First Passage Time for Stochastic Dynamic System and Climate Modeling

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- How long is a climate model (or ocean or atmospheric model) valid once being integrated from its initial state?

- This question should be answered before running any model.

Physical Reality

- $\boldsymbol{Y}$
- Physical Law: $d\boldsymbol{Y}/dt = \boldsymbol{h}(\boldsymbol{Y}, t)$
- Initial Condition: $\boldsymbol{Y}(t_0) = \boldsymbol{Y}_0$

- $\boldsymbol{X}$ is the prediction of $\mathbf{Y}$

- **Model:** $d\mathbf{X}/dt = \mathbf{f}(\mathbf{X},t) + q(t)\mathbf{X}$

- Initial Condition: $\boldsymbol{X}(t_0) = \boldsymbol{X}_0$

- Stochastic Forcing:

$$\langle q(t) \rangle = 0$$

$$\langle q(t)q(t') \rangle = q^2 \delta(t - t')$$

Model Error

$$Z = X - Y$$

Initial: $Z_0 = X_0 - Y_0$

One Missing Parameter

- Tolerance Level ε
- Maximum accepted error

FPT

- FPT is defined as the time period when the prediction error first exceeds a pre-determined criterion (i.e., the tolerance level ε).

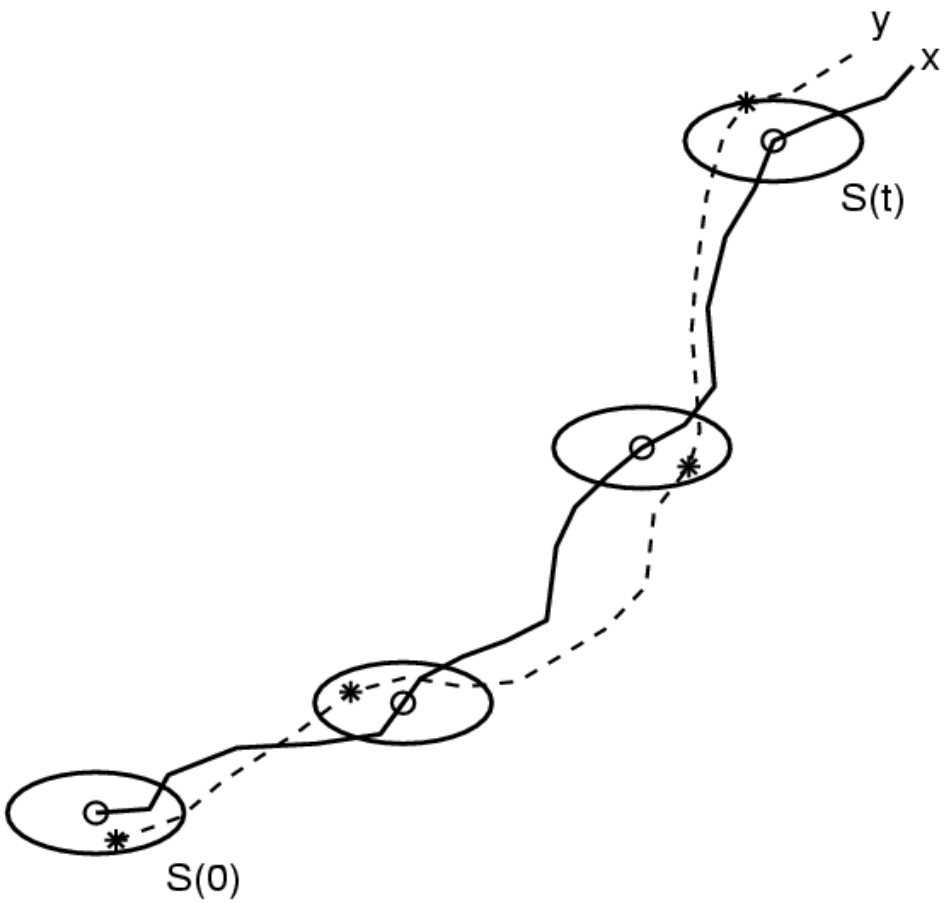
First-Passage Time (FPT)

ρ → Radius

FPT → Time when the particle first passes through the boundary.



FPT for Model Predictability



FPT is defined as the time period when the prediction error first exceeds a pre-determined criterion (i.e., the tolerance level ε).

Conditional Probability Density Function

- Initial Error: $\mathbf{Z}_0$
- $(t - t_0)$ → Random Variable
- Conditional PDF of $(t - t_0)$ with given $\mathbf{Z}_0$ →

$$P\left[(t - t_0) \mid \mathbf{z}_0\right]$$

Two Approaches

- Analytical (low dimension dynamical system)
- Practical (operational atmospheric or ocean model)

Backward Fokker-Planck Equation

(Pontrygin et al. 1962)

$$\frac{\partial P}{\partial t} - [\mathbf{f}(\mathbf{z}_0, t)]\frac{\partial P}{\partial \mathbf{z}_0} - \frac{1}{2} q^2 \mathbf{z}_0^2 \frac{\partial^2 P}{\partial \mathbf{z}_0 \partial \mathbf{z}_0} = 0 \qquad (1)$$

Moments

$$\tau_1(\mathbf{z}_0) = \int_{t_0}^{\infty} P(t_0, \mathbf{z}_0, t - t_0)(t - t_0)dt \qquad (2)$$

$$\tau_2(\mathbf{z}_0) = \int_{t_0}^{\infty} P(t_0, \mathbf{z}_0, t - t_0)(t - t_0)^2 dt \qquad (3)$$

Lorenz System (Lorenz 1963)

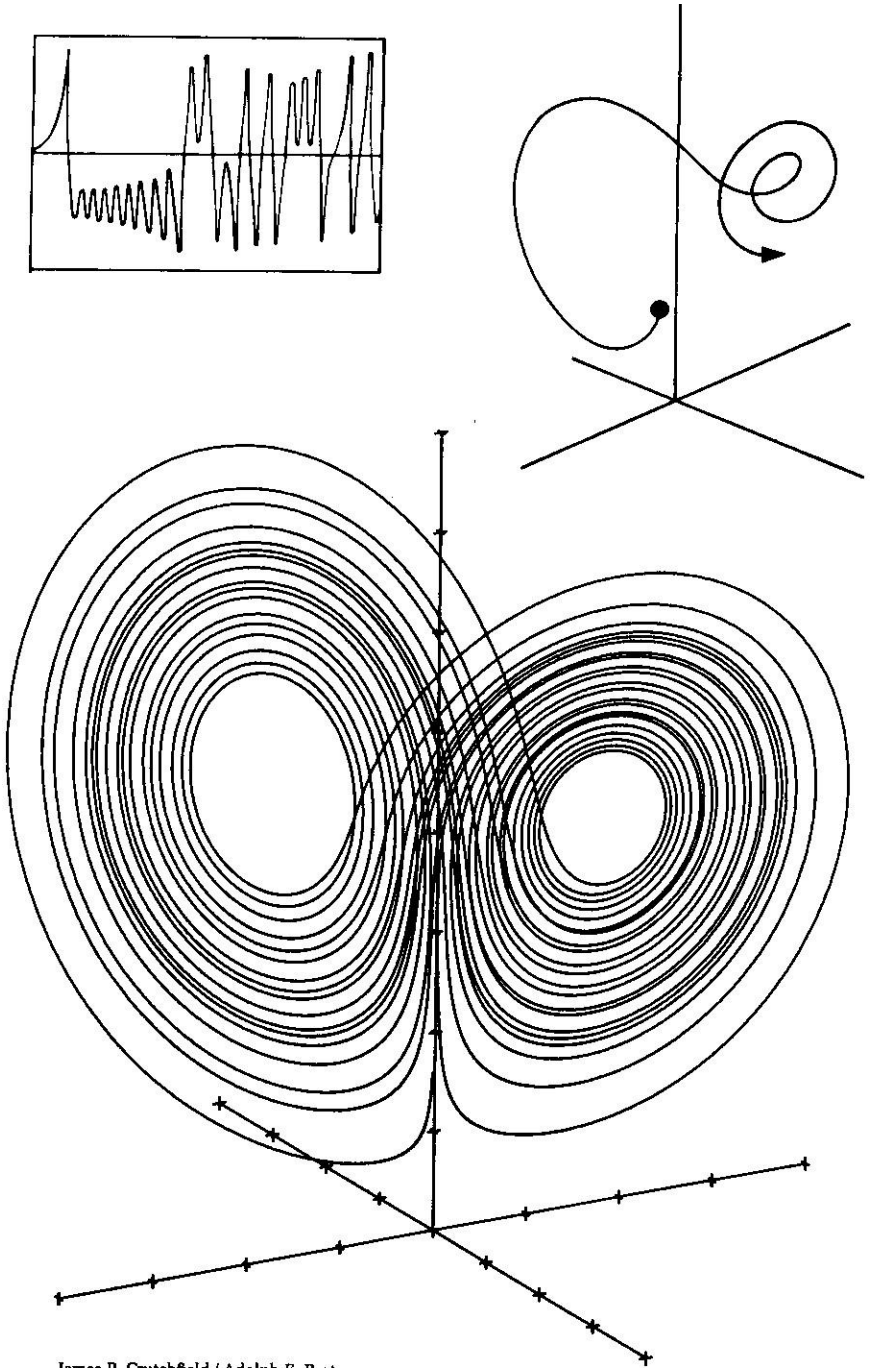
Fourier Truncated Atmospheric Convective System (three components) →

$$\frac{dX}{dt} = -\sigma X + \sigma Y$$

$$\frac{dY}{dt} = -XZ + rX - Y$$

$$\frac{dZ}{dt} = XY - bZ$$

Lorenz System (Lorenz 1963)



Example-1: Maximum Growing Maniford of Lorenz System (Nicolis, 1992)

$$\frac{d\xi}{dt} = (\sigma - g\xi^2) + v(t)\xi, \qquad 0 \le \xi < \infty$$

$$\langle v(t) \rangle = 0, \qquad \langle v(t)v(t') \rangle = q^2 \delta(t - t').$$

$$\sigma = 0.64, \qquad g = 0.3, \qquad q^2 = 0.2. \tag{4}$$

Backward Fokker-Planck Equation

$$\frac{\partial P}{\partial t} - (\sigma - g\xi_0^2)\frac{\partial P}{\partial \xi_0} - \frac{1}{2}q^2\xi_0^2\frac{\partial^2 P}{\partial \xi_0^2} = 0 \tag{5}$$

$$z_0 = \xi_0 \rightarrow \mathbf{f}(\mathbf{z}_0) \equiv \sigma - g\xi_0^2, \qquad \leftarrow \text{Time-independent}$$

- We multiply Equation (5) $(t - t_0)$ and $(t - t_0)^2$, then integrate with respect to t from t_0 to 1 and obtain the mean and variance of FPT

Mean and Variance of FPT

$$(\sigma\xi_0 - g\xi_0^2)\frac{d\tau_1}{d\xi_0} + \frac{q^2\xi_0^2}{2}\frac{d^2\tau_1}{d\xi_0^2} = -1$$

$$(\sigma\xi_0 - g\xi_0^2)\frac{d\tau_2}{d\xi_0} + \frac{q^2\xi_0^2}{2}\frac{d^2\tau_2}{d\xi_0^2} = -2\tau_1$$

$\tau_1 = 0, \quad \tau_2 = 0$ for $\xi_0 = \varepsilon$.

$\frac{d\tau_1}{d\xi_0} = 0, \quad \frac{d\tau_2}{d\xi_0} = 0$ for $\xi_0 = \xi_{noise}$.

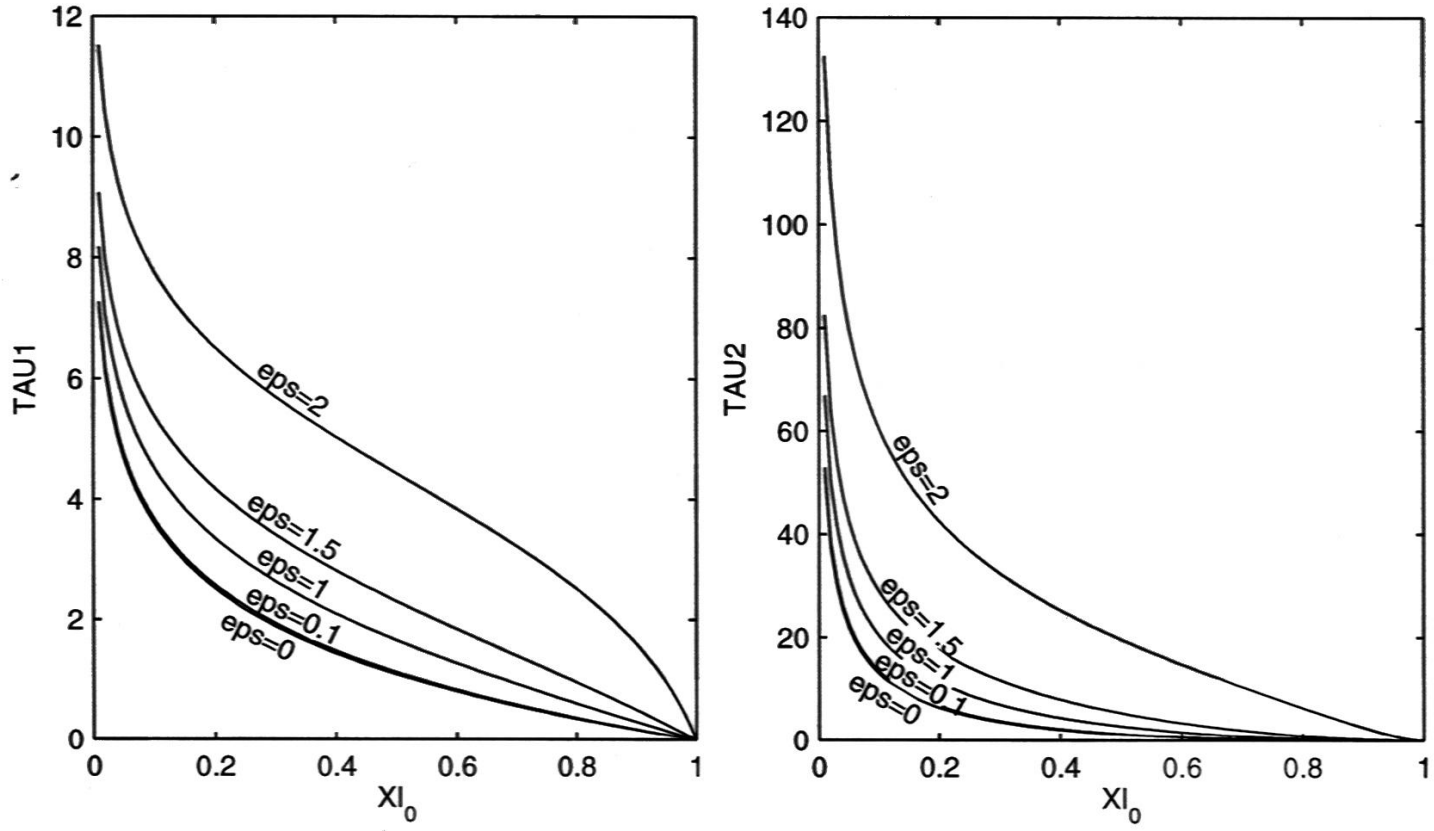
Analytical Solutions

$$\tau_1(\bar{\xi}_0, \bar{\xi}_{noise}, \varepsilon) = \frac{2}{q^2} \int_{\bar{\xi}_0}^{1} y^{-\frac{2\sigma}{q^2}} \exp\left(\frac{2\varepsilon g}{q^2} y\right) \left[\int_{\bar{\xi}_{noise}}^{y} x^{\frac{2\sigma}{q^2}-2} \exp\left(-\frac{2\varepsilon g}{q^2} x\right) dx \right] dy$$

$$\tau_2(\bar{\xi}_0, \bar{\xi}_{noise}, \varepsilon) = \frac{4}{q^2} \int_{\bar{\xi}_0}^{1} y^{-\frac{2\sigma}{q^2}} \exp\left(\frac{2\varepsilon g}{q^2} y\right) \left[\int_{\bar{\xi}_{noise}}^{y} \tau_1(x) x^{\frac{2\sigma}{q^2}-2} \exp\left(-\frac{2\varepsilon g}{q^2} x\right) dx \right] dy$$

$$\bar{\xi}_0 = \xi_0 / \varepsilon, \qquad \bar{\xi}_{noise} = \xi_{noise} / \varepsilon$$

Dependence of $(\tau_1 \quad \tau_2)$ on Initial Condition Error ξ_0 / ε



Example-2: Lorenz System (1984) Modified Hadley Circulation

$$\frac{dx_1}{dt} = -x_2^2 - x_3^2 - ax_1 + aF,$$

$$\frac{dx_2}{dt} = x_1 x_2 - bx_1 x_3 - x_2 + G,$$

$$\frac{dx_3}{dt} = bx_1 x_2 + x_1 x_3 - x_3,$$

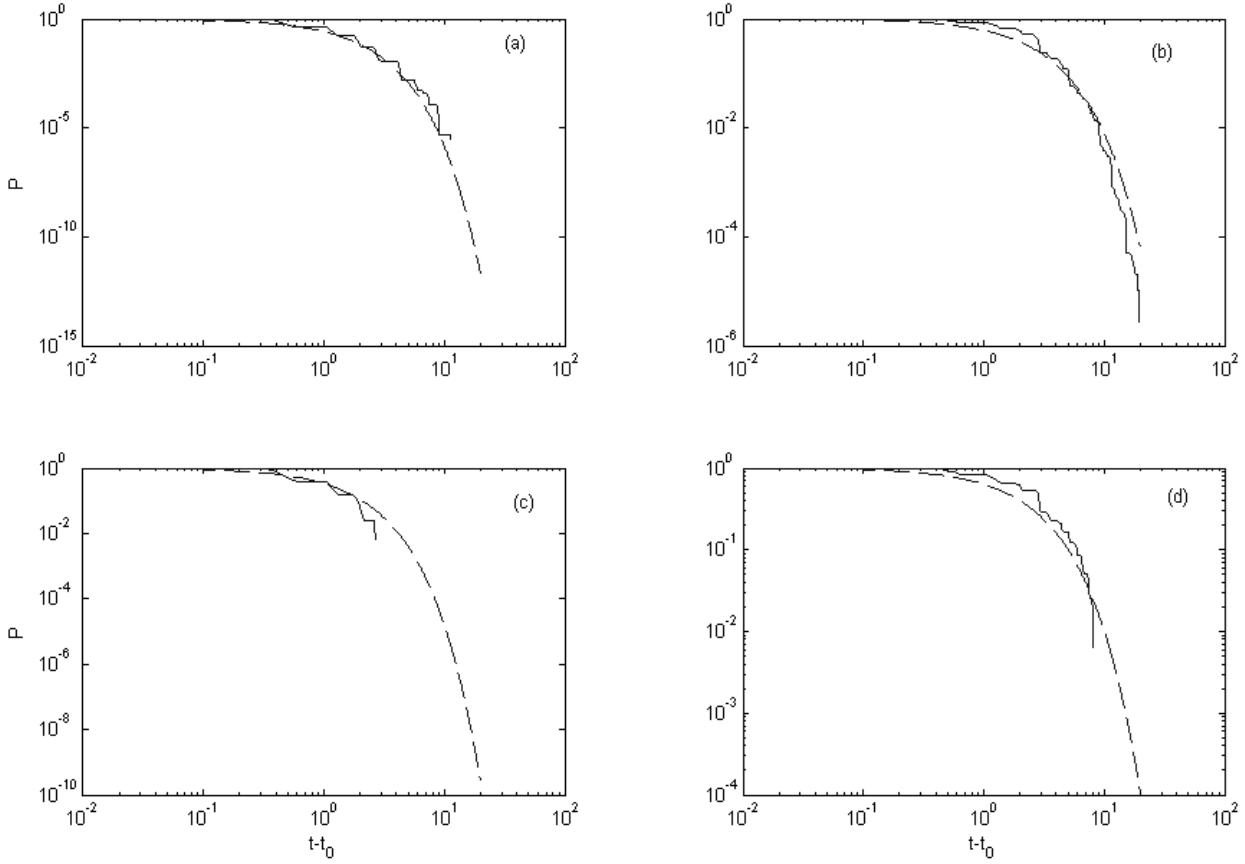
Sensitivity of PDF on the tolerance level and ensemble dimension: numerical (solid curve), analytic (dashed curve)

ε=0.1

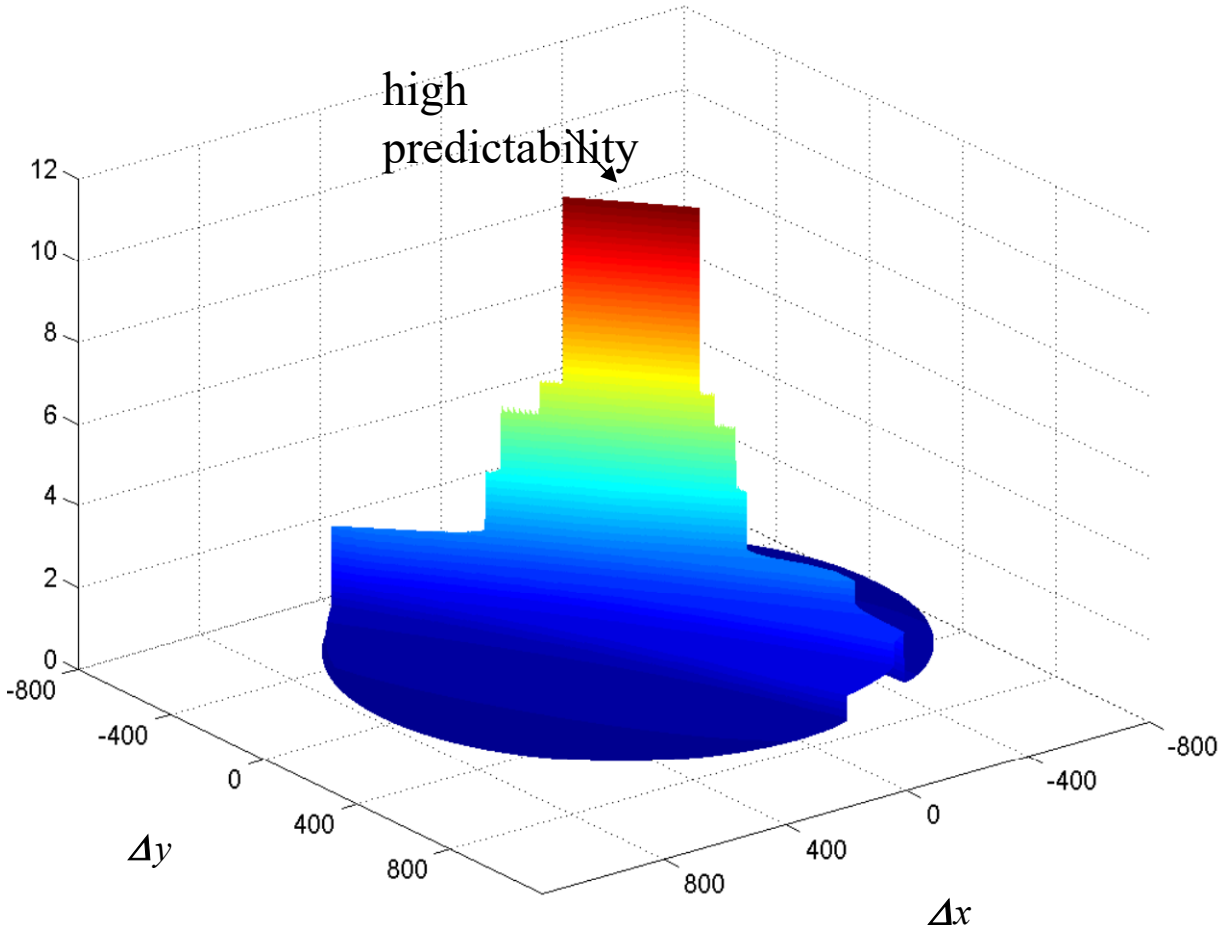
ε=0.5

N=250,000

N=100

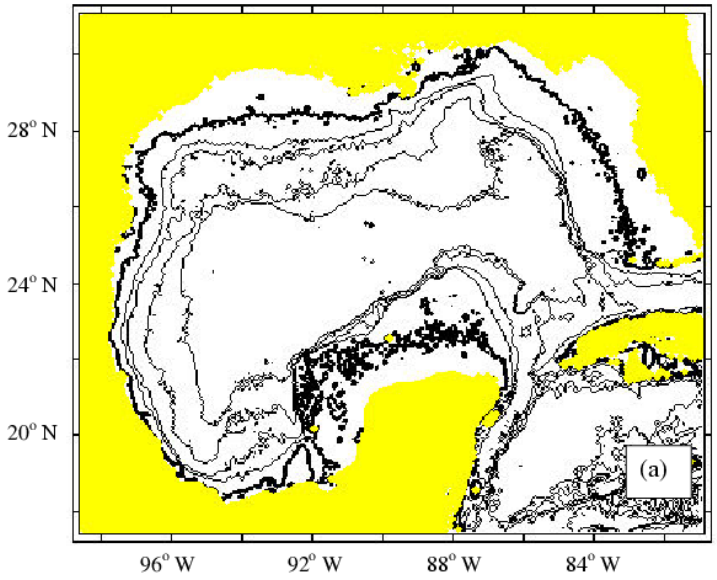


Dependence of FPT on Initial Error $(Dx_1\ Dx_2)_0$ for $\varepsilon = 0.1$



Example-3
Operational Ocean Model Predictability Skill

Gulf of Mexico Prediction System



Gulf of Mexico Bottom Topography

Basic Equations

$$\frac{d\rho}{dt} + \rho\Delta \cdot \vec{v} = 0$$

$$\rho \frac{d\vec{v}}{dt} + \rho(2\vec{\Omega} \times \vec{v}) = -\nabla p - \rho g\vec{k} + \nabla \cdot \mathbf{T}$$

$$\rho c_p \frac{dT}{dt} - \beta T \frac{dp}{dt} = -\nabla \cdot \vec{q} - \sigma$$

$$\rho = \rho(T, S, p)$$

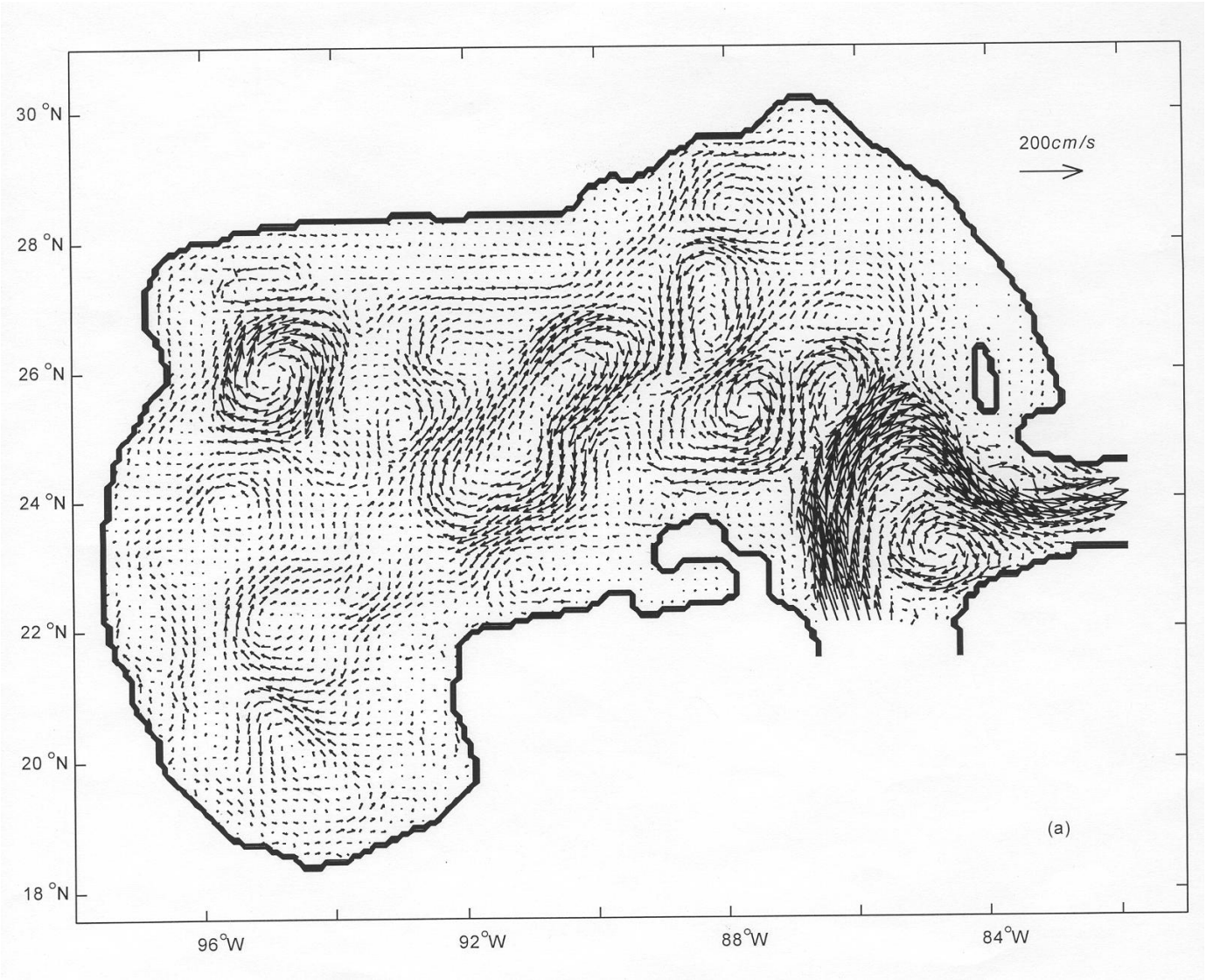
$$\frac{dS}{dt} = \kappa_s \nabla \cdot (\nabla S)$$

- $\mathbf{T} \equiv \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix}$
- $\vec{q} \equiv$ heat flux
- $c_p \equiv$ specific heat
- $\beta \equiv$ thermal expansion
- $\kappa_S \equiv$ molecular diffusion

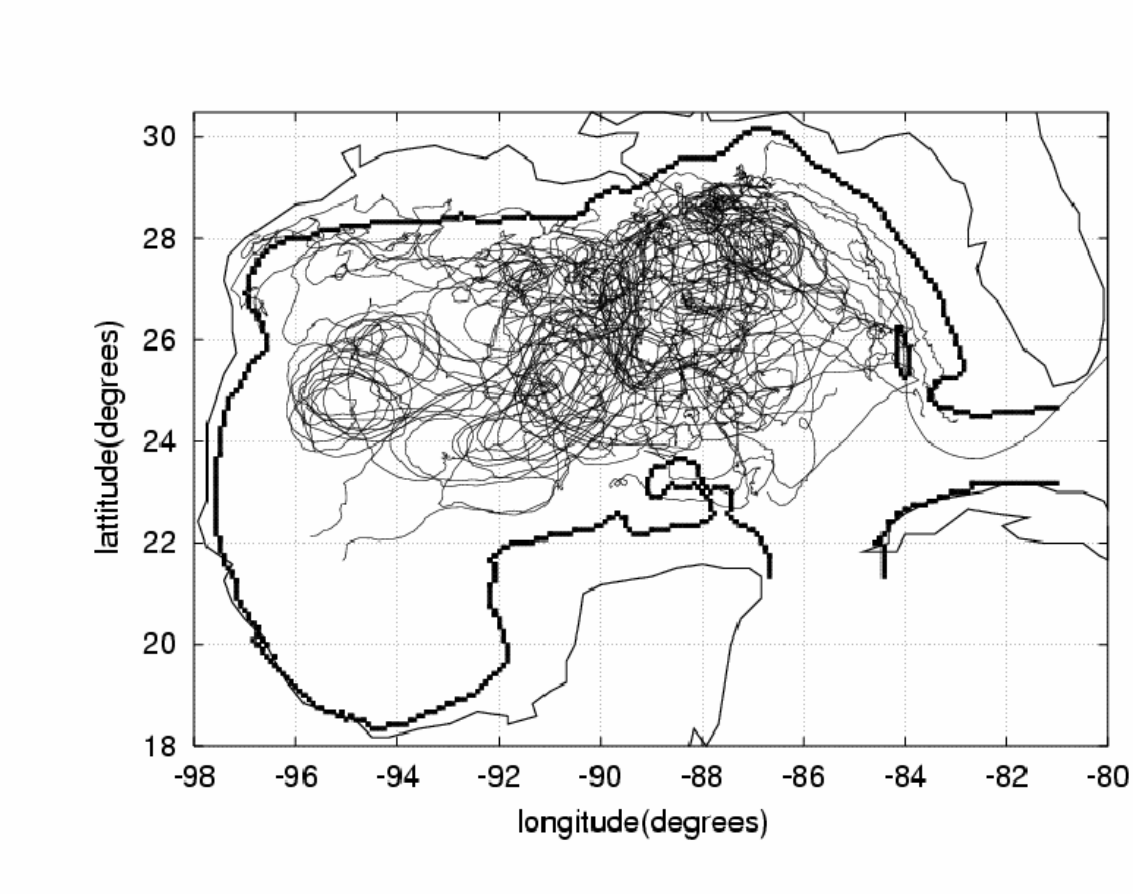
Gulf of Mexico Forecast System

- University of Colorado Version of POM
- 1/12^{o} Resolution
- Real-Time SSH Data (TOPEX, ESA ERS-1/2) Assimilated
- Real Time SST Data (MCSST, NOAA AVHRR) Assimilated
- Six Months Four-Times Daily Data From July 9, 1998 for Verification

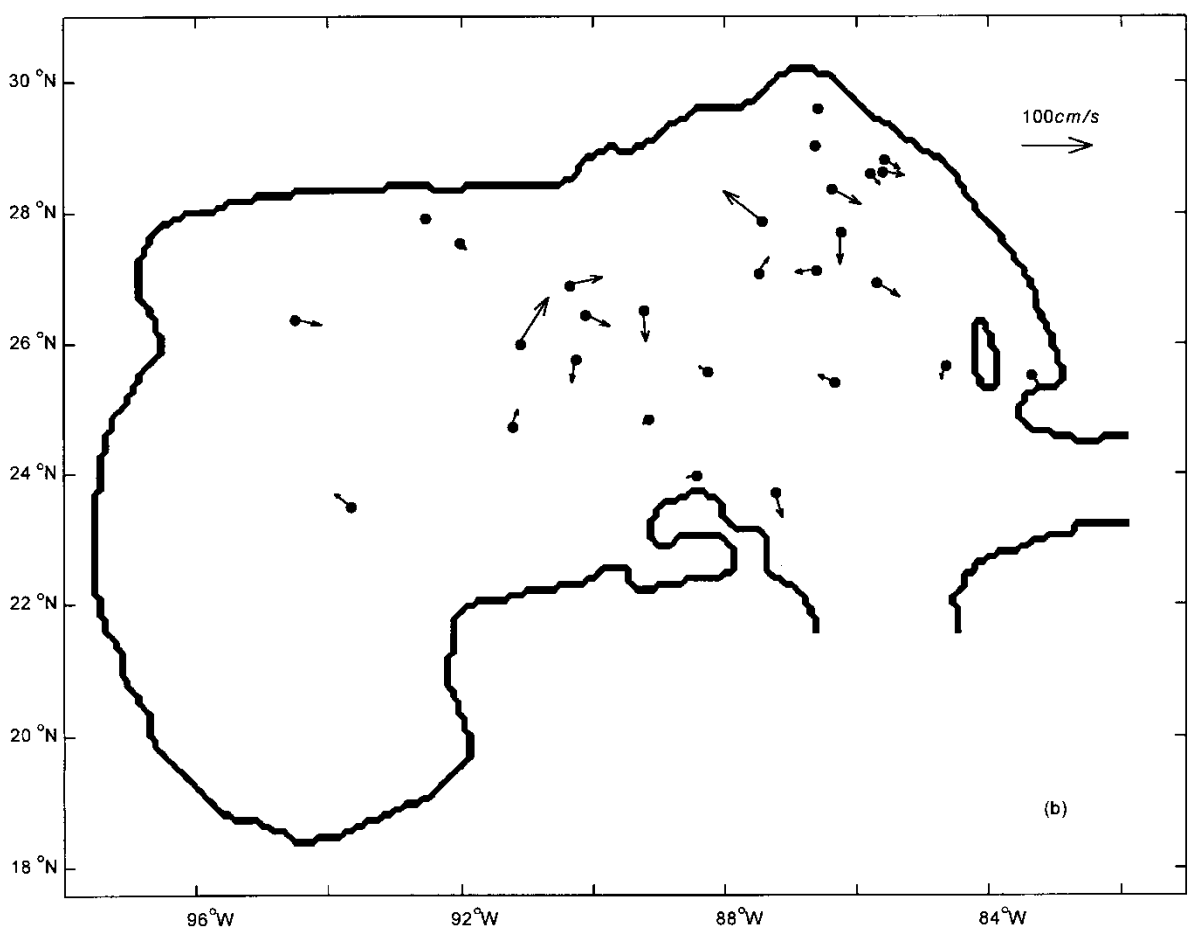
Model Generated Velocity Vectors at 50 m on 00:00 July 9, 1998



Fifteen Drifting Buoys

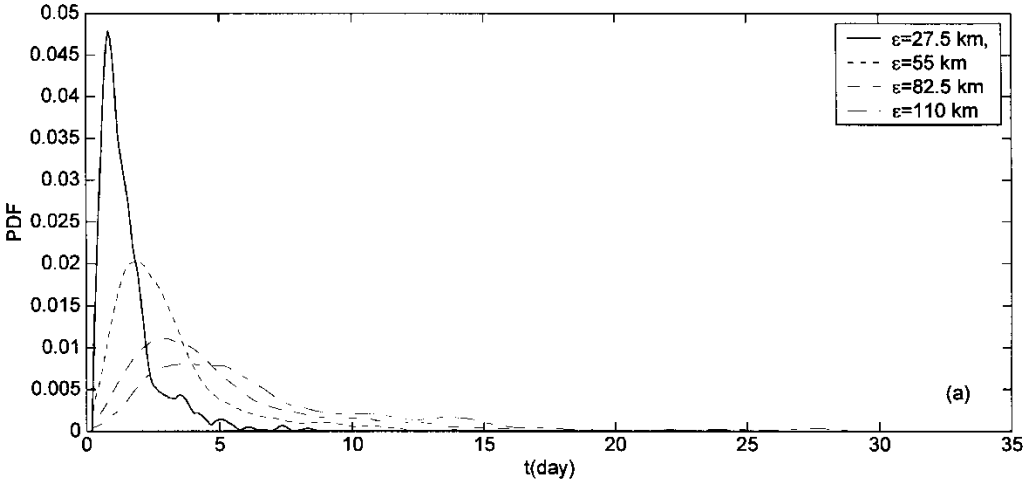


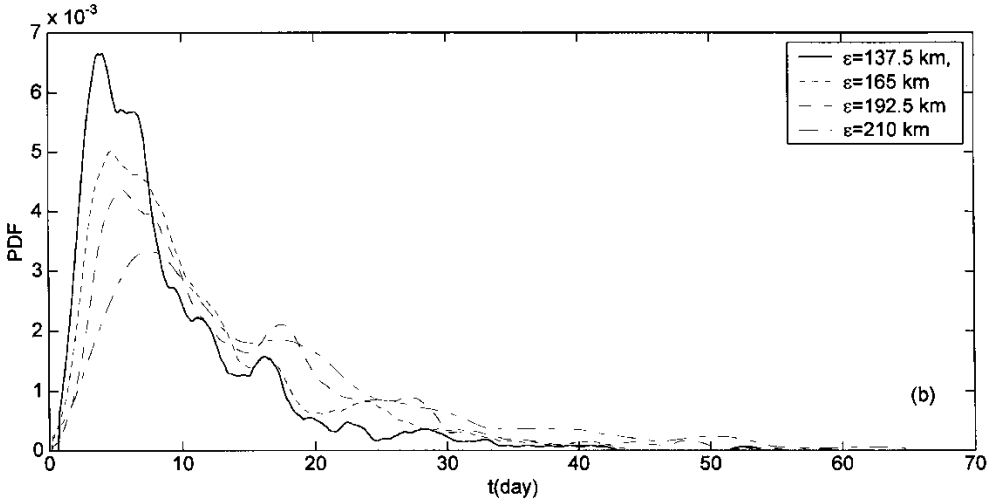
(Observational) Drifter Data at 50 m on 00:00 July 9, 1998



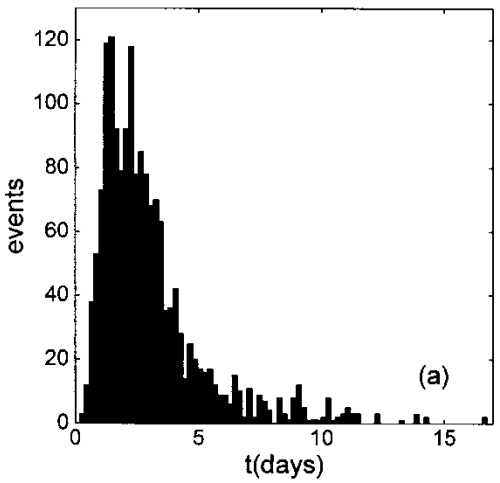
PDF of FPT calculated with different tolerance levels

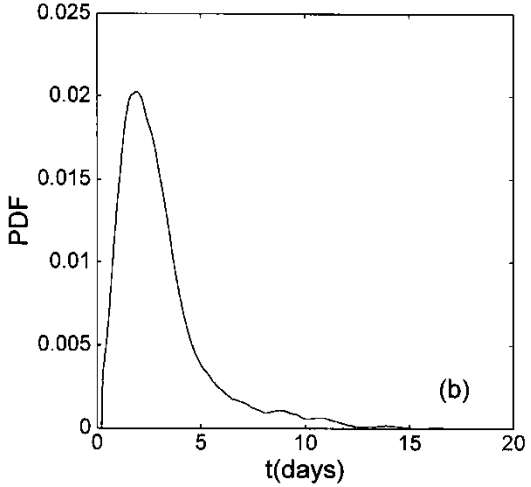
Non-Gaussian distribution with long tail toward large values of PFT (Long-term Predictability)

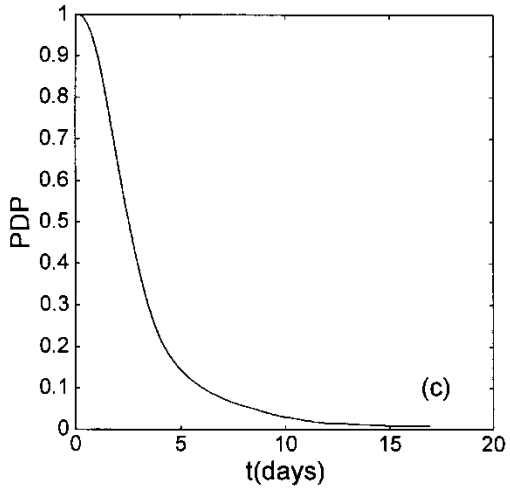


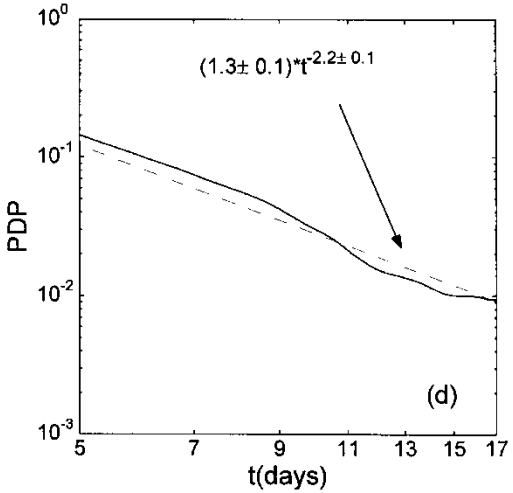


Statistical Characteristics of FPT for zero initial error and 22 km tolerance level (Non-Gaussion)









Error Mean and Variance

Error Mean $$\mathbf{L}_1 = \langle \mathbf{z} \rangle$$

Error Variance $$\mathbf{L}_2 = \left\langle \left(\mathbf{z} - \langle \mathbf{z} \rangle\right)^t \left(\mathbf{z} - \langle \mathbf{z} \rangle\right) \right\rangle$$

Exponential Error Growth

$$L_1 \propto e^{\sigma t}, \qquad L_2 \propto e^{\omega t},$$

Classical Linear Theory

No Long-Term Predictability

Power Law

$$L_1 \propto t^{\alpha}, \qquad L_2 \propto t^{\beta},$$

$$P(t_0, \mathbf{z}_0, \varepsilon, t - t_0) \sim t^{-\gamma} \quad \text{for large } t.$$

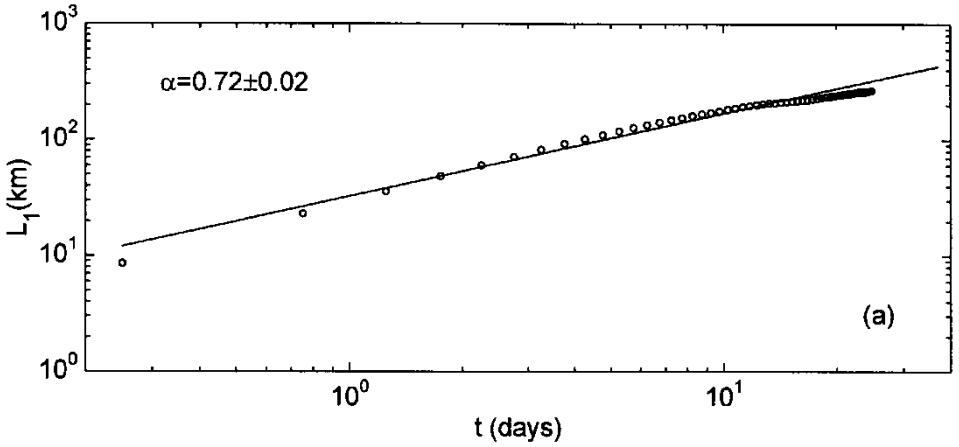
Long-Term Predictability May Occur

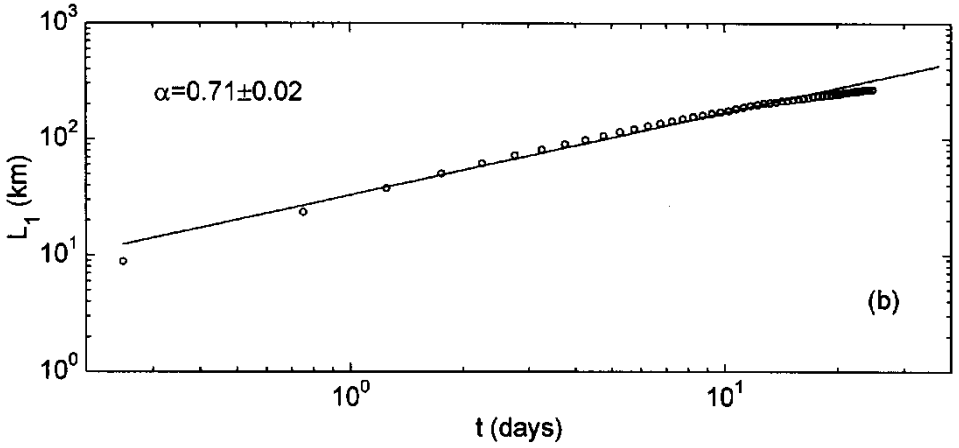
Scaling behavior of the mean error (L_1) growth for initial error levels:

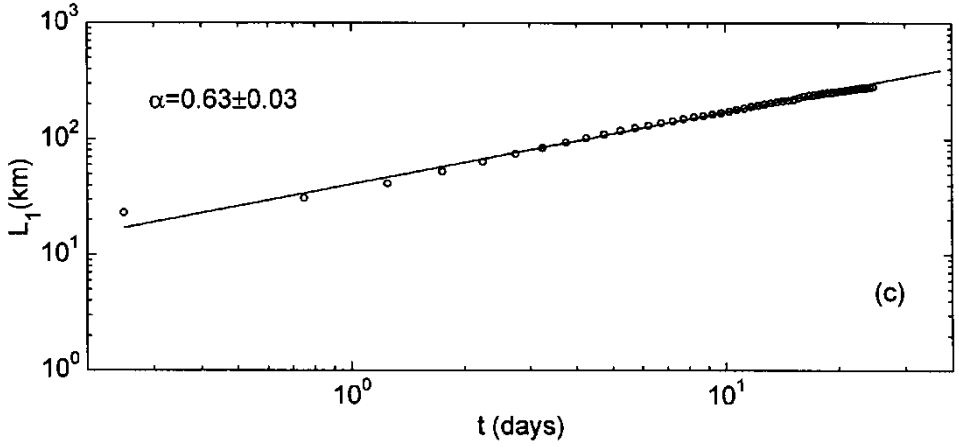
(a) 0

(b) 2.2 km

(c) 22 km







Scaling behavior of the

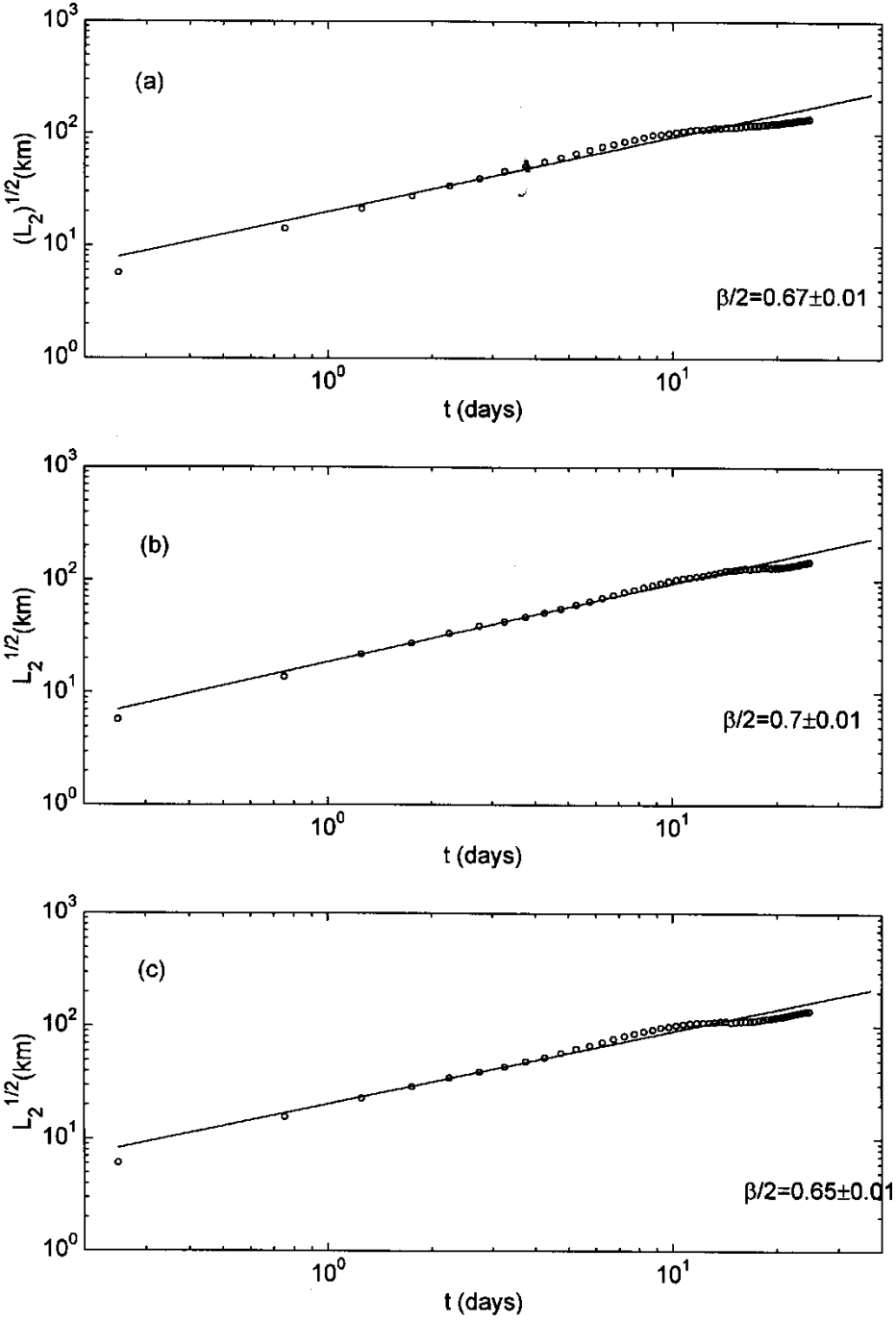
Error variance (L_2) growth

for initial error levels:

(a) 0

(b) 2.2 km

(c) 22 km



Predictability Tube

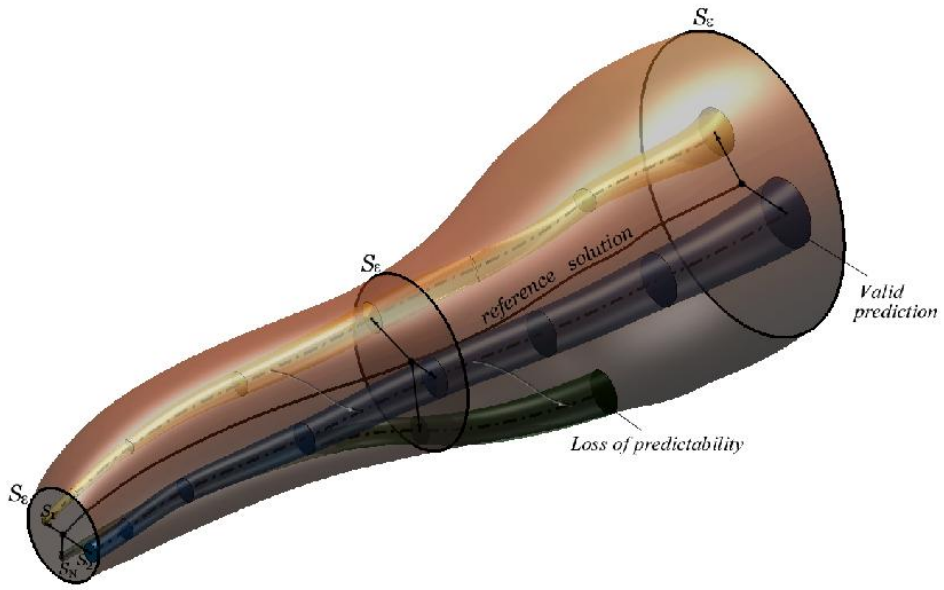


Fig. 31

Results

- (1) FPT is an effective prediction skill measure (scalar).
- (2) Theoretical framework for FPT (such as Backward Fokker-Planck equation) can be directly used for model predictability study.
- (3) Spectral method is an effective way to transfer the data from the physical space into phase space (theoretical and practical significances).