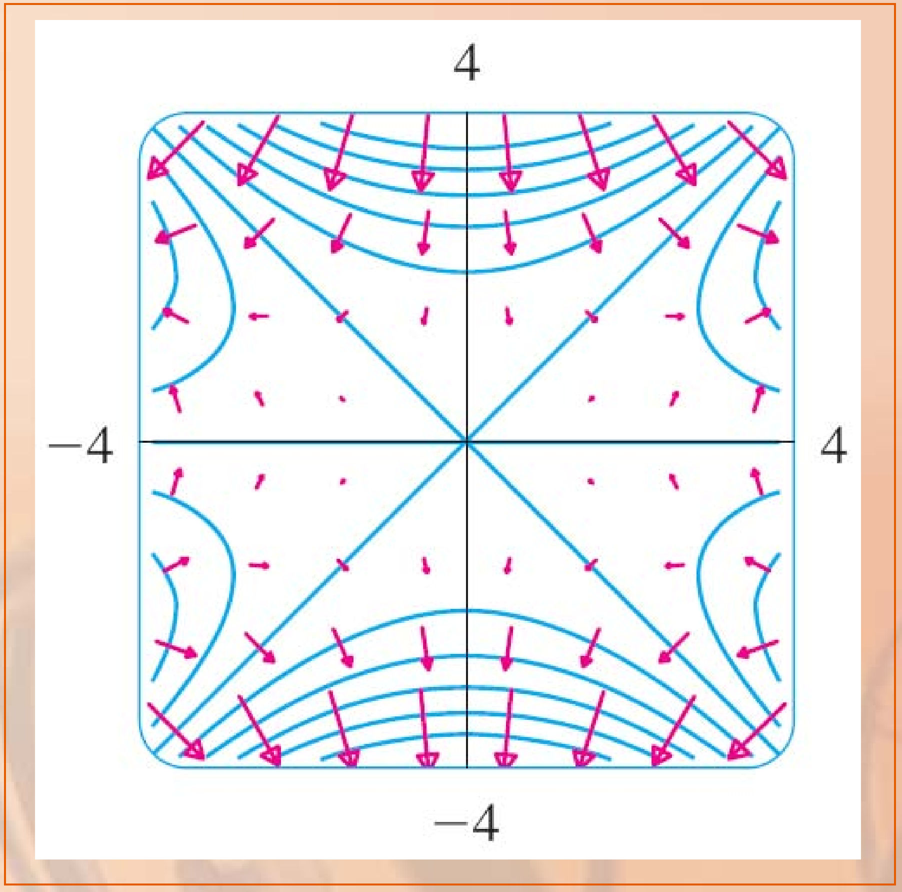


Notice also that the gradient vectors are:

- Long where the level curves are close to each other.
- Short where the curves are farther apart.



That's because the length of the gradient vector is the value of the directional derivative of *f* and closely spaced level curves indicate a steep graph.



CONSERVATIVE VECTOR FIELD

A vector field **F** is called a conservative vector field if it is the gradient of some scalar function—that is, if there exists a function *f* such that **F** = ∇f.

- In this situation, *f* is called a potential function for **F**.
- Let's assume that the object with mass *M* is located at the origin in R^3.
- For instance, *M* could be the mass of the earth and the origin would be at its center.

For example, the gravitational field **F** in Example 4 is conservative.

- Suppose we define:

$$f(x, y, z) = \frac{mMG}{\sqrt{x^2 + y^2 + z^2}}$$

CONSERVATIVE VECTOR FIELDS

Then,

$$\nabla f(x, y, z)$$

$$= \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

$$= \frac{-mMGx}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{i} + \frac{-mMGy}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{j} + \frac{-mMGz}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{k}$$

$$= \mathbf{F}(x, y, z)$$