NAVAL POSTGRADUATE SCHOOL

Why We Can Approximate Spheroidal Geopotential Surfaces as Spherical but Can't Approximate True Geopotential Surfaces as Spheroidal?

Peter C. Chu

Department of Oceanography

Naval Postgraduate School

pcchu@nps.edu

http://http://faculty.nps.edu/pcchu/

NCAR/CGD –9 October 2024

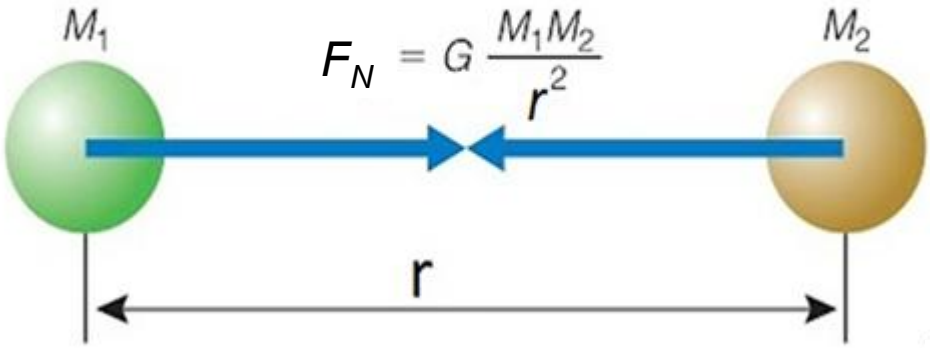
Distribution Statement A: Approved for Public Release

Outline

- (1) Earth Gravitation due to Inhomogeneous and Homogeneous Mass Densities
- (2) Spherical, Spheroidal, and True Geopotential Coordinates
- (3) Negligible Advective Metric Terms Among Spherical, Spheroidal, and True Geopotential Coordinates
- (4) Metric Coefficients in the Vector Differential Operator Among Spherical, Spheroidal, and True Geopotential Coordinates
- (5) Verification of SGA and EGA
- (6) Horizontal Equation of Motion Using the True Geopotential
- (7) Nonnegligible (Evidently Large) Horizontal Disturbance Vector $g_0 \nabla N$
- (8) Conclusions

(1) Earth Gravitation due to Inhomogeneous and Homogeneous Mass Densities

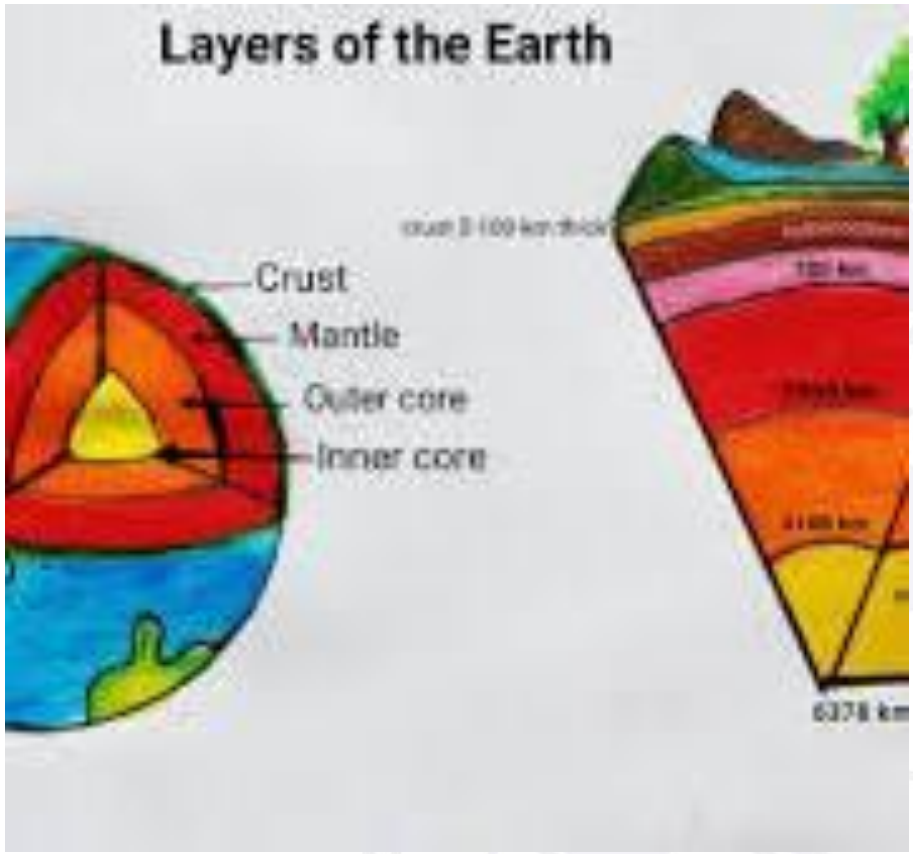
Newton's Universal Law of Gravitation



M_1 and M_2 are two point-masses (no volume) or far apart →
The distance between M_1 and M_2 are much larger than their radii.

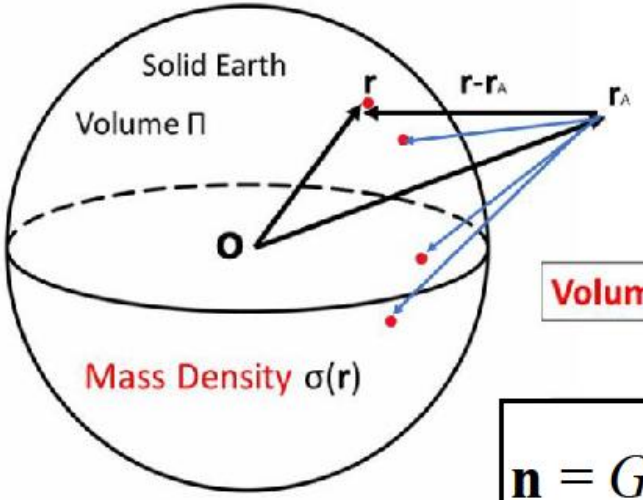
Newton's Gravitational Constant: $G = 6.67408 \times 10^{-11} Nm^2kg^{-2}$

Earth 5 Layered Structure



True Earth Gravitation (n)

Newton's gravitation is the volume integration over all the point masses inside the solid Earth to the point mass in atmosphere and oceans.



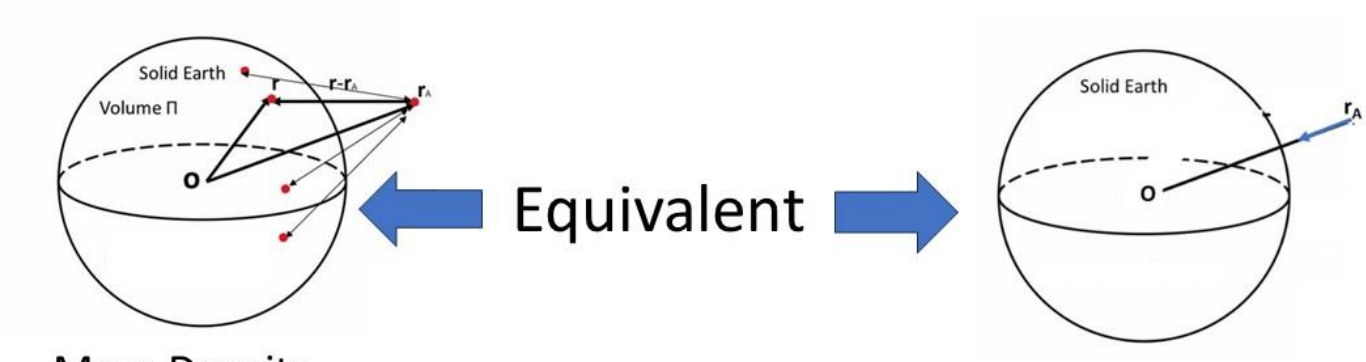
Newton's Gravitational Force of the Solid Earth on a Point Mass m_A at $\mathbf{r}_A$ in atmosphere

Volume Integration over the Solid Earth

$$\mathbf{n} = G\iiint_{\Pi} \frac{\sigma(\mathbf{r})}{|\mathbf{r}-\mathbf{r}_A|^3}(\mathbf{r}-\mathbf{r}_A)d\Pi$$

Standard Earth Gravitation ($\mathbf{n}_0$)

Standard Earth Gravitation ($\mathbf{n}_0$) is used in Meteorology and Oceanography



Mass Density
σ(r) = const

Solid Earth is treated as a point mass at the Earth center (**O**) with the whole Earth mass (M)

$$\mathbf{n}_0 \equiv -\frac{GM}{|\mathbf{r}_A|^3}\mathbf{r}_A$$

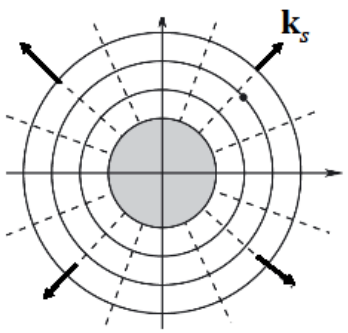
(2) Spherical, Spheroidal, and True Geopotential Coordinates

Spherical, Spheroidal, and True Geopotentials and Coordinates

Standard Gravity

$$\mathbf{g}_s = \mathbf{n}_0$$

(a)



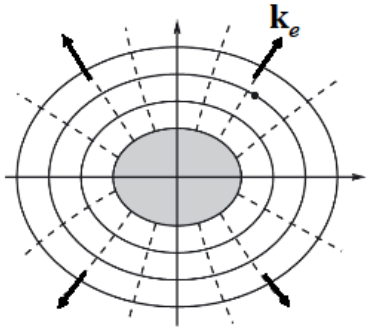
Standard Gravity
$\mathbf{g}_s = -g_0\mathbf{k}_s$

Spherical Geopotential Surfaces
$\Phi_s(r) = \text{const}$

Apparent Gravity

$$\mathbf{g}_a = \mathbf{n}_0 + \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r})$$

(b)



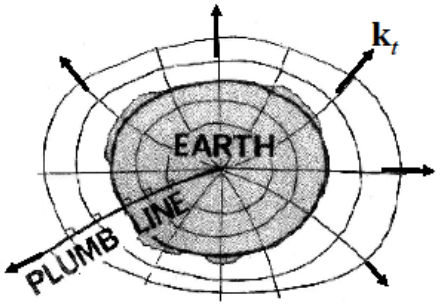
Apparent Gravity
$\mathbf{g}_a \approx -g_0\mathbf{k}_e$

Spheroidal Geopotential Surfaces
$\Phi_a(\varphi, r) = \text{const}$

True Gravity

$$\mathbf{g}_t = \mathbf{n} + \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r})$$

(c)



True Gravity
$\mathbf{g}_t \approx -g_0\mathbf{k}_t$

True Geopotential Surfaces
$\Phi_t(\lambda, \varphi, r) = \text{const}$

No any air-ocean model uses the true geopotential coordinates

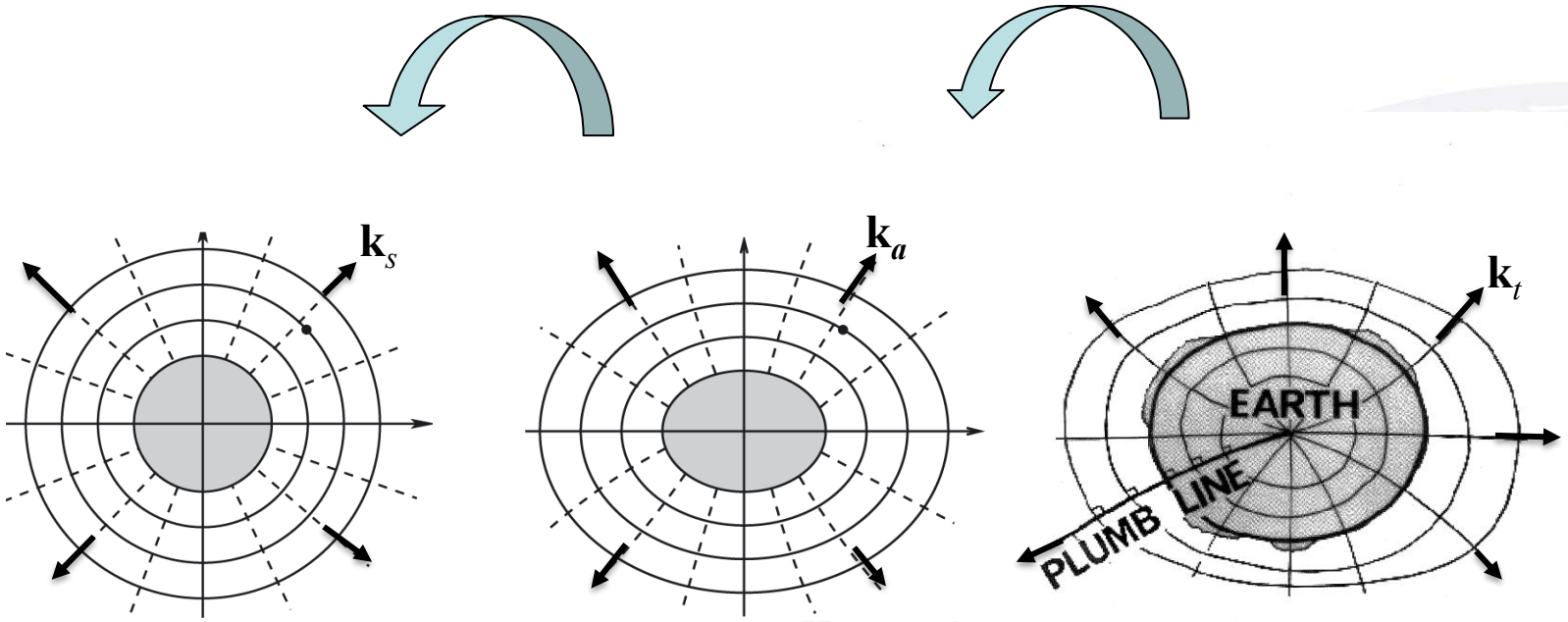
Gravitational Disturbance Vector δg

$$\delta\mathbf{g} \equiv \mathbf{g}_t - \mathbf{g}_a = \mathbf{n} - \mathbf{n}_0 = \nabla_3(\Phi_t - \Phi_a) = G\iiint_{\Pi} \frac{[\sigma(\mathbf{r}) - \sigma_0]}{|\mathbf{r} - \mathbf{r}_A|^3}(\mathbf{r} - \mathbf{r}_A)d\Pi$$

σ_0 = Average Mass Density of the Solid Earth

δ**g** is neglected in meteorology and oceanography through spheroidal geopotential approximation (EGA) and spherical geopotential approximation (SGA).

Spherical and Spheroidal Geopotential Approximations



SGA → Approximate the spheroidal geopotential surfaces as spherical

EGA → Approximate the true geopotential surfaces as spheroidal

EGA has never been verified.

Two Small Nondimensional Parameters

Ellipticity ε

Centrifugal-Gravitation Ratio μ

$$\varepsilon \equiv \frac{a-b}{a}, \quad \mu \equiv \frac{\Omega^2 a^3}{GM}$$

a = 6,378 km, equatorial semi-axis of Earth's assumed spheroidal surface
b = 6,356 km, polar semi-axis of Earth's assumed spheroidal surface
$G = 6.67408\times10^{-11}\text{Nm}^2\text{kg}^{-2}$, the Newtonian gravitational constant
$M = 5.98\times10^{24}$ kg, the total mass of the solid Earth
$\Omega = |\mathbf{\Omega}|$ [$= 2\pi/(86164\text{ s})$]

$$O(\varepsilon^2, \varepsilon\mu, \mu^2) = O(10^{-5})$$

Staniforth 2014 (QJRMS)

$$O(\varepsilon, \mu) \approx 10^{-5/2} = 0.003$$

Thinness of Combined Ocean and Troposphere

Thickness of combined troposphere and oceans is much thinner in comparison to the Earth radius R

$$\frac{r}{R} = 1 + \frac{r-R}{R} = 1 + O(\varepsilon, \mu)$$

$$O(\varepsilon, \mu) = 0.003$$

Negligible radial variation in the standard gravity $\mathbf{g}_s$ and gravitational disturbance vector $\delta\mathbf{g}$

$$|\mathbf{g}_s| = g_0, \quad g_0 = 9.81 \text{ m s}^{-2}$$

$$\frac{\left|\|\mathbf{g}_a| - g_0\right|}{g_0} < 10^{-2}, \quad \frac{\left|\|\mathbf{g}_t| - g_0\right|}{g_0} < 10^{-2}$$

$$|\mathbf{g}_a| = g_0, \quad |\mathbf{g}_t| = g_0$$

$$\delta\mathbf{g} = g_0 \nabla N$$

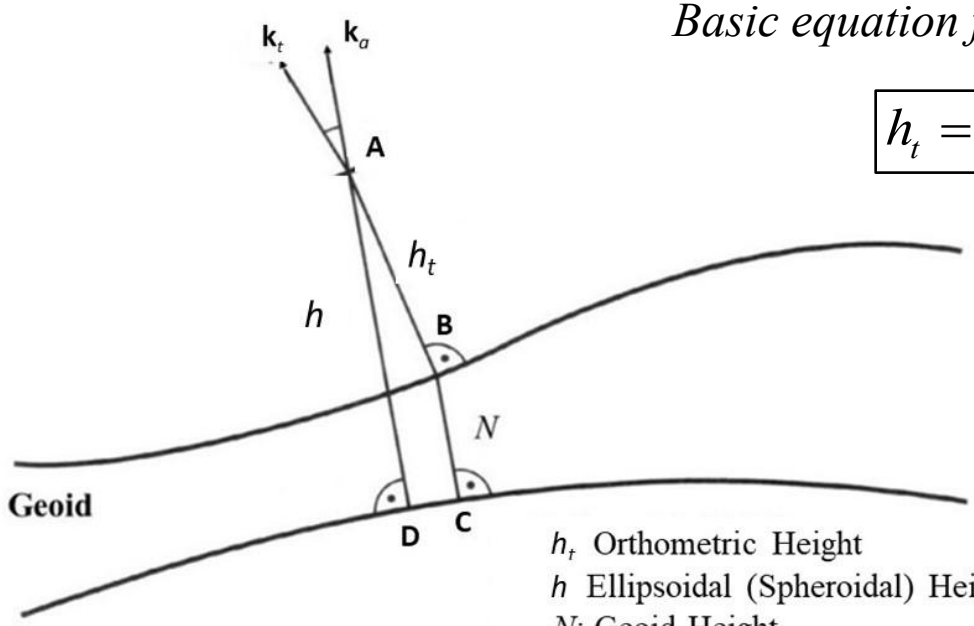
$N \rightarrow$ Geoid Height

$\nabla \rightarrow$ Horizontal Vector Differential Operator

Orthometric, Spheroidal & Geoid Heights

Basic equation for GPS levering

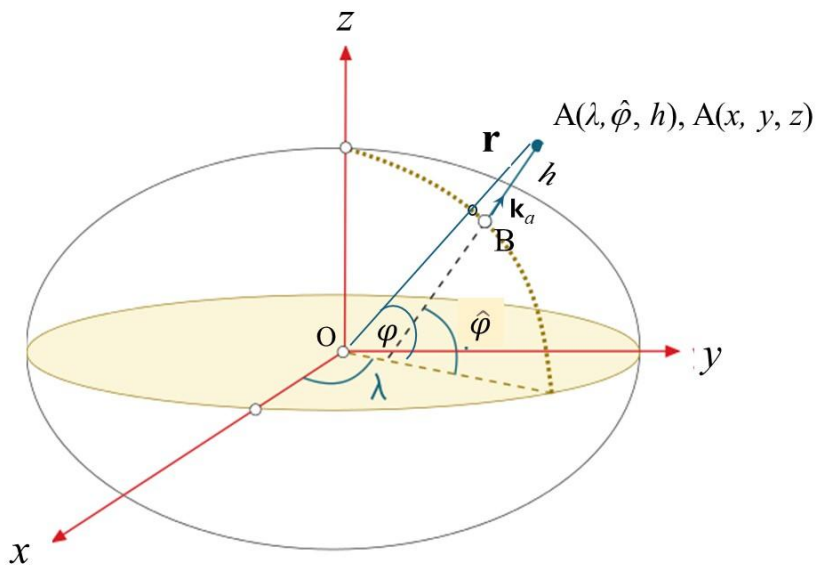
$$h_t = h - N$$



h_t Orthometric Height
h Ellipsoidal (Spheroidal) Height
N: Geoid Height
$\mathbf{k}_a$: Unit Vector for Ellipsoidal (Spheroidal) Normal
$\mathbf{k}_t$: Unit Vector for Geoid Normal

The angle between ($\mathbf{k}_a$, $\mathbf{k}_t$), defined as the deflection of vertical, is tiny (the figure is over exaggerated), e.g., only 5" in the Himalayas through astrogeodetic measurements [see Page 141 in *Physical Geodesy*, Hofmann-Wellenhof and Moritz (2006)].

Spheroidal and True Geopotential Coordinates vs Cartesian Coordinates



$\hat{\varphi}$ → Geodetic Latitude

φ → Geocentric Latitude

Spheroidal Geopotential Coordinates

$[\lambda, \hat{\varphi}, h]^a$

True Geopotential Coordinates

$[\lambda, \hat{\varphi}, h_t]^t$

$$[\lambda, \hat{\varphi}, h_t]^t \Leftrightarrow (\lambda, \hat{\varphi}) \Rightarrow \text{horizontal}, \quad h_t \Rightarrow \text{vertical}$$

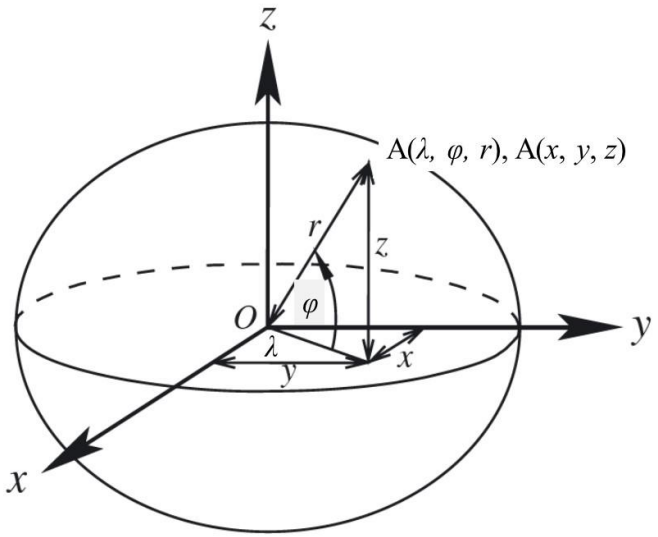
Cartesian Coordinates → (Oxyz)

Approximate Spheroidal Geopotential Coordinates

Thinness → $$\frac{r}{R} = 1 + \frac{r-R}{R} = 1 + O(\varepsilon, \mu)$$

Approximate Spheroidal Geopotential (Staniforth 2014 QJRMS)

$$\hat{\Phi}_a(\varphi, r) = \frac{GM}{r}\left\{1 - \left[\left(\frac{2\varepsilon - \mu}{2}\right)\frac{a^2}{R^2} + \frac{\mu}{2}\frac{R^3}{a^3}\right]\sin^2\varphi\right\} + \frac{GM}{R}\left[\left(\frac{2\varepsilon - \mu}{6}\right)\frac{a^2}{R^2} + \frac{\mu}{2}\frac{R^3}{a^3}\right]$$



Approximate Spheroidal Geopotential Coordinates

$$[\lambda, \varphi, r]^{\hat{a}}$$

(3) Negligible Advective Metric Terms Among Spherical, Spheroidal, and True Geopotential Coordinates

Advective Metric Terms

$[\lambda, \hat{\varphi}, h_t]^t$ True Geopotential Coordinates

$[\lambda, \hat{\varphi}, h]^a$ Spheroidal Geopotential Coordinates

$[\lambda, \varphi, r]^{\hat{a}}$ Approximate Spheroidal Geopotential Coordinates

$[\lambda, \varphi, r]^s$ Spherical Geopotential Coordinates

→ Curved & Quasi-Orthogonal Coordinates

$$\frac{D\mathbf{V}}{Dt} = \mathbf{i}\frac{Du}{Dt} + \mathbf{j}\frac{Dv}{Dt} + \mathbf{k}\frac{Dw}{Dt} + u\frac{D\mathbf{i}}{Dt} + v\frac{D\mathbf{j}}{Dt} + w\frac{D\mathbf{k}}{Dt}$$

The unit vectors (**i**, **j**, **k**) are for the four geopotential coordinates

$$\frac{d\mathbf{V}}{dt} \equiv \mathbf{i}\frac{Du}{Dt} + \mathbf{j}\frac{Dv}{Dt} + \mathbf{k}\frac{Dw}{Dt}$$

← Acceleration vector as if it is in the global Cartesian system

$$\mathbf{M} \equiv u\frac{D\mathbf{i}}{Dt} + v\frac{D\mathbf{j}}{Dt} + w\frac{D\mathbf{k}}{Dt}$$

← Advective Metric Terms

Negligible Advective Metric Terms in Spherical, Spheroidal, and True Geopotential Coordinates

Spherical (Gill 1982)

$$\mathbf{M}_s = \left[\frac{uw}{r} - \frac{uv}{r}\tan\varphi, \quad \frac{vw}{r} + \frac{u^2}{r}\tan\varphi, \quad -\frac{u^2+v^2}{r} \right]^s$$

$$|\mathbf{M}_s| \ll \left| {}^{d\mathbf{V}}/_{dt} \right|$$

Spheroidal (Gates 2004)

$$\mathbf{M}_a = \begin{bmatrix} \left(\frac{uw}{c\chi}\tanh\xi - \frac{uv}{c\chi}\tan\varphi \right), & \left(\frac{vw}{c\chi^3}\sinh\xi\cosh\xi + \frac{u^2}{c\chi}\tan\varphi - \frac{w^2}{c\chi^3}\sin\varphi\cos\varphi \right), \\ \left(-\frac{v^2}{c\chi^3}\sinh\xi\cosh\xi - \frac{u^2}{c\chi}\tanh\xi + \frac{vw}{c\chi^3}\sin\varphi\cos\varphi \right) \end{bmatrix}^a$$

$$\frac{O\left(|\mathbf{M}_a - \mathbf{M}_s|\right)}{O\left(|\mathbf{M}_s|\right)} \ll 1$$

The geoid height (N) varies from a minimum value of -106.20 m in the Indian Ocean, and a maximum of 85.83 m near Greenland, which is much smaller than the difference between the Earth spheroidal and spherical surfaces.

$$\frac{O\left(|\mathbf{M}_t - \mathbf{M}_a|\right)}{O\left(|\mathbf{M}_a|\right)} \ll 1$$

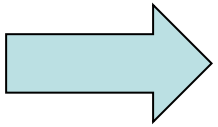
$$\left(|\mathbf{M}_s|, |\mathbf{M}_a|, |\mathbf{M}_t|\right) \ll \left| {}^{d\mathbf{V}}/_{dt} \right|$$

All the advective metric terms ($\mathbf{M}_s$, $\mathbf{M}_a$, $\mathbf{M}_t$) are negligible

Negligible Advective Metric Terms Are Not Sufficient for SGA and EGA

All the advective metric terms ($\mathbf{M}_s$, $\mathbf{M}_a$, $\mathbf{M}_t$) are negligible

$$(|\mathbf{M}_s|, |\mathbf{M}_a|, |\mathbf{M}_t|) \ll \left|{}^{d\mathbf{V}}/_{dt}\right|$$



NOT Sufficient for Verifying SGA and EGA

(4) Metric Coefficients in the Vector Differential Operator Among Spherical, Spheroidal, and True Geopotential Coordinates

Metric Coefficient Tensors in Vector Differential Operator

$$g_{ij} = \frac{\partial x^p}{\partial \xi^i}\frac{\partial x^q}{\partial \xi^j}\delta_{pq} = \frac{\partial x^p}{\partial \xi^i}\frac{\partial x^p}{\partial \xi^j}$$

$[x^1, x^2, x^3] = [x, y, z]$ → Cartesian Coordinates

$[\xi^1, \xi^2, \xi^3]$ →

$[\lambda, \hat{\varphi}, h_t]^t$	True Geopotential Coordinates
$[\lambda, \hat{\varphi}, h]^a$	Spheroidal Geopotential Coordinates
$[\lambda, \varphi, r]^{\hat{a}}$	Approximate Spheroidal Geopotential Coordinates
$[\lambda, \varphi, r]^s$	Spherical Geopotential Coordinates

Same Metric Coefficients between the True and Spheroidal Geopotential Coordinates

Spheroidal Geopotential Coordinates

$[\lambda, \hat{\varphi}, h]^a$

$$x = (R_N + h)\cos\hat{\varphi}\cos\lambda$$

$$y = (R_N + h)\cos\hat{\varphi}\sin\lambda$$

$$z = \left(\frac{b^2}{a^2}R_N + h\right)\sin\hat{\varphi}$$

$$R_N = \frac{a^2}{\sqrt{a^2\cos^2\hat{\varphi} + b^2\sin^2\hat{\varphi}}}$$

Hofmann-Wellenhof and Moritz (2006)

$$[\nabla_3]^a = \left(\frac{1}{R_N\cos\hat{\varphi}}\frac{\partial}{\partial\lambda}, \frac{1}{R_N}\frac{\partial}{\partial\hat{\varphi}}, \frac{\partial}{\partial h}\right)$$

$$[\nabla p(\lambda,\hat{\varphi},h)]^a = \left(\frac{1}{R_N\cos\hat{\varphi}}\frac{\partial p}{\partial\lambda}, \frac{1}{R_N}\frac{\partial p}{\partial\hat{\varphi}}\right) \quad (1)$$

$\nabla \rightarrow$ horizontal vector differential operator

R_N is the normal radius of the curvature

True Geopotential Coordinates

$[\lambda, \hat{\varphi}, h_t]^t$

$$x = (R_N + h_t + N)\cos\hat{\varphi}\cos\lambda$$

$$y = (R_N + h_t + N)\cos\hat{\varphi}\sin\lambda$$

$$z = \left(\frac{b^2}{a^2}R_N + h_t + N\right)\sin\hat{\varphi}$$

$$[\nabla_3]^t = \left(\frac{1}{R_N\cos\hat{\varphi}}\frac{\partial}{\partial\lambda}, \frac{1}{R_N}\frac{\partial}{\partial\hat{\varphi}}, \frac{\partial}{\partial h_t}\right)$$

$$h_t = h - N$$

$$[\nabla p(\lambda,\hat{\varphi},h_t)]^t = \left[1/(R_N\cos\hat{\varphi})\left(\partial p/\partial\lambda + \partial p/\partial h_t\ \partial h_t/\partial\lambda\right), 1/R_N\left(\partial p/\partial\hat{\varphi} + \partial p/\partial h_t\ \partial h_t/\partial\hat{\varphi}\right)\right] \quad (2)$$

Metric Coefficients for the Approximate Spheroidal and Spherical Geopotential Coordinates

Approximate Spheroidal Geopotential Coordinates

$[\lambda, \varphi, r]^{\hat{a}}$

$$x = R\left\{1-\left[\left(\frac{2\varepsilon-m}{2}\right)\frac{a^2}{R^2}+\frac{m}{2}\frac{R^3}{a^3}\right]\sin^2\varphi\right\}\cos\varphi\cos\lambda$$

$$y = R\left\{1-\left[\left(\frac{2\varepsilon-m}{2}\right)\frac{a^2}{R^2}+\frac{m}{2}\frac{R^3}{a^3}\right]\sin^2\varphi\right\}\cos\varphi\sin\lambda$$

$$z = R\left\{1-\left[\left(\frac{2\varepsilon-m}{2}\right)\frac{a^2}{R^2}+\frac{m}{2}\frac{R^3}{a^3}\right]\sin^2\varphi\right\}\sin\varphi$$

Staniforth (2014)

$$\hat{g}_{11} = R^2\left\{1-\left[\left(\frac{2\varepsilon-\mu}{2}\right)\frac{a^2}{R^2}+\frac{\mu}{2}\frac{R^3}{a^3}\right]\sin^2\varphi\right\}^2\cos^2\varphi$$

$$\hat{g}_{22} = R^2\left\{1+\left[\left(\frac{2\varepsilon-\mu}{2}\right)\frac{a^2}{R^2}+\frac{\mu}{2}\frac{R^3}{a^3}\right]\left(2-3\sin^2\varphi\right)\sin^2\varphi\right\}^2\sin^2\varphi$$

$$+R^2\left\{1+\left[\left(\frac{2\varepsilon-\mu}{2}\right)\frac{a^2}{R^2}+\frac{\mu}{2}\frac{R^3}{a^3}\right]\left(-3\sin^2\varphi_s\right)\sin^2\varphi\right\}^2\cos^2\varphi$$

$$\hat{g}_{33} = \left\{1+\left[\left(\frac{2\varepsilon-\mu}{2}\right)\frac{a^2}{R^2}-\frac{\mu}{2}\frac{R^3}{a^3}\right]\sin^2\varphi\right\}^2$$

$$[\nabla_3]^{\hat{a}} = \left(1/\sqrt{\hat{g}_{11}}\,\partial/\partial\lambda,\ 1/\sqrt{\hat{g}_{22}}\,\partial/\partial\varphi,\ \partial/\sqrt{\hat{g}_{33}}\partial r\right) \quad (3)$$

Spherical Geopotential Coordinates

$[\lambda, \varphi, r]^{s}$

$$[\nabla_3]^{s} = \left(1/R\cos\varphi\,\partial/\partial\lambda,\ 1/R\,\partial/\partial\varphi,\ \partial/\partial r\right) \quad (4)$$

(5) Verification of SGA and EGA

Horizontal Equation of Motion and Associate Hydrostatic Equilibrium

F → Frictional Force

Spherical $[\lambda, \varphi, r]^s$

$$[d\mathbf{U}_s/dt]^s + 2\boldsymbol{\Omega} \times [\mathbf{U}_s]^s = -\frac{1}{\rho}[\nabla p]^s + \mathbf{F} \quad (5a)$$

$$[\partial p / \partial r]^s = -\rho g_0 \quad (5b)$$

Approximate Spheroidal $[\lambda, \varphi, r]^{\hat{a}}$

$$[d\mathbf{U}_a/dt]^{\hat{a}} + 2\boldsymbol{\Omega} \times [\mathbf{U}_a]^{\hat{a}} = -\frac{1}{\rho}[\nabla p]^{\hat{a}} + \mathbf{F} \quad (6a)$$

$$[\partial p / \partial r]^a = -\rho g_0 \quad (6b)$$

Spheroidal $[\lambda, \hat{\varphi}, h]^a$

$$[d\mathbf{U}_a/dt]^a + 2\boldsymbol{\Omega} \times [\mathbf{U}_a]^a = -\frac{1}{\rho}[\nabla p]^a + \mathbf{F} \quad (7a)$$

$$[\partial p / \partial h]^a = -\rho g_0 \quad (7b)$$

True $[\lambda, \hat{\varphi}, h_t]^t$

$$[d\mathbf{U}_t/dt]^t + 2\boldsymbol{\Omega} \times [\mathbf{U}_t]^t = -\frac{1}{\rho}[\nabla p]^t + \mathbf{F} \quad (8a)$$

$$[\partial p / \partial h_t]^t = -\rho g_0 \quad (8b)$$

Mathematical Representation of the SGA

SGA is to use the equation of motion (5a) on the spherical geopotential surfaces for the approximate spheroidal geopotential

$$[d\mathbf{U}_a/dt]^s + 2\boldsymbol{\Omega}\times[\mathbf{U}_a]^s = -\frac{1}{\rho}[\nabla p]^s + \mathbf{F} + \boldsymbol{\varepsilon}_1 \qquad (9)$$

to get the same velocity and acceleration as if they are on the spheroidal geopotential surfaces

$$[\mathbf{U}_a]^{\hat{a}} = [\mathbf{U}_a]^s,\ [d\mathbf{U}_a/dt]^{\hat{a}} = [d\mathbf{U}_a/dt]^s \qquad (10)$$

Subtracting Eq. (9) from Eq. (6a) and using Eq. (10) →

$$\boldsymbol{\varepsilon}_1 = \frac{1}{\rho}[\nabla p(\lambda, \varphi, r)]^s - \frac{1}{\rho}[\nabla p(\lambda, \varphi, r)]^{\hat{a}} \qquad (11)$$

Negligible Relative SGA Errors

- SGA is Valid -

$$[\nabla_3]^{\hat{a}} = \left(1/\sqrt{\hat{g}_{11}}\,\partial/\partial\lambda,\ 1/\sqrt{\hat{g}_{22}}\,\partial/\partial\varphi,\ \partial/\sqrt{\hat{g}_{33}}\partial r\right) \tag{3}$$

$$[\nabla_3]^{s} = \left(1/R\cos\varphi\,\partial/\partial\lambda,\ 1/R\,\partial/\partial\varphi,\ \partial/\partial r\right) \tag{4}$$

Longitudinal

$$\gamma_\lambda \equiv \left(1/\sqrt{\hat{g}_{11}}\,\partial p/\partial\lambda - 1/R\cos\varphi\,\partial p/\partial\lambda\right)\Big/\left(1/\sqrt{\hat{g}_{11}}\,\partial p/\partial\lambda\right) = 1 - \sqrt{\hat{g}_{11}}/R\cos\varphi$$
$$= \left[(2\varepsilon-\mu)a^2/(2R^2) + \mu R^3/(2a^3)\right]\sin^2\varphi$$

Latitudinal

$$\gamma_\varphi \equiv \left(1/\sqrt{\hat{g}_{22}}\,\partial p/\partial\varphi - 1/R\,\partial p/\partial\varphi\right)\Big/\left(1/\sqrt{\hat{g}_{22}}\,\partial p/\partial\varphi\right) = 1 - \sqrt{\hat{g}_{22}}/R$$
$$= -\left[(2\varepsilon-\mu)a^2/(2R^2) + \mu R^3/(2a^3)\right](2-6\sin^2\varphi)\sin^2\varphi$$

$$O(|\gamma_\lambda|) = O(|\gamma_\varphi|) = O(\varepsilon,\mu) \approx 10^{-2.5} = 0.003$$

Mathematical Representation of the EGA

EGA is to use the equation of motion (7a) on the spheroidal geopotential surfaces for the true geopotential

$$[d\mathbf{U}_t/dt]^a + 2\boldsymbol{\Omega}\times[\mathbf{U}_t]^a = -\frac{1}{\rho}[\nabla p]^a + \mathbf{F} + \boldsymbol{\varepsilon}_2 \qquad (12)$$

to get the same velocity and acceleration as if they are on the spheroidal geopotential surfaces

$$[\mathbf{U}_t]^t = [\mathbf{U}_t]^a,\ [d\mathbf{U}_t/dt]^t = [d\mathbf{U}_t/dt]^a \qquad (13)$$

Subtracting Eq. (12) from Eq. (8a) and using Eq. (13) →

$$\boldsymbol{\varepsilon}_2 = \frac{1}{\rho}[\nabla p]^a - \frac{1}{\rho}[\nabla p]^t \qquad (14)$$

EGA Error = $g_0 \nabla N$

- EGA is Invalid -

$$[\nabla p(\lambda, \hat{\varphi}, h)]^a = \left(\frac{1}{R_N \cos\hat{\varphi}} \frac{\partial p}{\partial \lambda}, \quad \frac{1}{R_N} \frac{\partial p}{\partial \hat{\varphi}} \right) \qquad (1)$$

$$h_t = h - N$$

$$[\nabla p(\lambda, \hat{\varphi}, h_t)]^t = \left[{}^{1}/_{(R_N cos\hat{\varphi})} \left({}^{\partial p}/_{\partial \lambda} + {}^{\partial p}/_{\partial h_t} {}^{\partial h_t}/_{\partial \lambda} \right), {}^{1}/_{R_N} \left({}^{\partial p}/_{\partial \hat{\varphi}} + {}^{\partial p}/_{\partial h_t} {}^{\partial h_t}/_{\partial \hat{\varphi}} \right) \right] \qquad (2)$$

$$[\partial p / \partial h_t]^t = -\rho g_0$$

$$\nabla h_t = -\nabla N$$

$$\boldsymbol{\varepsilon}_2 = \frac{1}{\rho}[\nabla p]^a - \frac{1}{\rho}[\nabla p]^t$$

$$\boldsymbol{\varepsilon}_2 = g_0 \nabla N$$

(6) Horizontal Equation of Motion Using the True Geopotential

Horizontal Equation of Motion with the True Geopotential

Spheroidal Geopotential Coordinates →

$$\left[d\mathbf{U}_t / dt\right]^a + 2\boldsymbol{\Omega} \times \left[\mathbf{U}_t\right]^a = -\frac{1}{\rho}\left[\nabla p\right]^a + g_0\left[\nabla N\right]^a + \mathbf{F}$$

Spherical Geopotential Coordinates Using SGA →

$$\left[d\mathbf{U}_t / dt\right]^s + 2\boldsymbol{\Omega} \times \left[\mathbf{U}_t\right]^s = -\frac{1}{\rho}\left[\nabla p\right]^s + g_0\left[\nabla N\right]^s + \mathbf{F}$$

δg in Atmospheric Dynamics

Horizontal Momentum Equation in the Spheroidal Coordinates with Pressure as Vertical Coordinate

$$\frac{D\mathbf{U}}{Dt} + f\mathbf{k}\times\mathbf{U} = -g_0\nabla Z + g_0\nabla N + \mathbf{F}$$

(7) Nonnegligible (Sufficiently Large) Horizontal Disturbance Vector $g_0 \nabla N$

Gravity Disturbance Vector δg vs Pressure Gradient and Coriolis Forces

$$B \equiv \frac{O(|\,\delta \mathbf{g}|)}{O(|\,\text{Pressure Gradient Force}|)}$$
$$= \frac{\mathrm{O}(|\nabla \mathrm{N}|)}{\mathrm{O}\left[|\nabla \mathrm{Z}|\right]} = \frac{\text{mean}(|\,\nabla N\,|)}{\text{mean}\left[|\,\nabla Z\,|\right]}$$

$$C \equiv \frac{O(|\,\delta \mathbf{g}|)}{O(|\,\text{Coriolis Force}|)}$$
$$= \frac{g_0 O(|\,\nabla N\,|)}{O(|\,f\mathbf{U}\,|)} = g_0 \frac{\text{mean}(|\,\nabla N\,|)}{\text{mean}(|\,f\mathbf{U}\,|)}$$

Datasets to identify the importance of δg

(a) International Center for Global Earth Models (ICGEM) static gravity field model EIGEN-6C4 (http://icgem.gfz-potsdam.de/home) for $N(\lambda, \varphi)$

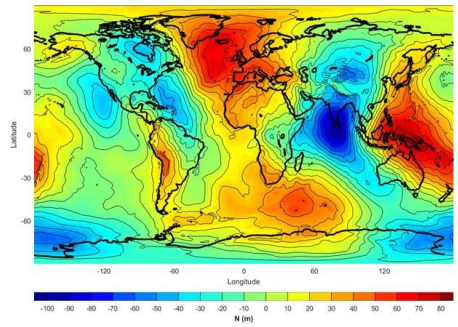
(b) NCEP/NCAR Reanalysed Monthly long-term mean (effective) geopotential height (Z), wind velocity (u, v), and temperature (T) at 12 pressure levels 1,000, 925, 850, 700, 600, 500, 400, 300, 250, 200, 150, and 100 hPa (https://psl.noaa.gov/data/gridded/data.ncep.reanalysis.derived.pressue.html)

(c) Climatological annual mean temperature and salinity from the NOAA/NCEI WOA18 for the sea water density (ρ) data (from https://www.ncei.noaa.gov/access/world-ocean-atlas-2018/)

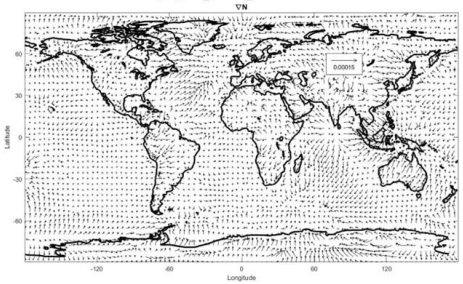
(d) Climatological annual mean surface wind stress (τ_λ, τ_φ) from the Atlas of Surface Marine Data 1994 (SMD94) (from https://iridl.ldeo.columbia.edu/SOURCES/.DASILVA/.)

Geoid Height *N* from EIGEN-6C4

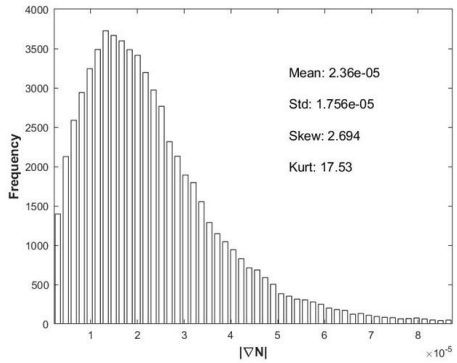
(a) $N(\lambda, \phi)$



(b) $[\nabla N]^S$



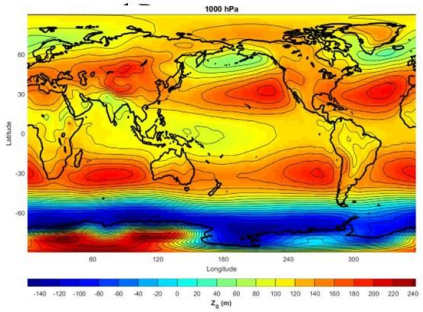
(b) Histogram of $|[\nabla N]^S|$



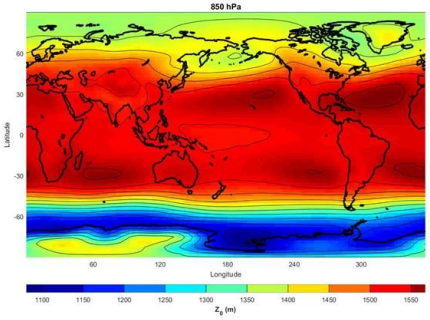
NAVAL POSTGRADUATE SCHOOL

Long-term Annual Mean Z at (1000, 850, 500) hPa

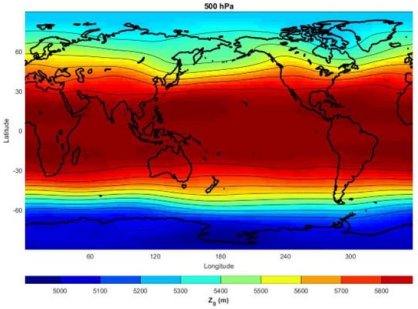
(a) Z at 1,000



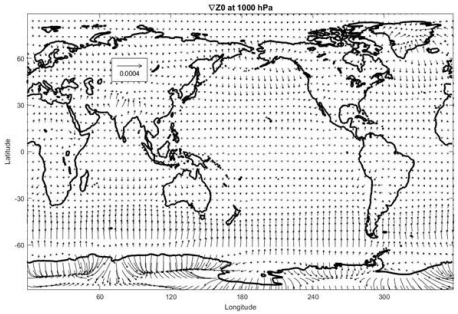
(b) Z at 850 hPa



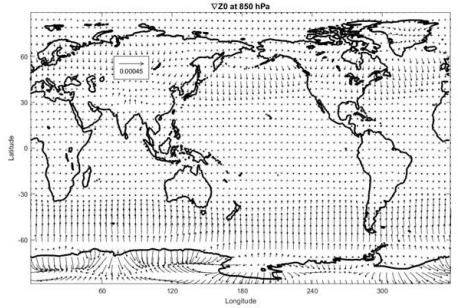
(c) Z at 500 hPa



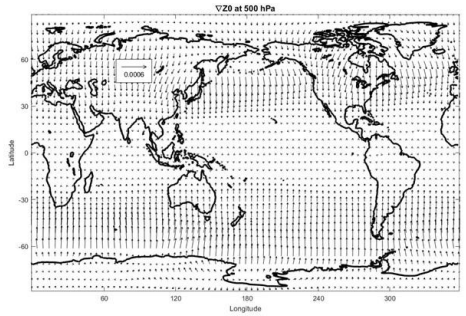
(d) $[\nabla Z]^s$ at 1,000 hPa



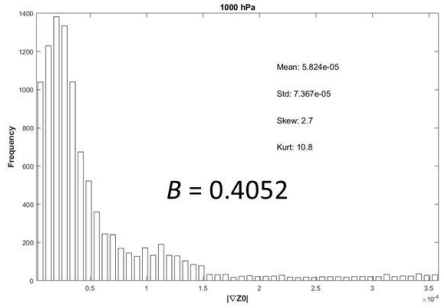
(e) $[\nabla Z]^s$ at 850 hPa



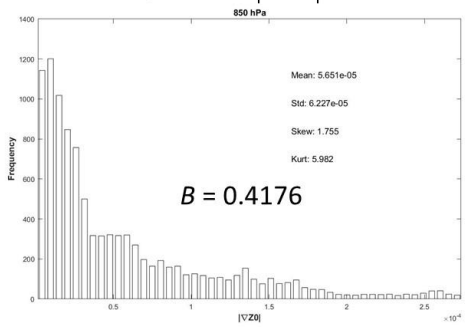
(f) $[\nabla Z]^s$ at 500 hPa



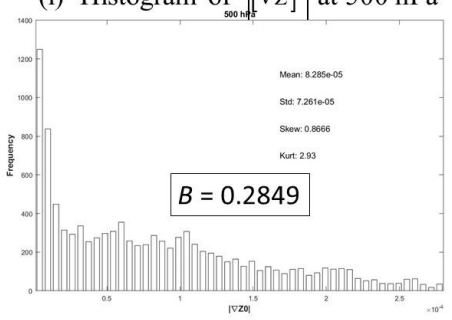
(g) Histogram of $\left|[\nabla Z]^s\right|$ at 1,000 hPa



(h) Histogram of $\left|[\nabla Z]^s\right|$ at 850 hPa

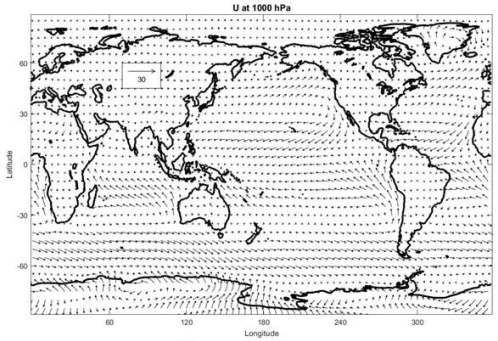


(i) Histogram of $\left|[\nabla Z]^s\right|$ at 500 hPa

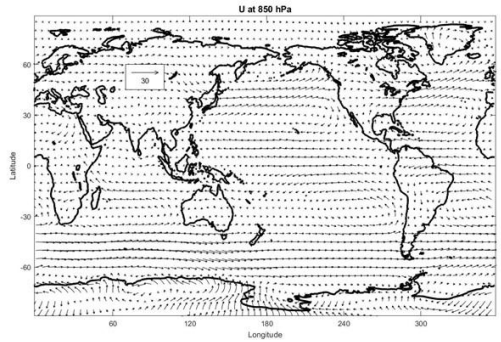


Long-term Annual Mean Wind Vector at (1000, 850, 500) hPa

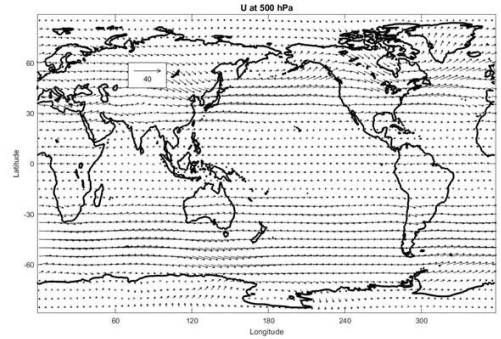
(a) 1,000 hPa

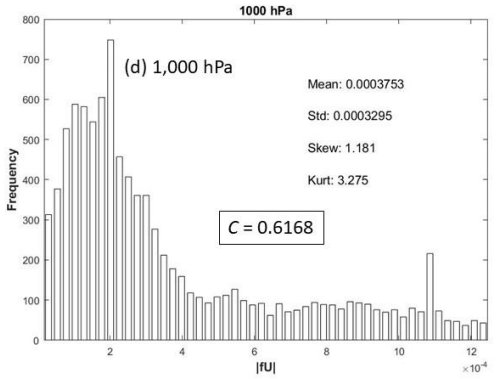


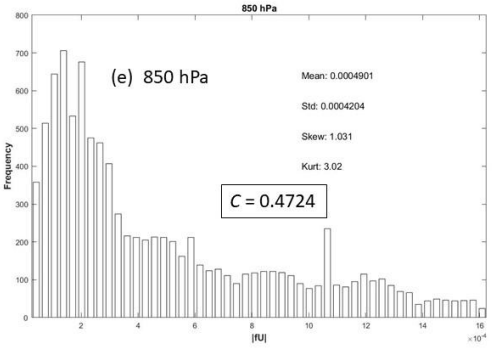
(b) 850 hPa

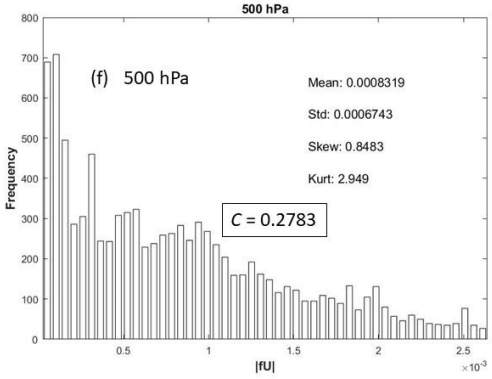


(c) 500 hPa









D-Number (Effect on the Ekman Pumping)

Geostrophic Vorticity

$$\zeta = \frac{1}{f}\nabla^2\Phi = \zeta_{eff} + \zeta_{gd}, \quad \zeta_{eff} = \frac{1}{f}\nabla^2\Phi_{eff} = \frac{g_0}{f}\nabla^2 Z, \quad \zeta_{gd} = -\frac{g_0}{f}\nabla^2 N$$

Ekman Pumping Velocity

$$w_{Ekman} = \frac{\zeta}{2\gamma} = \frac{1}{2\gamma}(\zeta_{eff} + \zeta_{gd}), \quad \gamma \equiv \left|\frac{f}{2K}\right|^{1/2}\left(\frac{f}{|f|}\right)$$

G-Number

$$G \equiv \frac{O(|\zeta_{gd}|)}{O(|\zeta_{eff}|)} = \frac{O[|\nabla^2 N|]}{O[|\nabla^2 Z|]}$$

Q Vector (q_1, q_2) and Omega Equation

$$\sigma\nabla^2\omega + f^2\frac{\partial^2\omega}{\partial p^2} = -2\nabla\bullet\mathbf{Q} - \frac{\kappa}{p}\nabla^2 J$$

$$\nabla\bullet\mathbf{Q}^{eff} = \frac{R_a g_0}{pf}\nabla\bullet\mathbf{q}^{eff},\quad \nabla\bullet\mathbf{Q}^{gd} = \frac{R_a g_0}{pf}\nabla\bullet\mathbf{q}^{gd}$$

Non-Dimensional (E_1, E_2, E_3) Numbers

$$E_1 \equiv \frac{O\left(\left|q_1^{gd}\right|\right)}{O\left(\left|q_1^{eff}\right|\right)} = \text{mean}\left(\left|J\left(\frac{\partial N}{\partial x}, T_a\right)\right|\right) / \text{mean}\left(\left|J\left(\frac{\partial Z}{\partial x}, T_a\right)\right|\right)$$

$$E_2 \equiv \frac{O\left(\left|q_2^{gd}\right|\right)}{O\left(\left|q_2^{eff}\right|\right)} = \text{mean}\left(\left|J\left(\frac{\partial N}{\partial y}, T_a\right)\right|\right) / \text{mean}\left(\left|J\left(\frac{\partial Z}{\partial y}, T_a\right)\right|\right)$$

$$E_3 \equiv \frac{O\left(\left|\nabla\bullet\mathbf{q}^{gd}\right|\right)}{O\left(\left|\nabla\bullet\mathbf{q}^{eff}\right|\right)}$$

Non-Dimensional Numbers in Atmosphere

Pressure Level (hPa)	B Pressure Gradient Force	C Coriolis Force	G Geostrophic Vorticity	E_1 Q_1	E_2 Q_2	E_3 Omega Equation
1,000	0.4052	0.6168	0.6712	1.4268	0.5482	2.584
925	0.4151	0.5086	0.7539	1.6708	0.6205	3.366
850	0.4176	0.4724	0.8534	1.9850	0.7247	4.510
700	0.3836	0.3829	1.1078	2.8620	0.9097	6.911
600	0.3435	0.3327	1.3344	3.3175	1.0570	8.622
500	0.2849	0.2783	1.3381	3.3550	1.0077	8.797
400	0.2298	0.2241	1.2529	3.3121	0.9395	8.834
300	0.1861	0.1797	1.1169	3.0013	0.8091	7.495
250	0.1713	0.1645	1.0701	3.2204	0.7786	7.590
200	0.1630	0.1573	1.0564	3.1943	0.6829	7.667
150	0.1639	0.1611	1.0955	3.4567	0.7789	9.165
100	0.1869	0.1837	1.2109	4.3834	0.9965	11.713
Mean	0.2792	0.3052	1.0718	2.9313	0.8211	7.271

δg in Ocean Dynamics

Horizontal Momentum Equation With the Boussinesq Approximation

$$\rho_0 \left[\frac{D\mathbf{U}}{Dt} + f\mathbf{k} \times \mathbf{U} \right] = -\nabla \hat{p} + (\rho - \rho_0) g_0 \nabla N + \rho_0 (\mathbf{F}_h + \mathbf{F}_v)$$

Reference Density $\rho_0 = 1{,}028\ \text{kg/m}^3$

Reference Pressure $p_0 = -\rho_0 g_0 (z - N)$

Dynamic Pressure $\hat{p} = p - p_0$

Hydrostatic Balance $\dfrac{\partial \hat{p}}{\partial z} = -(\rho - \rho_0) g_0$

$\mathbf{F}_h$ and $\mathbf{F}_v$ are the horizontal and vertical frictional forces

Geostrophic Current and Thermal Wind Relation

$$f\mathbf{k}\times\mathbf{U} = -\frac{1}{\rho_0}\nabla\hat{p} + \frac{\rho-\rho_0}{\rho_0}g_0\nabla N$$

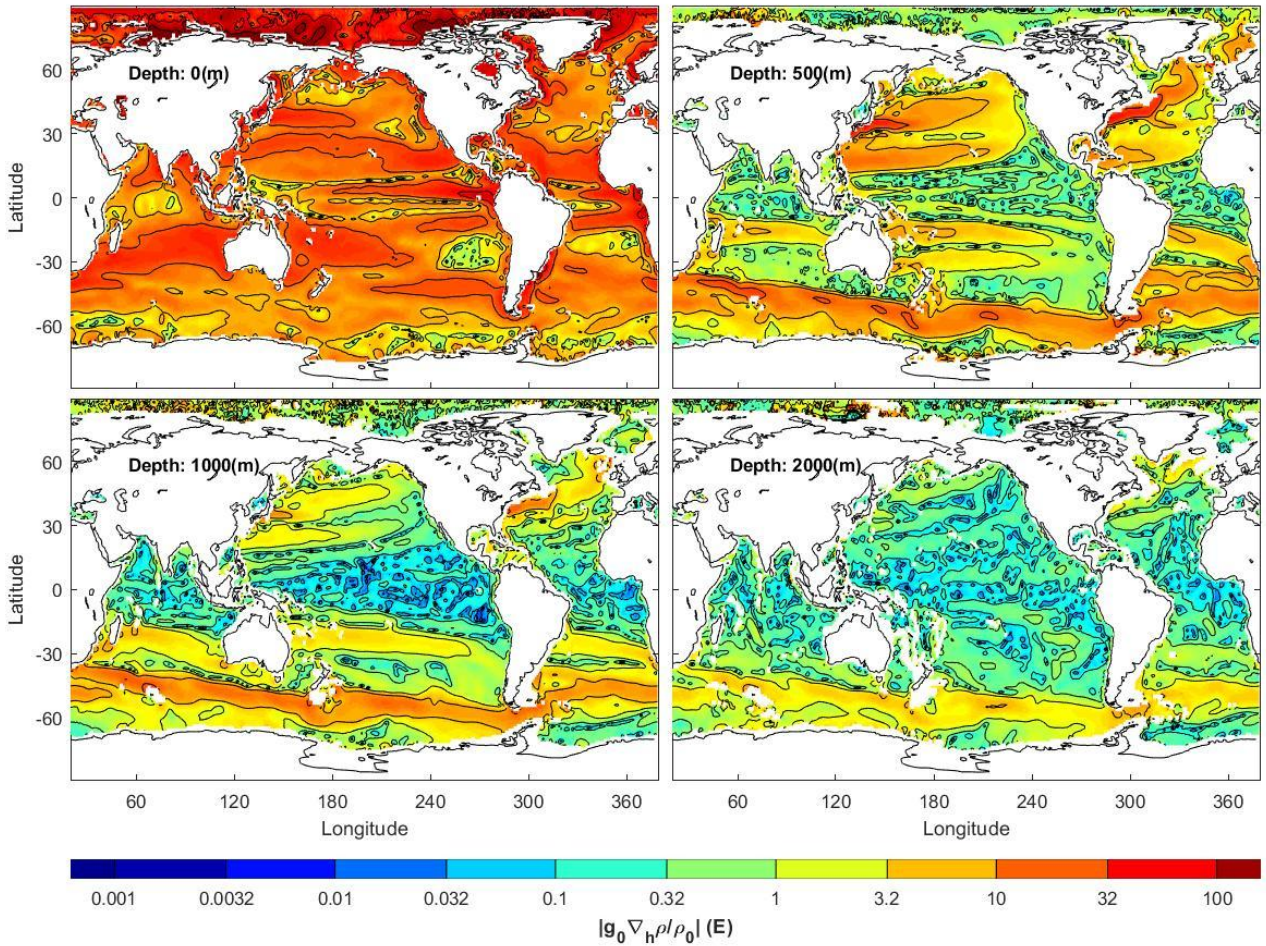
$$f\frac{\partial\mathbf{U}}{\partial z} = \mathbf{k}\times\left[-(g_0/\rho_0)\nabla\rho + \Theta^2\nabla N\right],\quad \Theta^2 \equiv -\left(\frac{g_0}{\rho_0}\frac{\partial\rho}{\partial z}\right)$$

Depth-dependent non-dimensional G number

$$D(z) = \frac{O\left(\left|\Theta^2\nabla N\right|\right)}{O\left(\left|(g_0/\rho_0)\nabla\rho\right|\right)} \approx \frac{\text{mean}\left(\left|\Theta^2\nabla N\right|\right)}{\text{mean}\left(\left|(g_0/\rho_0)\nabla\rho\right|\right)}$$

Thermal Wind due to Density

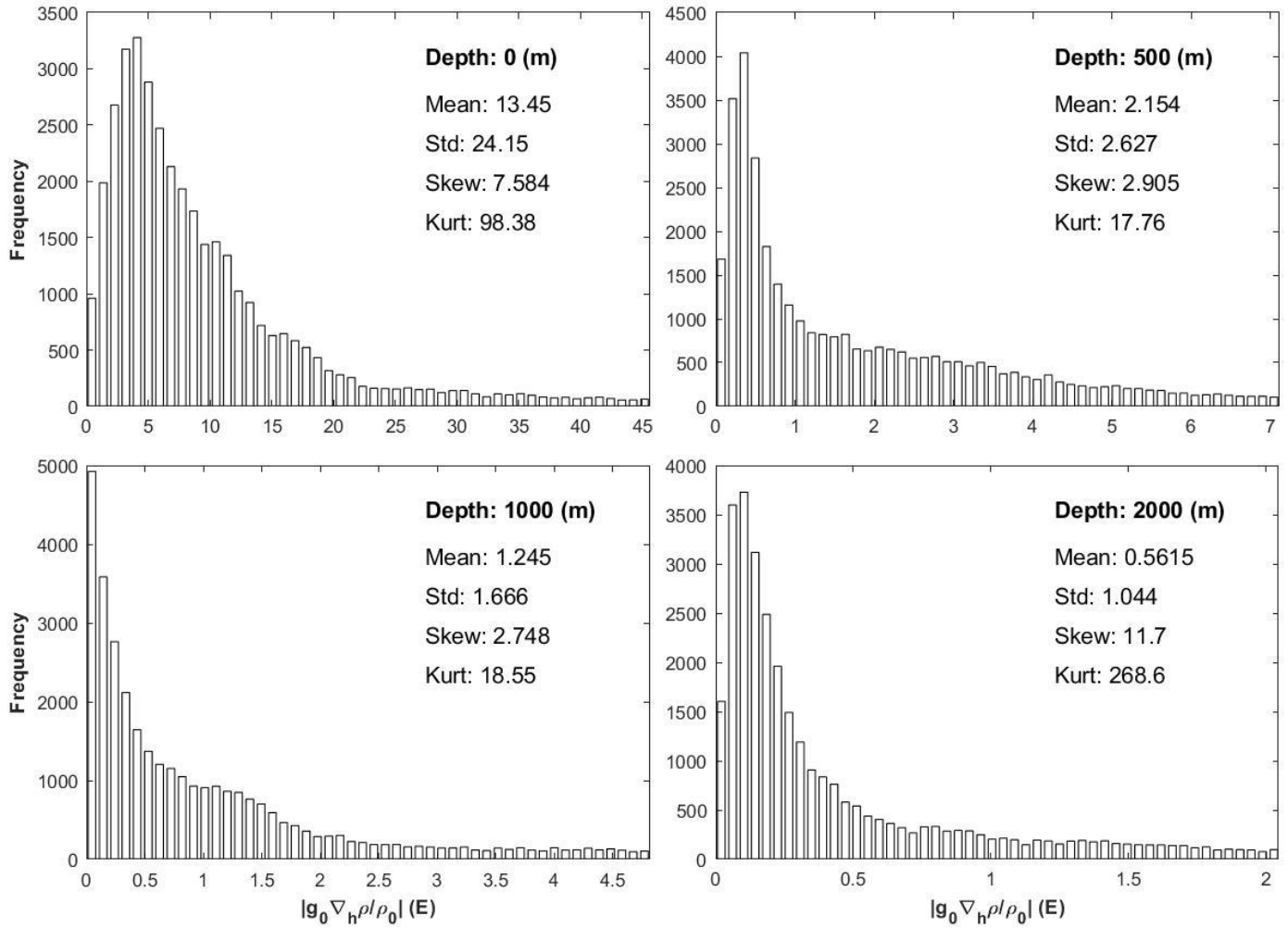
$|(g_0 / \rho_0)\nabla\rho|$ in the unit of Eotvos (E) (1 E = 10^{-9} s^{-2})



From NOAA/NCEI WOA18 Data

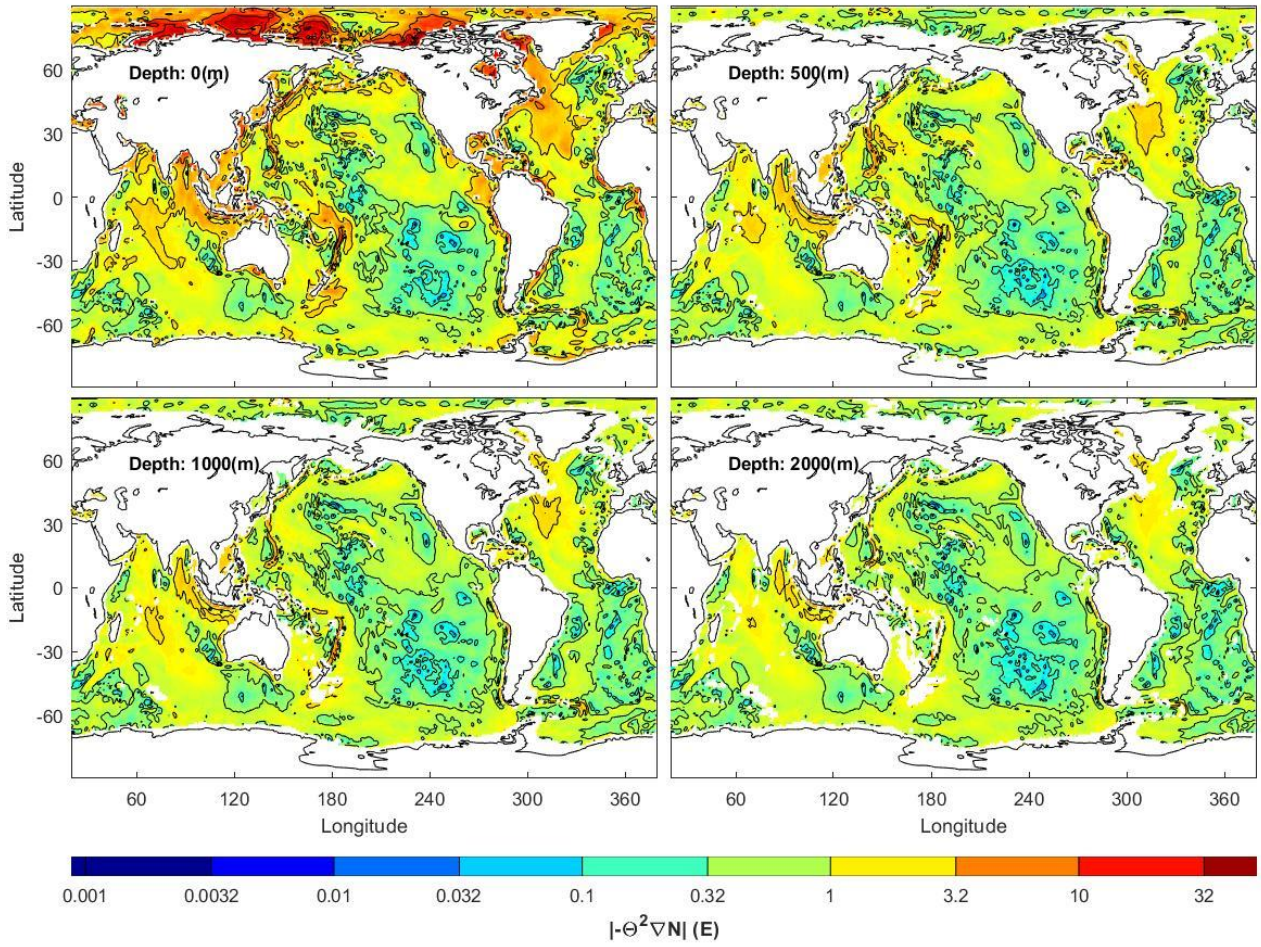
Histograms

$$|(g_0 / \rho_0)\nabla\rho|$$



Thermal Wind due to δg

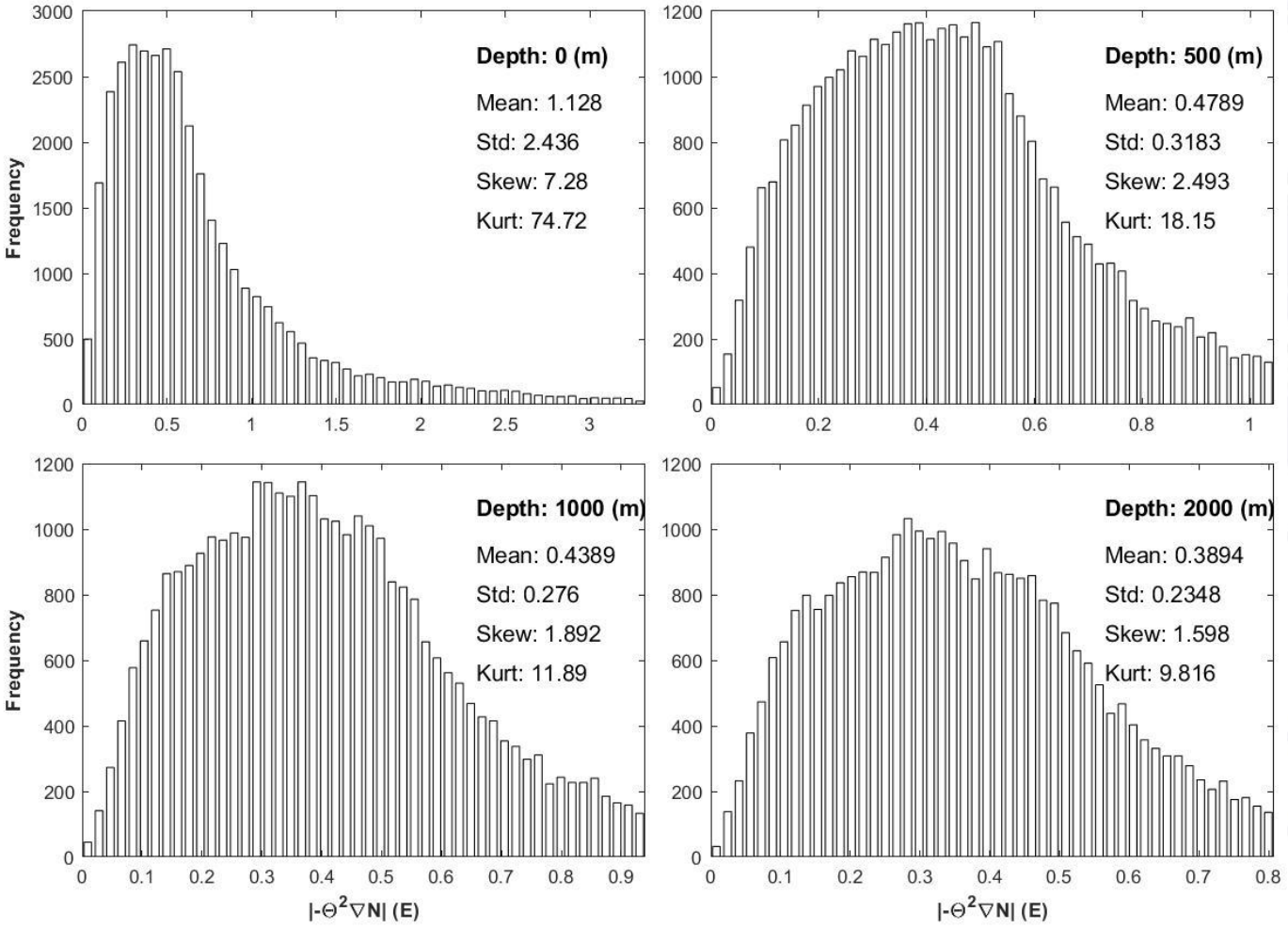
$| \Theta^2 \nabla N |$ in the unit of Eotvos (E)



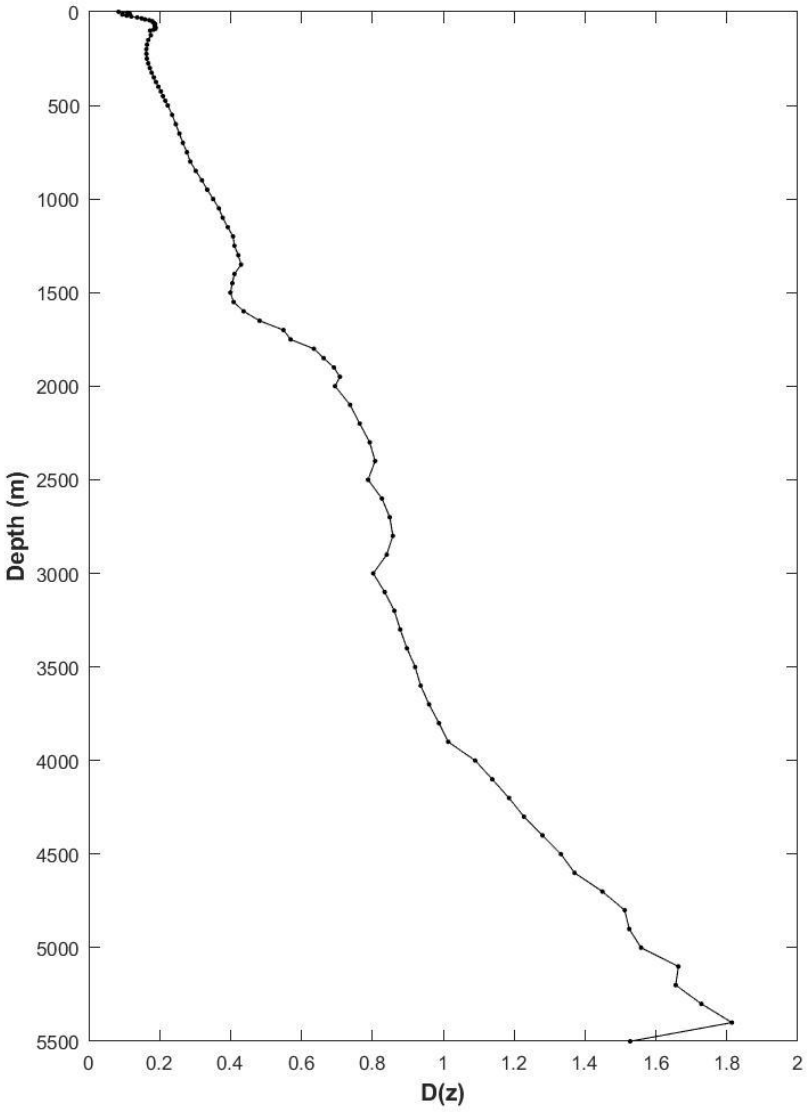
From EIGEN6C4

Histograms

$$| \Theta^2 \nabla N |$$



Depth-Dependent D Number



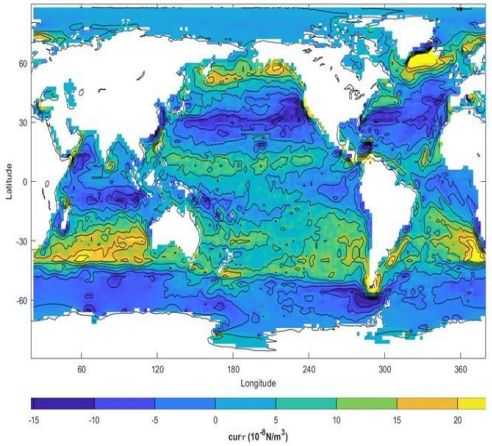
Wind-Driven Ocean Circulation
Combined Sverdrup-Stommel-Munk Dynamics

$$-A\nabla^4\Psi + \gamma\nabla^2\Psi + \beta\frac{\partial\Psi}{\partial x} = \frac{1}{\rho_0}\left[\text{curl}\,\boldsymbol{\tau} + g_0\int_{-H}^{0}\mathbf{k}\bullet\left(\nabla\rho\times\nabla N\right)dz\right]$$

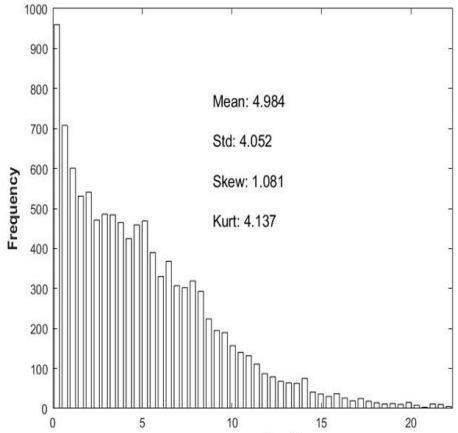
$$J = g_0\int_{-H}^{0}\mathbf{k}\bullet\left(\nabla\rho\times\nabla N\right)dz$$

Wind-Driven Ocean Circulation
Combined Sverdrup-Stommel-Munk Dynamics

Wind Forcing

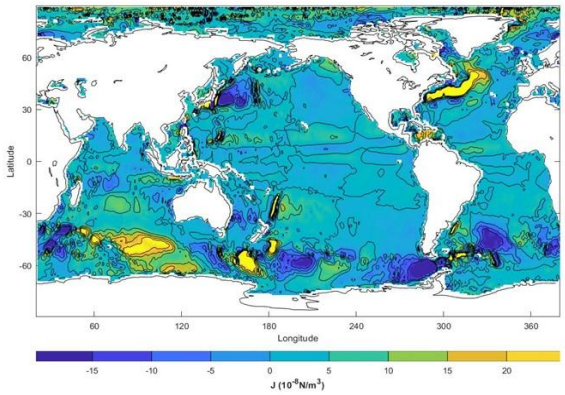


(b)

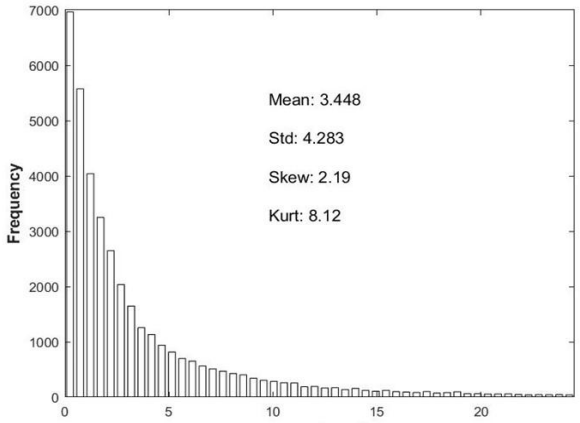


$|\text{curl } \boldsymbol{\tau}|$ (10^{-8} N/m^3)

Forcing due to $\delta\mathbf{g}$



(b)



$|J|$ (10^{-8} N/m^3)

$$F = \frac{\delta\mathbf{g}\text{-forcing}}{\text{wind-forcing}} = \frac{O\left[|J|\right]}{O\left[|\text{curl } \boldsymbol{\tau}|\right]}$$

$$= \frac{3.448\times10^{-8}\ \text{N/m}^3}{4.984\times10^{-8}\ \text{N/m}^3} = 0.69$$

8. Conclusions

- (1) Gravity used in meteorology and oceanography is not the true gravity.
- (2) The effect of gravity disturbance vector $g_0 \nabla N$ in atmospheric and oceanic dynamics is important.
- (3) It is urgent and easy to include $g_0 \nabla N$ in atmospheric and oceanic dynamics and numerical models.

References

- Chu, P.C., 2021: True gravity in ocean dynamics Part-1 Ekman transport. *Dynamics of Atmospheres and Oceans*, **96**, 101268, https://doi.org/10.1016/j.dynatmoce.2021.101268.

- Chu, P.C., 2023: Horizontal gravity disturbance vector in Atmospheric dynamics. *Dynamics of Atmospheres and Oceans*, **102**, 101369, https://www.sciencedirect.com/science/article/pii/S0377026523000209.

- Thank you very much.
- Any questions?