Horizontal Gravity Disturbance Vector in Atmospheric and Oceanic Dynamics

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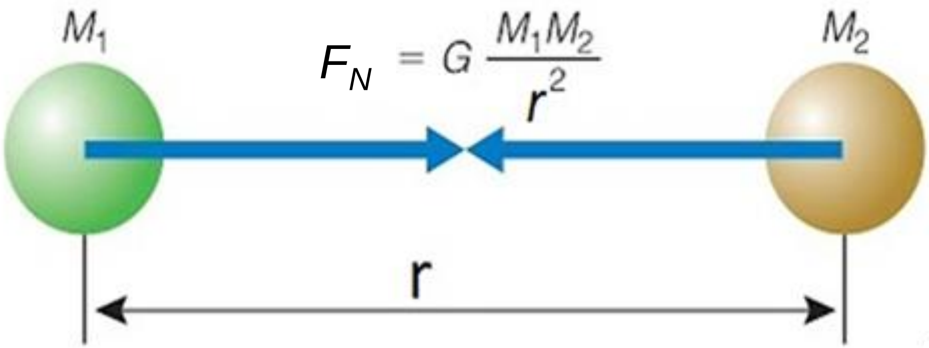
Background

- (1) Gravity = Newton's Gravitational Acceleration
- + Centrifugal Acceleration
- (2) Untrue gravitation in METOC.
- (3) True gravitation in geodesy
- (4) Gravity Disturbance Vector,

  δg = True gravitation – Untrue gravitation
- (5) δg is the most important variable in Geodesy.
- (6) δg has never been considered in METOC.
- (7) δg is important in atmospheric and oceanic dynamics.

Newton’s Law of Universal Gravitation



M_1 and M_2 are two point-masses (no volume).

Newton’s Gravitational Constant:

$G = 6.67408 \times 10^{-11} Nm^2kg^{-2}$

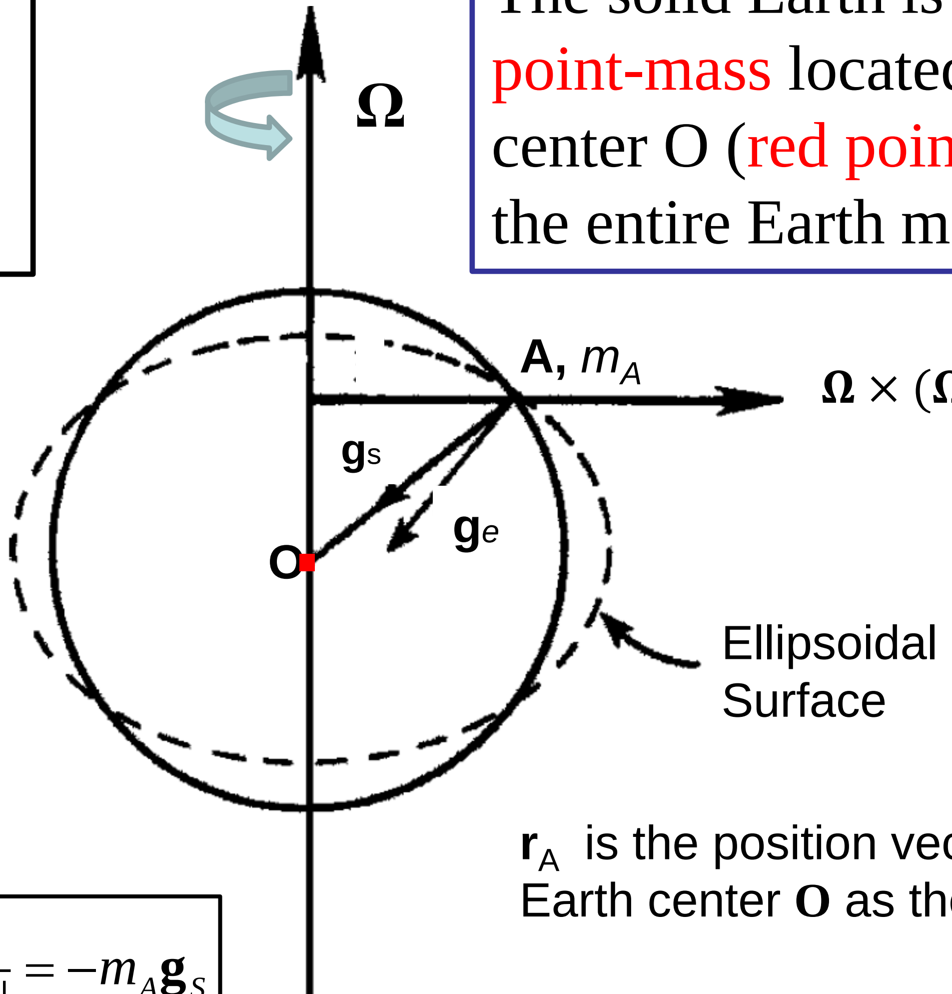
Earth Gravitation in METOC

Untrue gravitational Acceleration

$$\mathbf{g}_s = -\frac{GM}{|\mathbf{r}_A|^2}\frac{\mathbf{r}_A}{|\mathbf{r}_A|}$$

$|\mathbf{g}_s| = g_0 = 9.81\ \text{m/s}^2$

The solid Earth is treated as a point-mass located at the Earth center O (red point) containing the entire Earth mass (M)



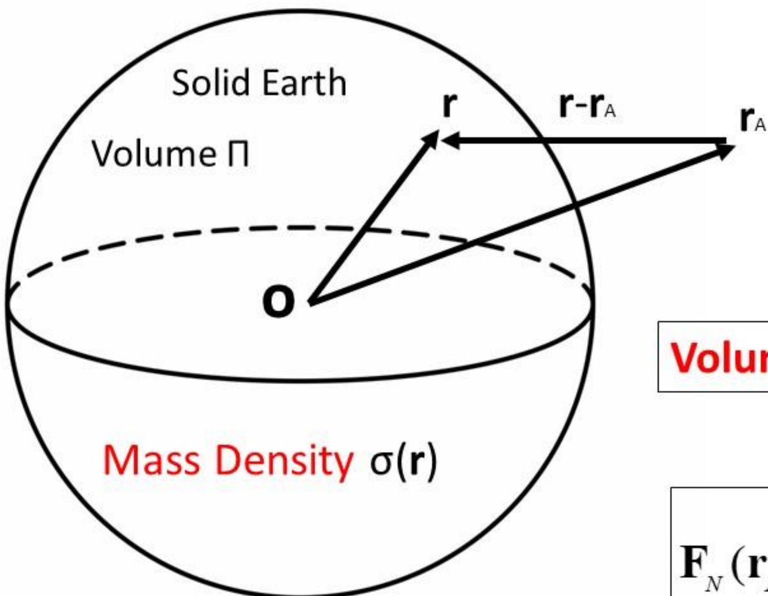
$$\mathbf{F}_N^{(O)} = -\frac{GMm_A}{|\mathbf{r}_A|^2}\frac{\mathbf{r}_A}{|\mathbf{r}_A|} = -m_A\mathbf{g}_S$$

$\mathbf{r}_A$ is the position vector with the Earth center **O** as the origin

$M = 5.98 \times 10^{24}$ kg (Mass of the Earth)

True Earth Gravitation

Newton’s gravitation is the volume integration over all the point masses inside the solid Earth to the point mass in atmosphere and oceans.



Newton’s Gravitational Force of the Solid Earth on a Point Mass m_A at $\mathbf{r}_A$ in atmosphere

↓

Volume Integration over the Solid Earth

↓

$$\mathbf{F}_N(\mathbf{r}_A) = Gm_A \iiint_{\Pi} \frac{\sigma(\mathbf{r})}{|\mathbf{r}-\mathbf{r}_A|^3}(\mathbf{r}-\mathbf{r}_A)\,d\Pi$$

Earth Gravitational Correction Vector δF_N

$$\delta \mathbf{F}_N = \left[\mathbf{F}_N(\mathbf{r}_A) - \mathbf{F}_N^{(O)}(\mathbf{r}_A) \right] = m_A G \iiint_{\Pi} \frac{\left[\sigma(\mathbf{r}) - \sigma_0\right]}{\left|\mathbf{r} - \mathbf{r}_A\right|^3} (\mathbf{r} - \mathbf{r}_A) d\Pi$$

σ_0 = Average Mass Density of the Solid Earth

$$\delta \mathbf{g} = \frac{\delta \mathbf{F}_N}{m_A} = \text{Gravity Disturbance Vector}$$

$$\delta \mathbf{g} = 0 \quad \text{if} \quad \sigma(\mathbf{r}) = \sigma_0 \ \text{(const)}$$

Uniform mass density → $\delta \mathbf{g} = 0$

Ultimate Cause of Using Gravity g

$$\mathbf{g} = \mathbf{F}_N + \mathbf{A}_C$$

$\mathbf{F}_N$ → Newton's gravitational acceleration

$\mathbf{A}_C$ → Centrifugal Acceleration

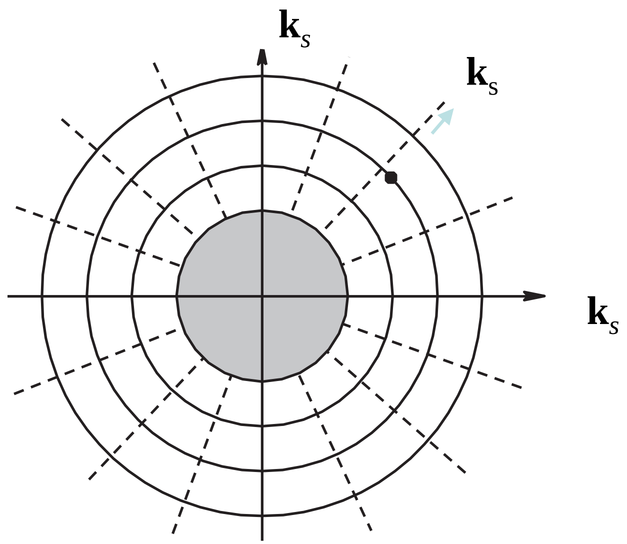
Make the centrifugal acceleration ($\mathbf{A}_C$) vanish in atmospheric and oceanic equations of motion

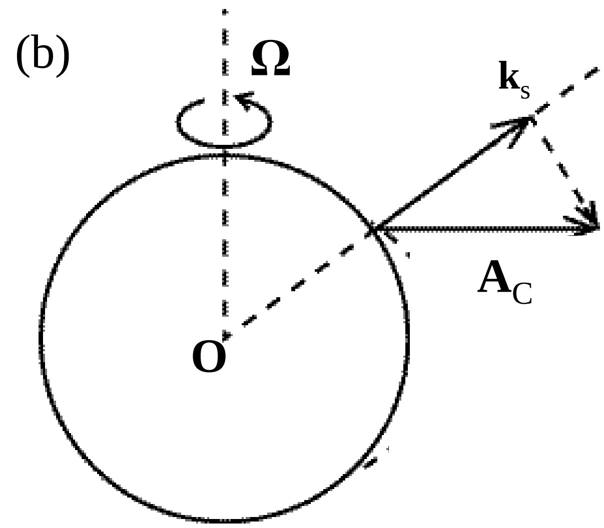
Two Basic Principles in the Use of Gravity g

- (1) The gravity should never be split into Newton's gravitational acceleration and centrifugal acceleration.

- (2) The centrifugal acceleration never occurs in any dynamic equations of atmosphere and oceans.

Spherical Geopotential





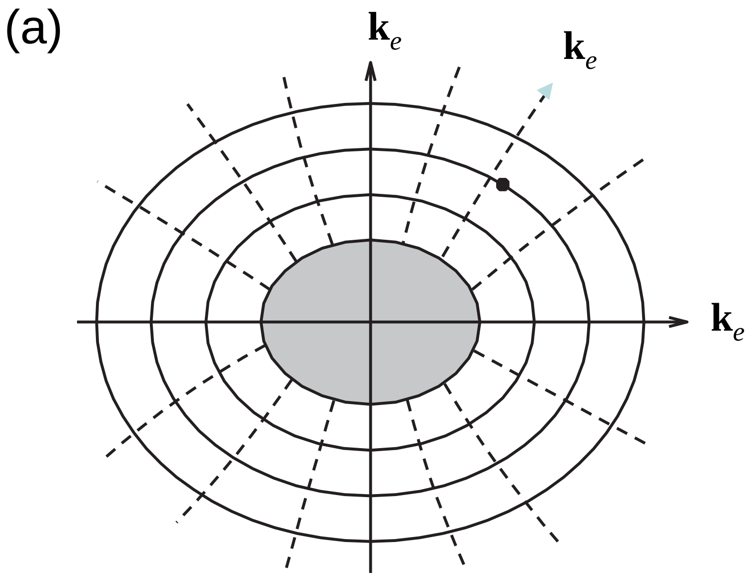
$\mathbf{r} = r\,\mathbf{k}_s$

Spherical Geopotential Surfaces

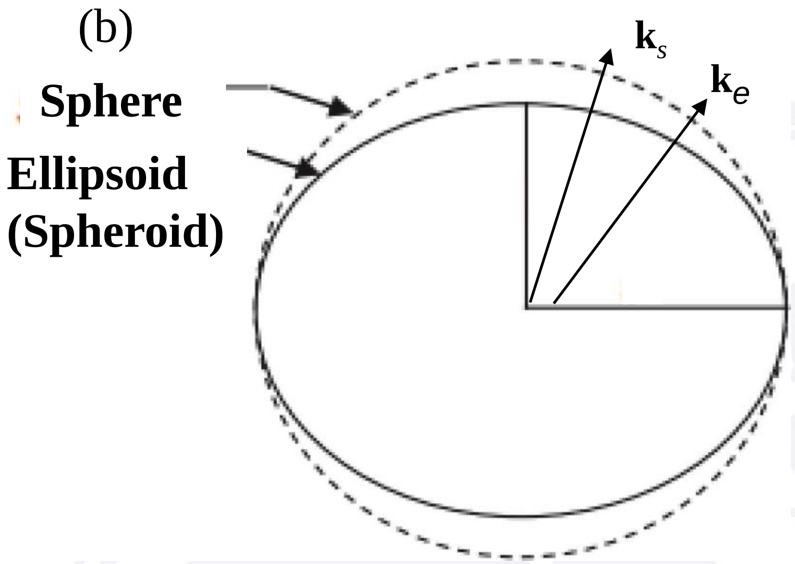
(1) Spherical Geopotential → No component of $\mathbf{A}_c$ on the Spherical Geopotential Surfaces → Error in $\mathbf{A}_c$

(2) Error in Newton's Gravitational Acceleration →Uniform Earth Mass Density

Spheroid Geopotential



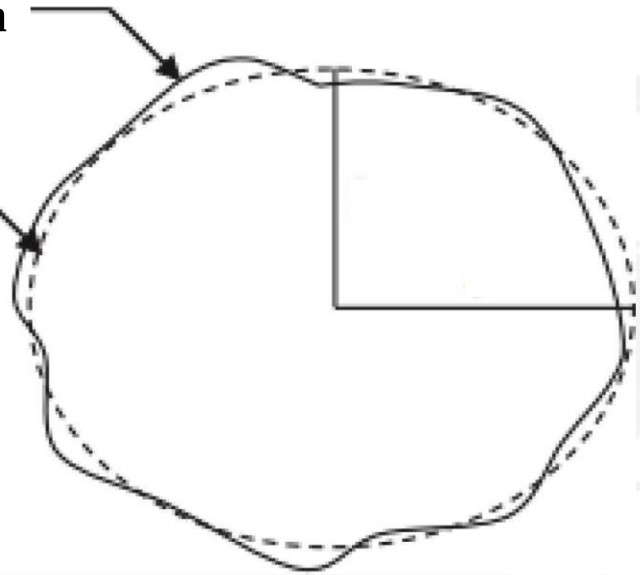
Spheroidal Geopotential Surfaces



(1) No Error in Centrifugal Acceleration ($\mathbf{A}_C$)

(2) Error in Newton's Gravitational Acceleration → Uniform Earth Mass Density

True Geopotential Surfaces

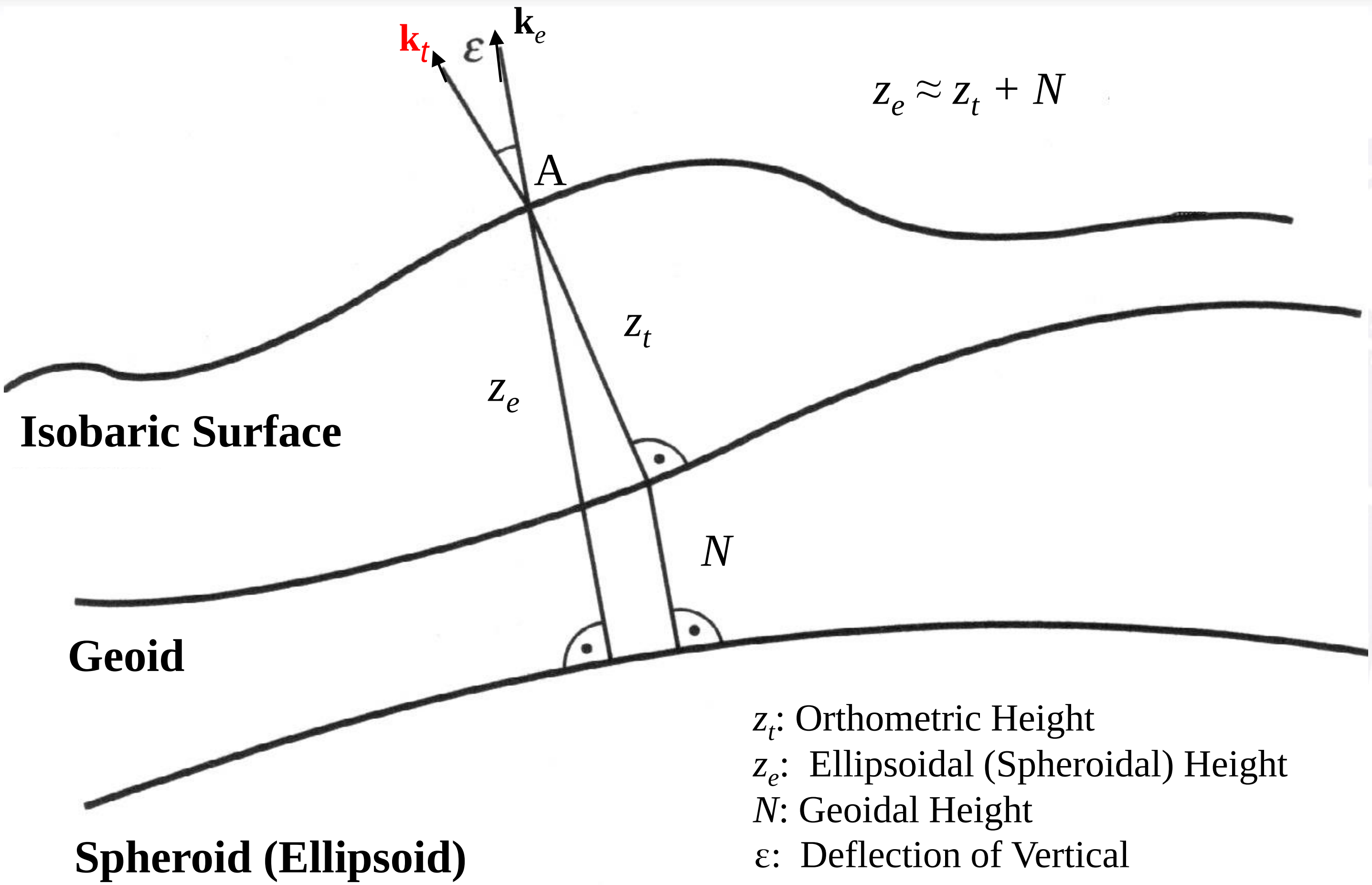


True Geopotential Surfaces → Composition of Spheroidal and Geoidal Surfaces

(1) No Error in Newton's Gravitational Acceleration → Non-uniform Earth Mass Density

(2) No Error in Centrifugal Acceleration ($\mathbf{A}_C$)

Deflection of Vertical



$$z_e \approx z_t + N$$

z_t: Orthometric Height
z_e: Ellipsoidal (Spheroidal) Height
N: Geoidal Height
ε: Deflection of Vertical
$\mathbf{k}_e$: Unit Vector for Ellipsoidal (Spheroidal) Normal
$\mathbf{k}_t$: Unit Vector for Vertical

Two Difference Vectors Among ($\mathbf{k}_s$, $\mathbf{k}_e$, $\mathbf{k}_t$)

- (1) Gravitational Correction → Horizontal Gravity Disturbance Vector

$$\delta\mathbf{g} = g_0(\mathbf{k}_t - \mathbf{k}_e) = g_0\nabla N$$

$g_0 = 9.81\text{m/s}^2$

$\delta\mathbf{g}$ occurs in the equations of motion in the spheroidal geopotential coordinates.

- (2) Centrifugal Acceleration Error Vector

$$\Delta\mathbf{g} = g_0(\mathbf{k}_s - \mathbf{k}_e)$$

$\Delta\mathbf{g}$ shouldn't occur in any dynamic equations with the spherical coordinates due to "vanish of $\mathbf{A}_C$ in all equations of motion"

Evaluation of Δg

(1) Analytical estimation using metrics since metrics for both spherical and spheroidal geopotential coordinates are available: "… which is less than 0.17% in the neighborhood of the Earth's surface. If this approximation is used, the equations are the same as written in spherical polar coordinates (Gill 1982)"

(2) Solutions of the spheroidal (spheroidal geopotential coordinates) and spherical (spherical geopotential coordinates) equations were obtained. The difference between the solutions is likely to be small except perhaps in long-term simulations in which small systematic differences may accumulate (Gates 2004, Beńard 2015, Staniforth 2014).

No Evaluation on δg

Currently, **δg is neglected** in METOC dynamics with the spheroidal geopotential coordinates

Question arises:

What is the effect of δg?

Objective

Identify the importance of $\delta \mathbf{g}$ in METOC through comparison to other forces such as Coriolis force, pressure gradient force, wind stress (for oceans) using 4 publicly available datasets:

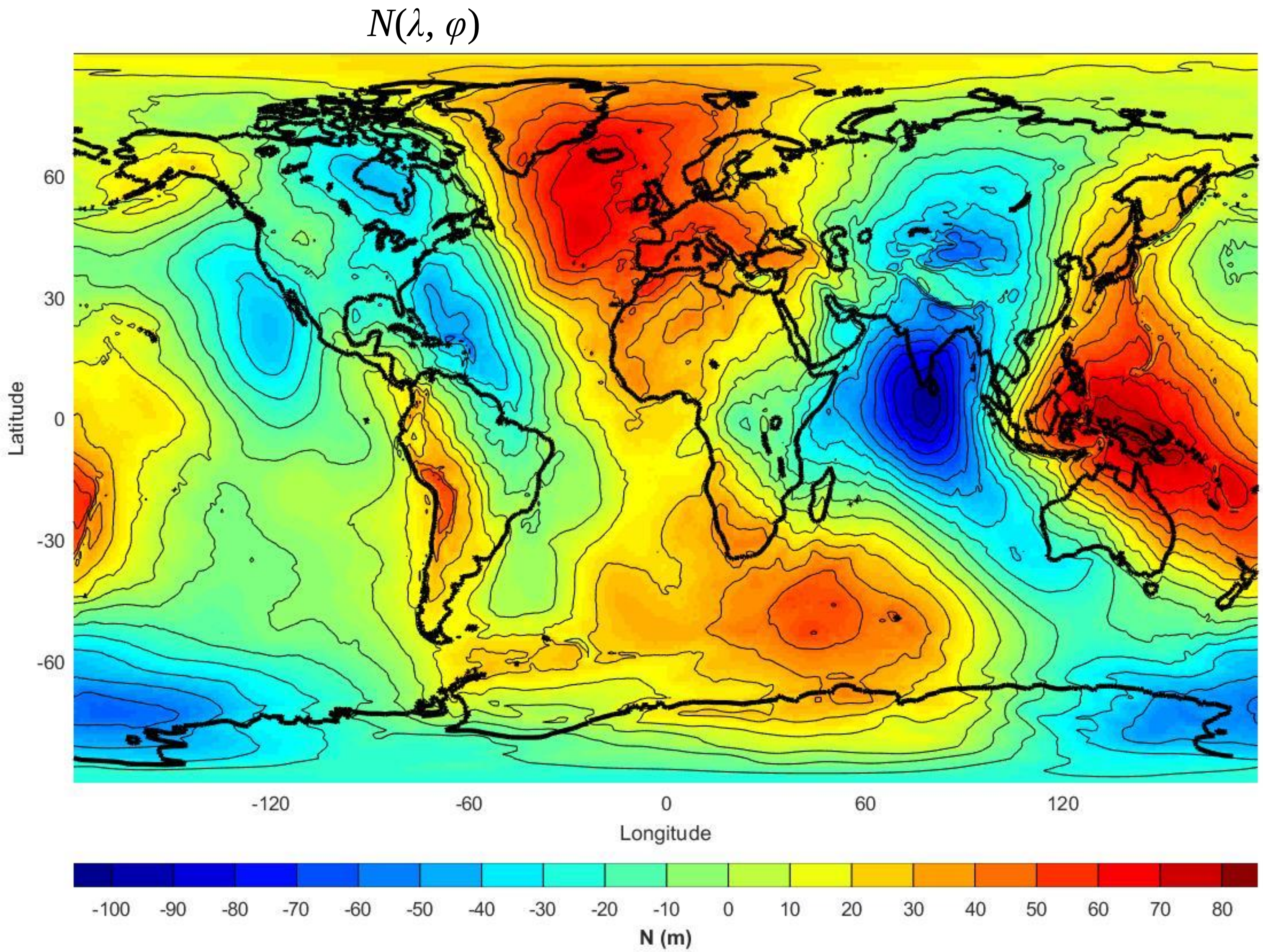
(a) International Center for Global Earth Models (ICGEM) static gravity field model EIGEN-6C4 (http://icgem.gfz-potsdam.de/home) for $N(\lambda, \varphi)$

(b) NCEP/NCAR Reanalysed Monthly long-term mean (effective) geopotential height (Z), wind velocity (u, v), and temperature (T) at 12 pressure levels 1,000, 925, 850, 700, 600, 500, 400, 300, 250, 200, 150, and 100 hPa (https://psl.noaa.gov/data/gridded/data.ncep.reanalysis.derived.pressue.html)

(c) Climatological annual mean temperature and salinity from the NOAA/NCEI WOA18 for the sea water density (ρ) data (from https://www.ncei.noaa.gov/access/world-ocean-atlas-2018/)

(d) Climatological annual mean surface wind stress (τ_λ, τ_φ) from the Atlas of Surface Marine Data 1994 (SMD94) (from https://iridl.ldeo.columbia.edu/SOURCES/.DASILVA/.)

Geoid Undulation N from EIGEN-6C4



δg in Atmospheric Dynamics

Horizontal Momentum Equation in the Spheroidal Coordinates with Pressure as Vertical Coordinate

$$\frac{D\mathbf{U}}{Dt} + f\mathbf{k}\times\mathbf{U} = -g_0\nabla Z + \delta\mathbf{g} + \mathbf{F}$$

⇓

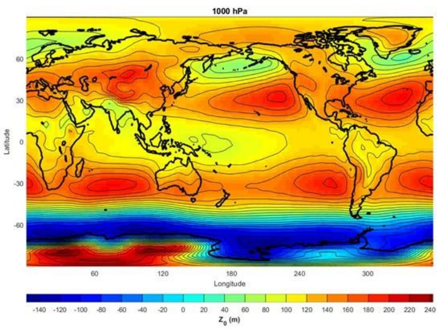
$$\frac{D\mathbf{U}}{Dt} + f\mathbf{k}\times\mathbf{U} = -g_0\nabla Z + g_0\nabla N + \mathbf{F}$$

F is the frictional force

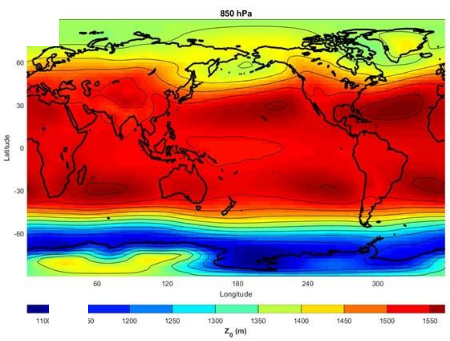
NCEP/NCAR reanalyzed global long-term atmospheric annual mean (Z, T, u, v) data

Z

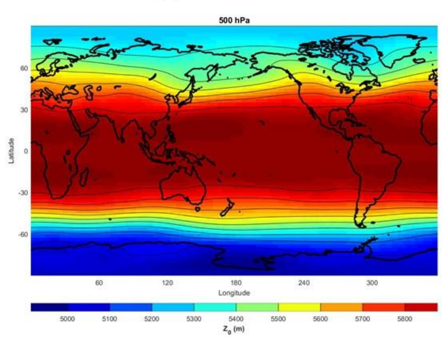
(a) Z at 1,000 hPa



(b) Z at 850 hPa

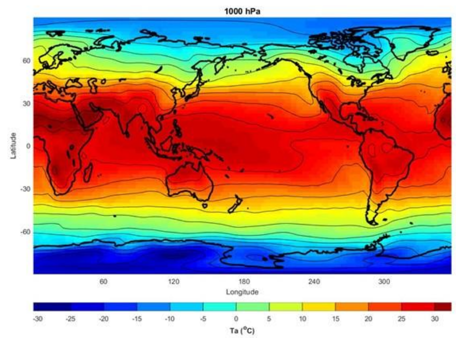


(c) Z at 500 hPa

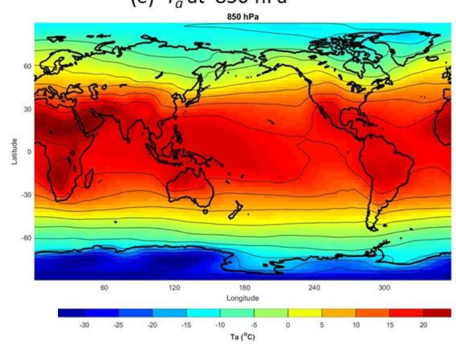


T (^{o}C)

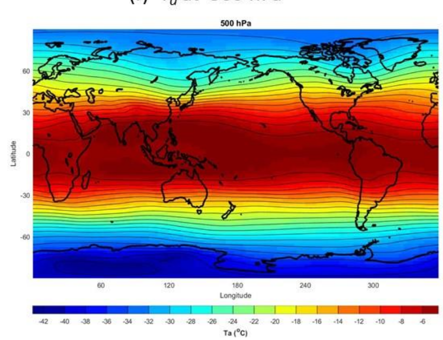
(d) T_a at 1,000 hPa



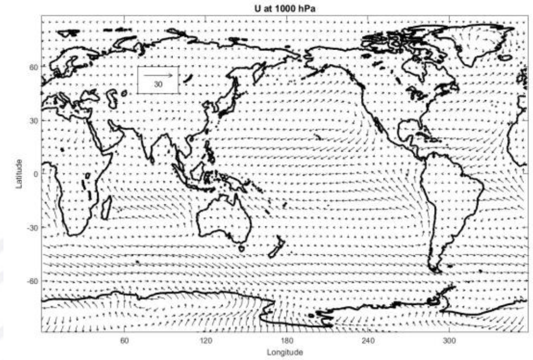
(e) T_a at 850 hPa



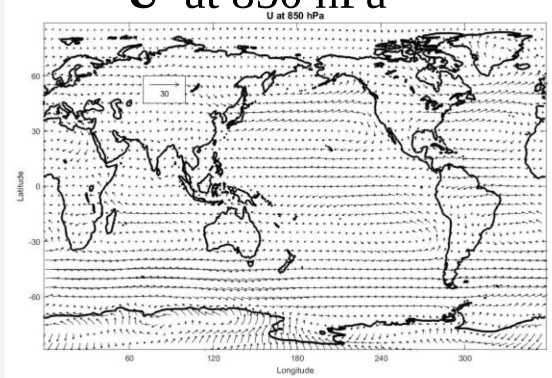
(f) T_a at 500 hPa



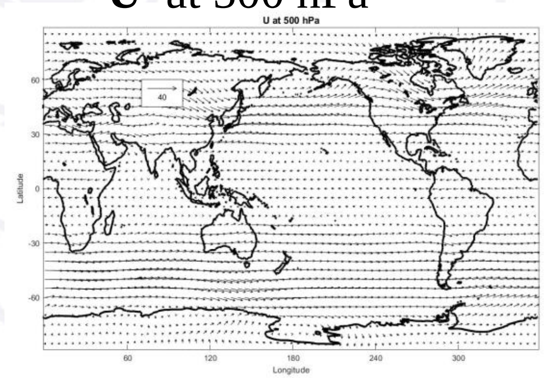
U



U at 850 hPa



U at 500 hPa



Geostrophic Wind

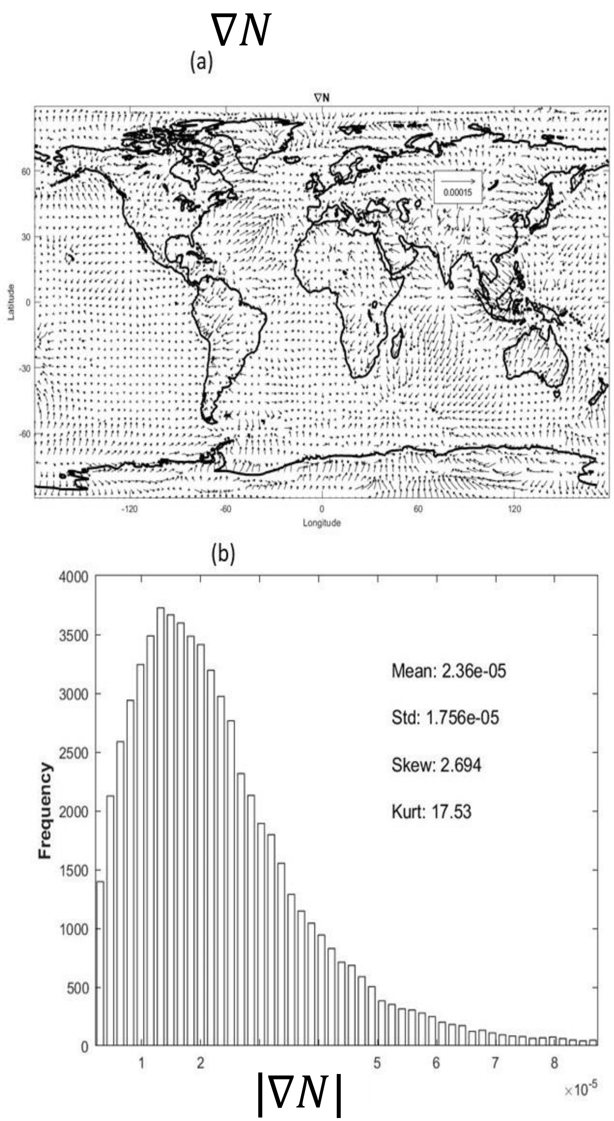
$$f\mathbf{k}\times\mathbf{U} = -g_0\nabla Z + g_0\nabla N$$

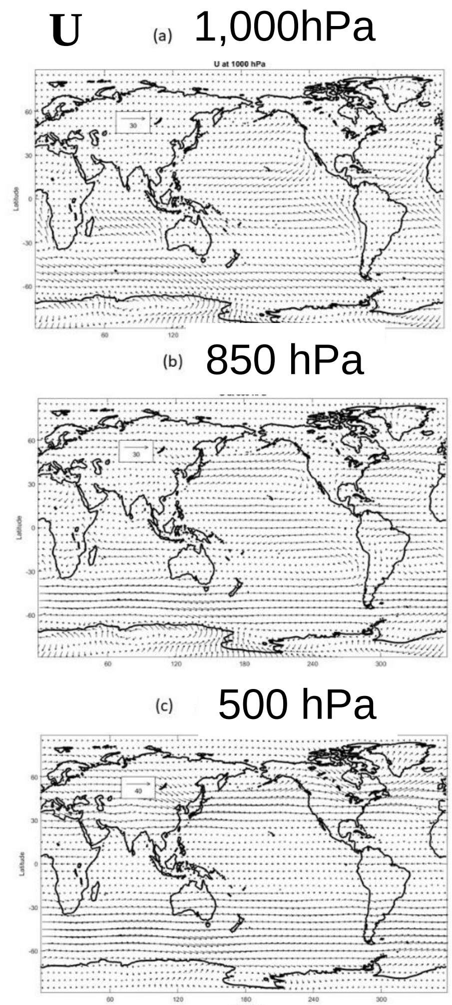
$$B \equiv \frac{O(|\delta\mathbf{g}|)}{O(|\text{Pressure Gradient Force}|)} = \frac{\mathrm{O}(|\nabla \mathrm{N}|)}{\mathrm{O}[|\nabla \mathrm{Z}|]} = \frac{\text{mean}(|\nabla N|)}{\text{mean}[|\nabla Z|]}$$

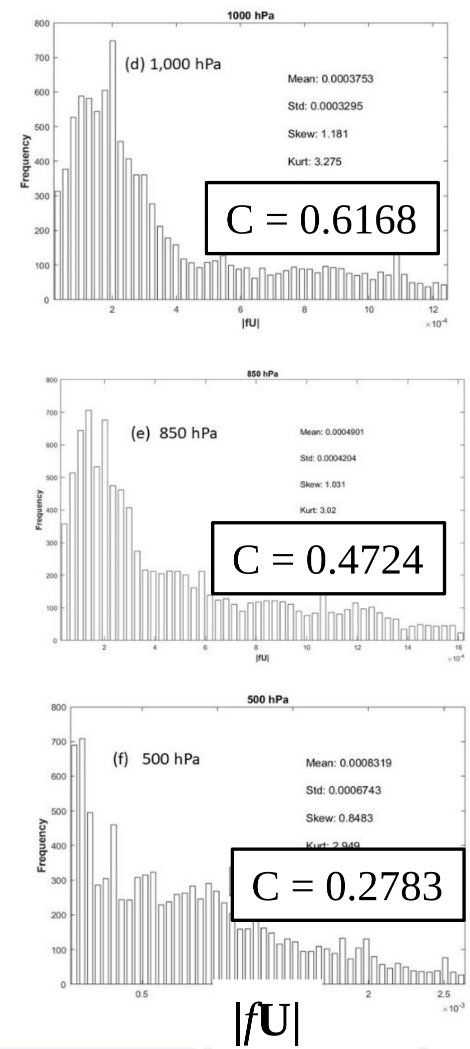
$$C \equiv \frac{O(|\delta\mathbf{g}|)}{O(|\text{Coriolis Force}|)} = \frac{g_0 O(|\nabla N|)}{O(|f\mathbf{U}|)} = g_0\frac{\text{mean}(|\nabla N|)}{\text{mean}(|f\mathbf{U}|)}$$

Nondimensional *C* number

$$C = g_0 \frac{\text{mean}(|\nabla N|)}{\text{mean}(|f\mathbf{U}|)}$$







D-Number (Effect on the Ekman Pumping)

Geostrophic Vorticity

$$\zeta = \frac{1}{f}\nabla^2\Phi = \zeta_{eff} + \zeta_{gd}, \quad \zeta_{eff} = \frac{1}{f}\nabla^2\Phi_{eff} = \frac{g_0}{f}\nabla^2 Z, \quad \zeta_{gd} = -\frac{g_0}{f}\nabla^2 N$$

Ekman Pumping Velocity

$$w_{Ekman} = \frac{\zeta}{2\gamma} = \frac{1}{2\gamma}(\zeta_{eff} + \zeta_{gd}), \quad \gamma \equiv \left|\frac{f}{2K}\right|^{1/2}\left(\frac{f}{|f|}\right)$$

G-Number

$$G \equiv \frac{O\left(\left|\zeta_{gd}\right|\right)}{O\left(\left|\zeta_{eff}\right|\right)} = \frac{O\left[\left|\nabla^2 N\right|\right]}{O\left[\left|\nabla^2 Z\right|\right]}$$

Q Vector (q_1, q_2) and Omega Equation

$$\sigma\nabla^2\omega + f^2\frac{\partial^2\omega}{\partial p^2} = -2\nabla\bullet\mathbf{Q} - \frac{\kappa}{p}\nabla^2 J$$

$$\nabla\bullet\mathbf{Q}^{eff} = \frac{R_a g_0}{pf}\nabla\bullet\mathbf{q}^{eff}, \quad \nabla\bullet\mathbf{Q}^{gd} = \frac{R_a g_0}{pf}\nabla\bullet\mathbf{q}^{gd}$$

Non-Dimensional (E_1, E_2, E_3) Numbers

$$E_1 \equiv \frac{O\left(\left|q_1^{gd}\right|\right)}{O\left(\left|q_1^{eff}\right|\right)} = \text{mean}\left(\left|J\left(\frac{\partial N}{\partial x}, T_a\right)\right|\right) / \text{mean}\left(\left|J\left(\frac{\partial Z}{\partial x}, T_a\right)\right|\right)$$

$$E_2 \equiv \frac{O\left(\left|q_2^{gd}\right|\right)}{O\left(\left|q_2^{eff}\right|\right)} = \text{mean}\left(\left|J\left(\frac{\partial N}{\partial y}, T_a\right)\right|\right) / \text{mean}\left(\left|J\left(\frac{\partial Z}{\partial y}, T_a\right)\right|\right)$$

$$E_3 \equiv \frac{O\left(\left|\nabla\bullet\mathbf{q}^{gd}\right|\right)}{O\left(\left|\nabla\bullet\mathbf{q}^{eff}\right|\right)}$$

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Non-Dimensional Numbers in Atmosphere

Pressure Level (hPa)	B Pressure Gradient Force	C Coriolis Force	G Geostrophic Vorticity	E_1 Q_1	E_2 Q_2	E_3 Omega Equation
1,000	0.4052	0.6168	0.6712	1.4268	0.5482	2.584
925	0.4151	0.5086	0.7539	1.6708	0.6205	3.366
850	0.4176	0.4724	0.8534	1.9850	0.7247	4.510
700	0.3836	0.3829	1.1078	2.8620	0.9097	6.911
600	0.3435	0.3327	1.3344	3.3175	1.0570	8.622
500	0.2849	0.2783	1.3381	3.3550	1.0077	8.797
400	0.2298	0.2241	1.2529	3.3121	0.9395	8.834
300	0.1861	0.1797	1.1169	3.0013	0.8091	7.495
250	0.1713	0.1645	1.0701	3.2204	0.7786	7.590
200	0.1630	0.1573	1.0564	3.1943	0.6829	7.667
150	0.1639	0.1611	1.0955	3.4567	0.7789	9.165
100	0.1869	0.1837	1.2109	4.3834	0.9965	11.713
Mean	0.2792	0.3052	1.0718	2.9313	0.8211	7.271

δg in Ocean Dynamics

Horizontal Momentum Equation in z Coordinate With the Boussinesq Approximation

$$\rho_0\left[\frac{D\mathbf{U}}{Dt}+f\mathbf{k}\times\mathbf{U}\right]=-\nabla\hat{p}+(\rho-\rho_0)g_0\nabla N+\rho_0(\mathbf{F}_h+\mathbf{F}_v)$$

Reference Density $\rho_0 = 1{,}028\ \mathrm{kg/m^3}$

Reference Pressure $p_0=-\rho_0 g_0(z-N)$

Dynamic Pressure $\hat{p}=p-p_0$

Hydrostatic Balance $\frac{\partial\hat{p}}{\partial z}=-(\rho-\rho_0)g_0$

$\mathbf{F}_h$ and $\mathbf{F}_v$ are the horizontal and vertical frictional forces

Geostrophic Current and Thermal Wind Relation

$$f\mathbf{k}\times\mathbf{U} = -\frac{1}{\rho_0}\nabla\hat{p} + \frac{\rho-\rho_0}{\rho_0}g_0\nabla N$$

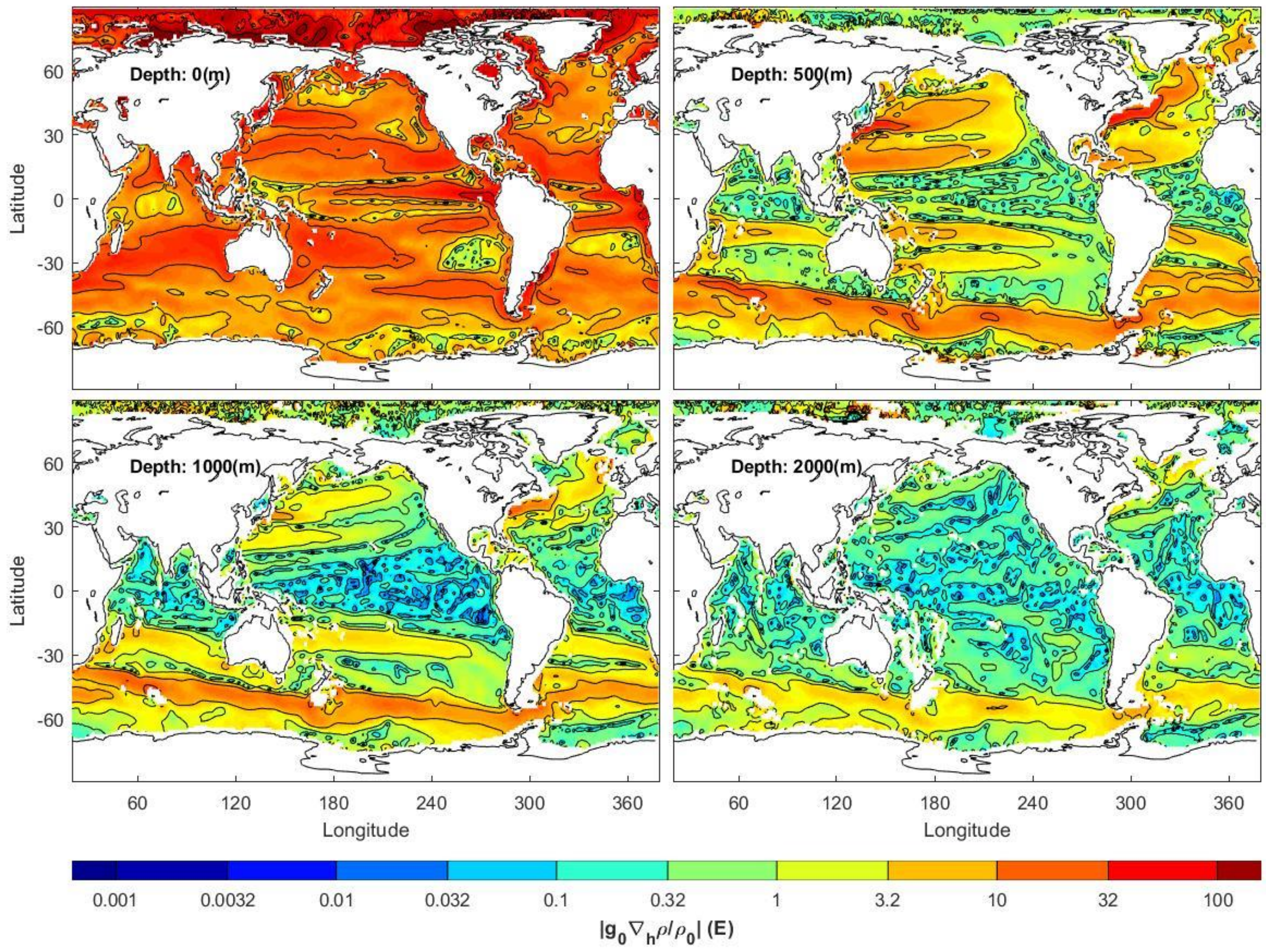
$$f\frac{\partial\mathbf{U}}{\partial z} = \mathbf{k}\times\left[-(g_0/\rho_0)\nabla\rho + \Theta^2\nabla N\right], \quad \Theta^2 \equiv -\left(\frac{g_0}{\rho_0}\frac{\partial\rho}{\partial z}\right)$$

Depth-dependent non-dimensional *G* number

$$D(z) = \frac{O\left(\left|\Theta^2\nabla N\right|\right)}{O\left(\left|(g_0/\rho_0)\nabla\rho\right|\right)} \approx \frac{\text{mean }\left(\left|\Theta^2\nabla N\right|\right)}{\text{mean }\left(\left|(g_0/\rho_0)\nabla\rho\right|\right)}$$

Thermal Wind due to Density

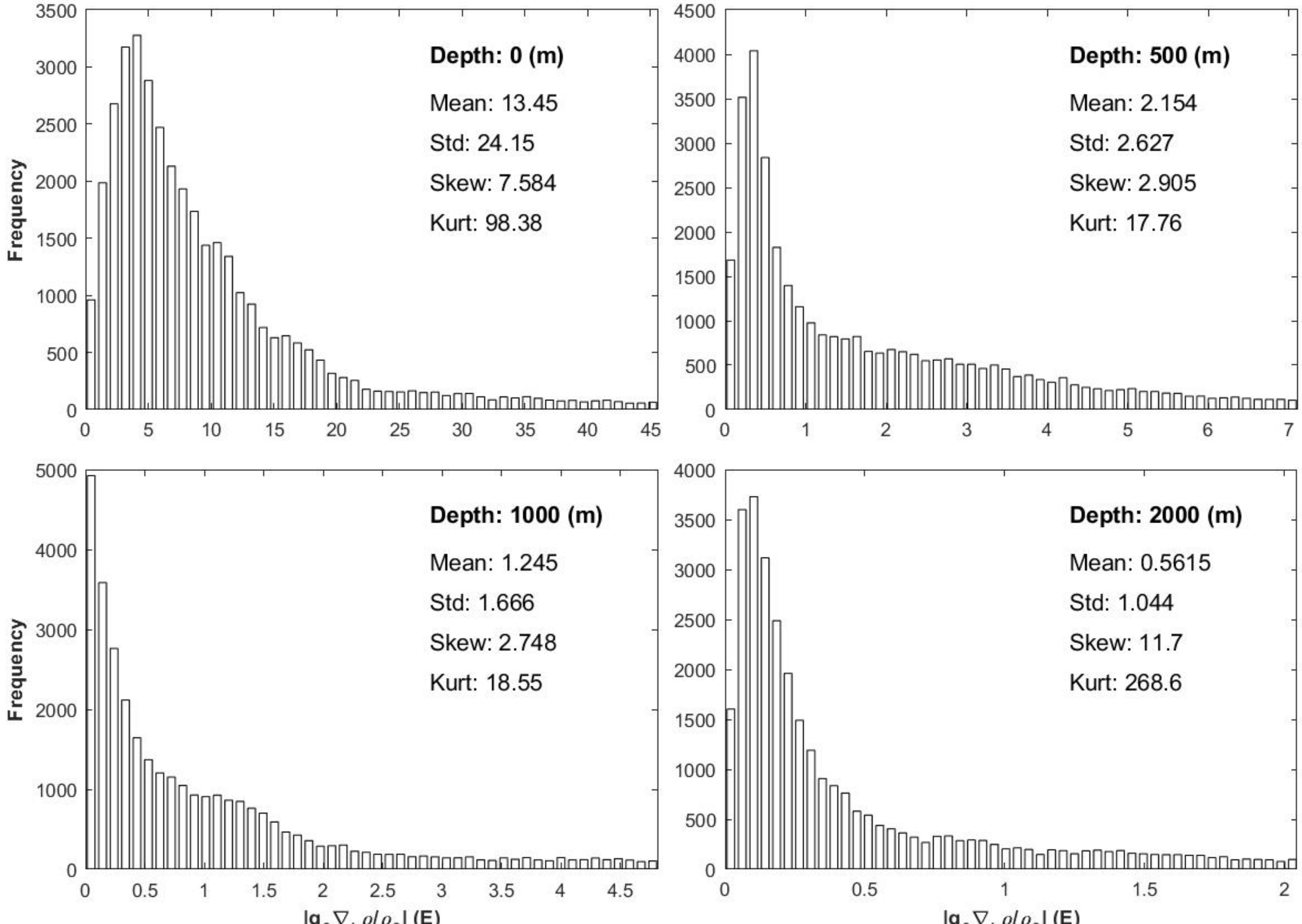
$|(g_0/\rho_0)\nabla\rho|$ in the unit of Eotvos (E) (1 E = 10^{-9} s^{-2})



From NOAA/NCEI WOA18 Data

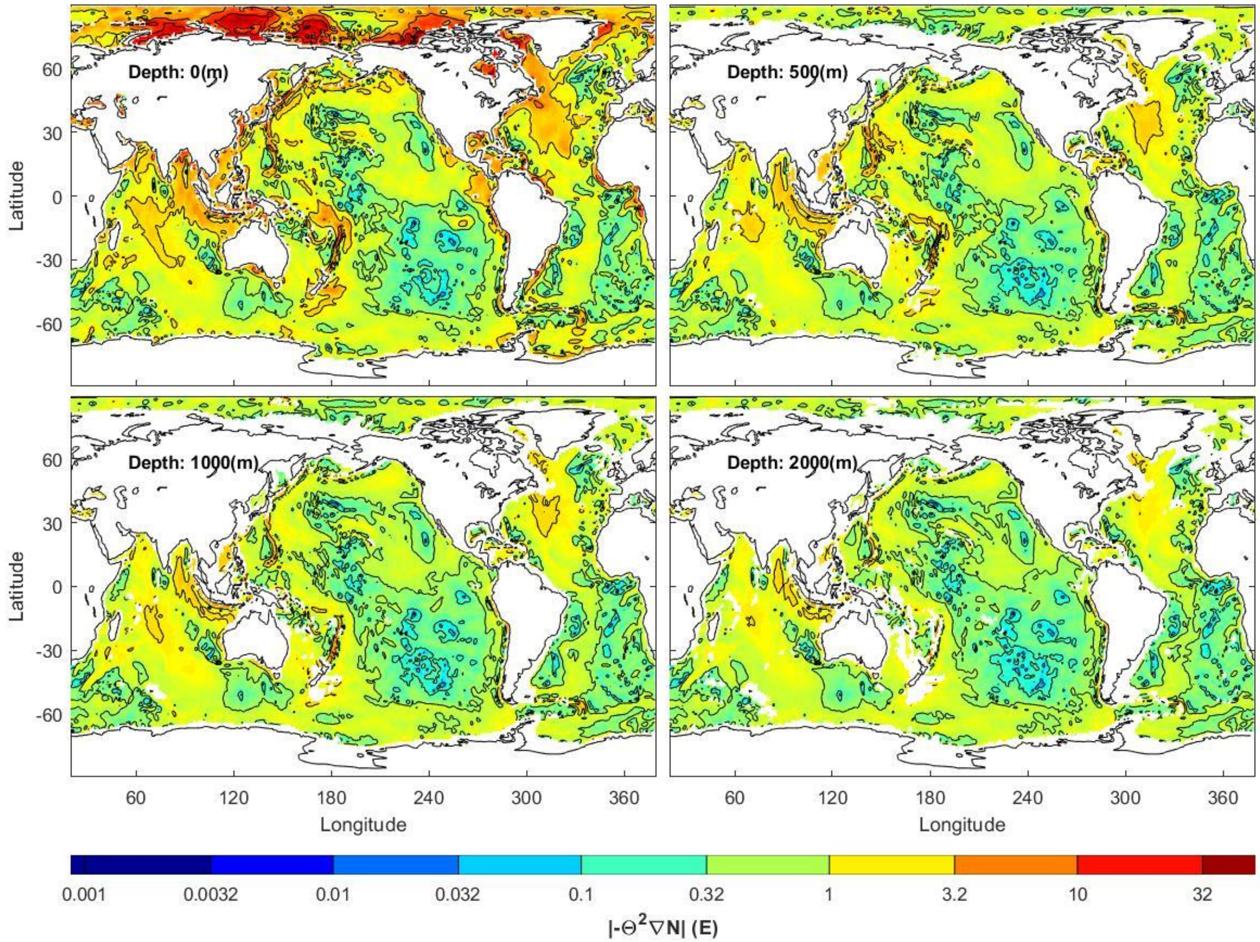
Histograms

$$|(g_0/\rho_0)\nabla\rho|$$



Thermal Wind due to δg

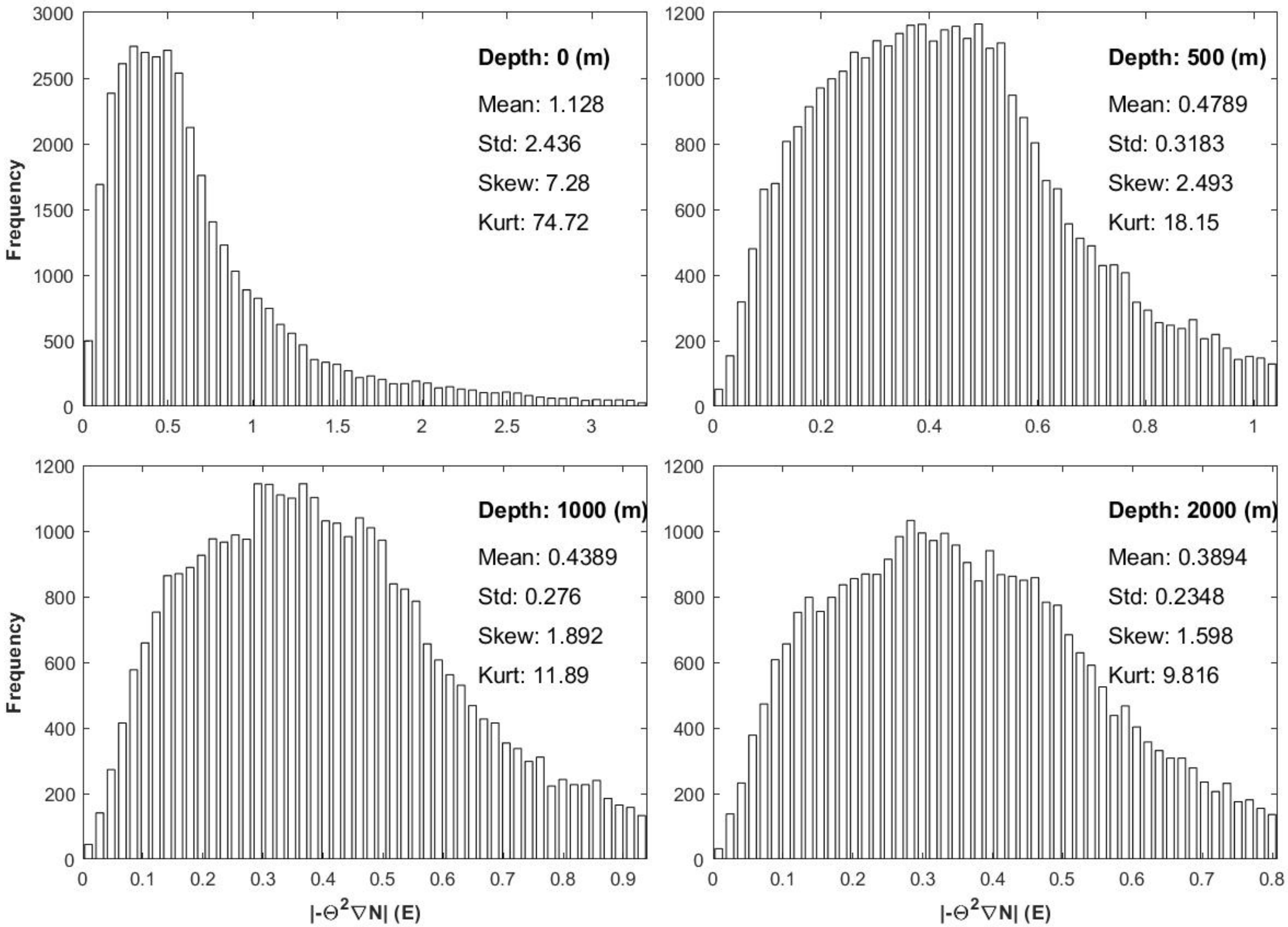
$|\Theta^2 \nabla N|$ in the unit of Eotvos (E)



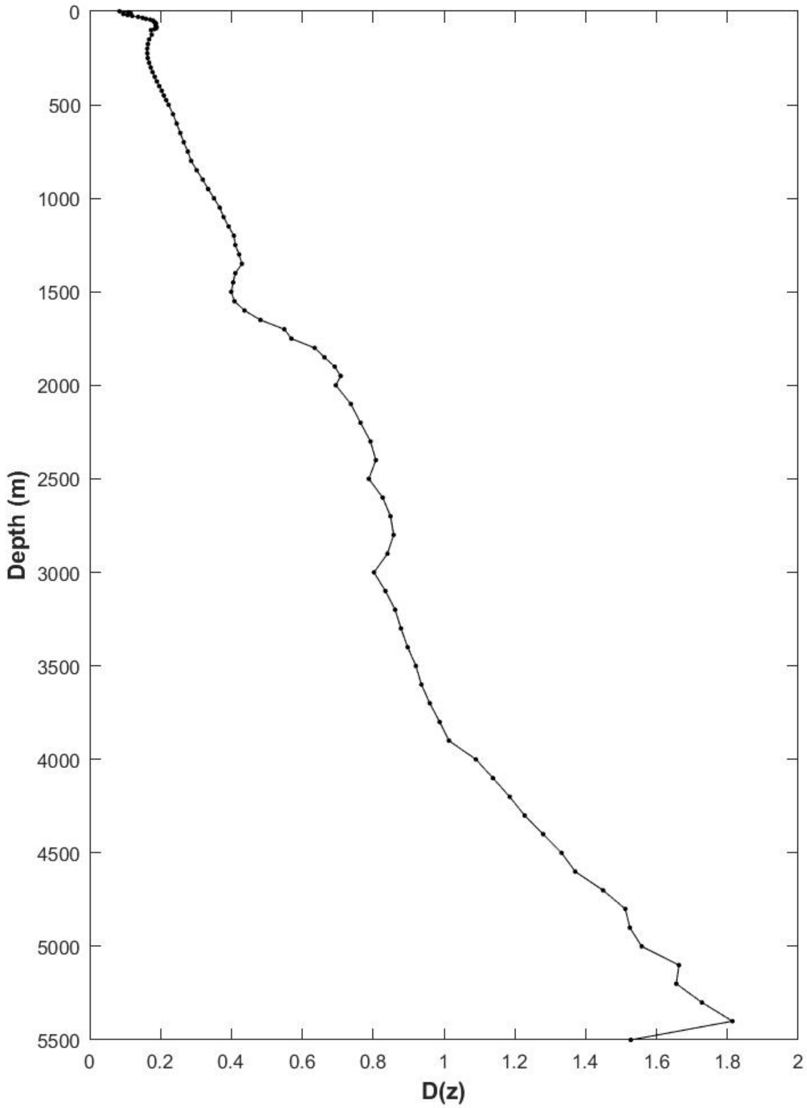
From EIGEN6C4

Histograms

$$|\Theta^2 \nabla N|$$



Depth-Dependent D Number



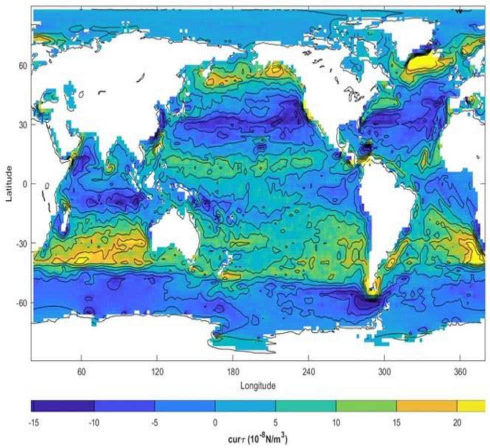
Wind-Driven Ocean Circulation
Combined Sverdrup-Stommel-Munk Dynamics

$$-A\nabla^4\Psi + \gamma\nabla^2\Psi + \beta\frac{\partial\Psi}{\partial x} = \frac{1}{\rho_0}\left[\text{curl}\,\boldsymbol{\tau} + g_0\int_{-H}^{0}\mathbf{k}\bullet\left(\nabla\rho\times\nabla N\right)dz\right]$$

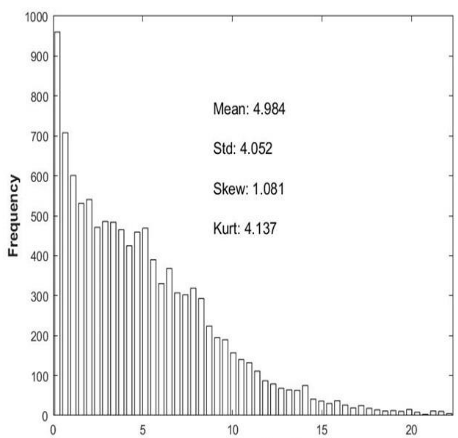
$$J = g_0\int_{-H}^{0}\mathbf{k}\bullet\left(\nabla\rho\times\nabla N\right)dz$$

Wind-Driven Ocean Circulation
Combined Sverdrup-Stommel-Munk Dynamics

Wind Forcing

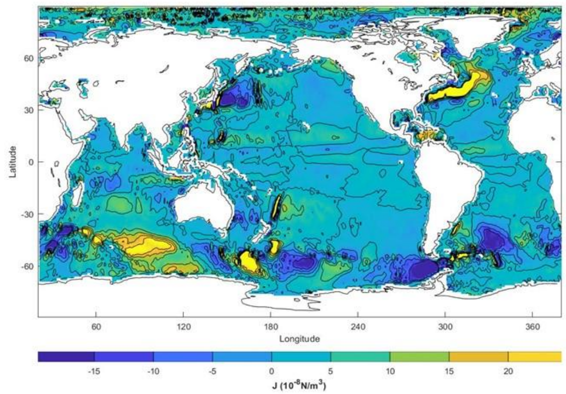


(b)

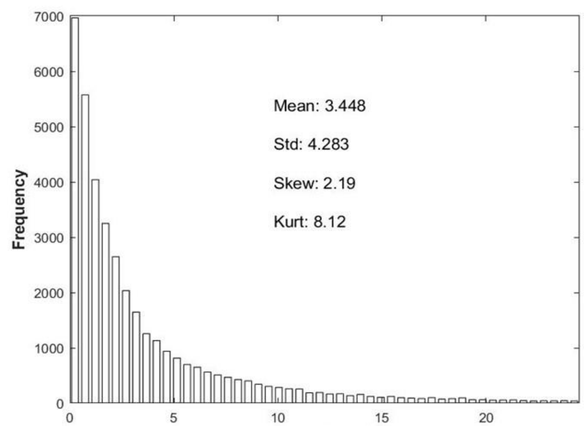


|curl **τ**| (10^{-8} N/m^3)

Forcing due to δ**g**



(b)



|J| (10^{-8} N/m^3)

$$F = \frac{\delta \mathbf{g}\text{-forcing}}{\text{wind-forcing}} = \frac{O\left[|J|\right]}{O\left[|\text{curl}\ \boldsymbol{\tau}|\right]} = \frac{3.448\times10^{-8}\ \text{N/m}^3}{4.984\times10^{-8}\ \text{N/m}^3} = 0.69$$

Results

- (1) Gravity used in meteorology and oceanography is not the true gravity.

- (2) The effect of gravity disturbance vector δ**g** in atmospheric and oceanic dynamics is important.
- (3) It is urgent and easy to include δ**g** in atmospheric and oceanic dynamics and numerical models.

References

- Chu, P.C., 2021: True gravity in ocean dynamics Part-1 Ekman transport. *Dynamics of Atmospheres and Oceans*, **96**, 101268, https://doi.org/10.1016/j.dynatmoce.2021.101268.

- Chu, P.C., 2023: Horizontal gravity disturbance vector in Atmospheric dynamics. *Dynamics of Atmospheres and Oceans*, **102**, 101369, https://www.sciencedirect.com/science/article/pii/S0377026523000209.

Thank you very much for listening.

Any comments and questions?