Resource Allocation in Two-Layered Cyber-Defense

Michael P. Atkinson*, Moshe Kress†

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Abstract

A common network security approach is to create a De-Militarized Zone (DMZ) comprising two layers of network defense. The DMZ structure provides an extra layer of security between the sensitive information in a network (e.g., research and development files) and the component of the network that must interface with the general internet (e.g., the mail server). We consider a cyber-attack on a DMZ network where both attacker and defender have limited resources and capabilities to attack and defend, respectively. We study two optimization problems and one game-theoretic problem. Given that the attacker (defender) knows the potential capabilities of the defender (attacker) in the two layers, we obtain the optimal allocation of resources for the attacker (defender). The two optimization problems are not symmetrical. Absent any knowledge regarding the allocation of the adversary's resources, we solve a game-theoretic problem and obtain some operational insights regarding the effect of combat (e.g., cyber) capabilities and their optimal allocation.

Keywords: stochastic duel, cyber, DMZ, allocation game

*mpatkins@nps.edu Operations Research Department, Naval Postgraduate School

†mkress@nps.edu Operations Research Department, Naval Postgraduate School

1 Introduction

Layered defense is a key concept in computer networks defense [1]. Specifically, a common network security approach is to utilize a De-Militarized Zone (DMZ) structure [2][3], which generates two layers of network defense. The DMZ itself consists of the portions of the enterprise network between the internet and the enterprise's intranet [3]. The intranet contains sensitive files such as personal information, financial records, and research and development plans. The DMZ contains the parts of the network that must interface with the internet (e.g., mail server). We focus on the two defensive layers of the DMZ that border the DMZ – the outer layer facing the internet, and the inner layer facing the intranet. A cyber-attacker, attempting to penetrate a computer network of the enterprise and access its intranet, needs to successfully breach these two layers of defense, without being detected by the defender, in order to successfully achieve its attacking goal. More generally, we consider a conflict situation in which the attacker (Red) proceeds to sequentially infiltrate the defender's (Blue) two layers of defense. Red prevails as the victor if it wins both battles. Otherwise, Blue wins. This conflict situation is modeled as a one-on-two combat model, where a single Red attacker engages two layers of Blue defense and Red must sequentially beat them both in order to win. Given that Red (Blue) has limited attack (defense) resources, the question is how should the two sides allocate their respective resources, where Red wants to maximize the probability of a win, and Blue wishes to minimize it.

While we focus on the cyber domain as our motivating case in this paper, our model is also appropriate for other scenarios. For example, physical locations (e.g., military bases, banks, museums) protected by layers of security that require different skills and/or tools to penetrate. In the museum scenario the attacker would need to first breach the exterior defenses of the museum (e.g., locks, patrollers), and then would need to avoid detection by guards, cameras, and sensors in the interior of the museum to successfully steal the artifact.

Mathematical models representing related armed conflicts comprise a large body of research that ranges between aggregate combat models, i.e., *Lanchester models*, which address large-formation engagements [4][5][6][7], and more detailed probabilistic models, i.e., *stochastic duels*, which describe small-scale engagements [8],[9],[10][11][12].

Colonel Blotto games consider a similar scenario where Red and Blue allocate resources across multiple battlefields [13, 14, 15]. While both our model and Blotto games are resource allocation models, there are several important differences. In contrast to our setting, most Blotto models assume the battlefields are homogenous and contested simultaneously, Red and Blue have equivalent capabilities, and the resources are discrete (e.g., military units). While Blotto assumes all battlefields are engaged in parallel, our setting can be viewed as a series-system from Red's perspective as Red must succeed in both layers to prevail. [16] examines game theoretic interactions in a series-system, however there are many differences between our scenario and the one in [16]. For example, only Red would choose their resource allocation under the framework in [16], whereas both Red and Blue make resource allocation decisions in our model.

Our setting is similar to missile defense where Red fires at Blue targets and Blue responds by launching a series of salvos to intercept the Red threats [17, 18, 19]. Red must sequentially penetrate several layers of Blue defense (interception salvos at long, medium, short range) to hit the targets. Blue only needs to successfully intercept Red in one of the salvos. While most of the work in missile defense is prescriptive, there are some descriptive models that analyze the number of threats that survive each layer [20, 21, 22, 23]. There are some crucial differences between our setting and the missile defense scenario. The missile defense problem is usually analyzed from Blue's perspective. While Red may have a decision about which targets to fire at, Red does not allocate resources across the defensive layers. Furthermore, most work in missile defense examines the weapon target assignment (WTA) problem, which considers the assignment of specific interceptors against specific threats at specific ranges [24, 25, 26]. The WTA is a nonlinear integer optimization problem, and most research focuses on developing heuristics [27]. Our model is much simpler and provides insight into the resource allocation of both Red and Blue across the two layers, and how that allocation varies with key input parameters.

Traditionally, combat models have been applied to violent "kinetic" conflicts where attacks are conducted with lethal weapons and attrition is physical. However, combat models can also be applied to "soft kill" settings, such as cyber-warfare, where missiles and bullets are replaced by lines of code. In such situations attrition is manifested in loss of valuable information and/or disruptions in the operation of the computer network. Cyber warfare has drawn the attention of the research community [28][29], and in particular, its potential impact on kinetic warfare [30][31]. Moreover, the operations-research community has addressed cyber-related modeling challenges by combining combat and epidemic models [31][32][33], analyzing the development and employment of munitions against exploits [34], and applying exploration-exploitation models [35]. A recent survey paper [36] reviews studies that apply optimization to the design of cyber infrastructure.

Game-theoretic approaches for modeling cyber warfare are reported in [37] and references therein. The setting in [38] is similar to ours with two layers of a cyber defense. However, the model in [38] considers many discrete attack and defense options with varying costs, which leads to an intractable non-zero-sum game that is analyzed with various heuristics. In contrast, we derive analytic results that provide insight into how the inputs drive the results.

Unlike most of the work reported in the cyber-warfare literature, we explicitly address the layered-defense feature that characterizes many computer networks in the form of the DMZ structure. The question we study in this paper is that of resource allocation, both by the Red and Blue, between the two layers of defense. This study also naturally leads to game-theoretic situations.

The rest of the paper is organized as follows. We describe the model in Section 2 and present results in 3–5 for various scenarios where either Blue or Red or both make resource allocation decisions. Sections 6–7 consider extensions to the model where Red does not need to necessarily penetrate both layers to accomplish its objectives. Section 8 expands the game theoretic results from Section 5 to N layers

2 Model

We base our model on the fundamental stochastic duel, where one Blue shooter and one Red shooter repeatedly fire at each other until one is hit [8]. There are many extensions to the basic model, including multiple shooters and tactical considerations [9],[10][11][12]. In most duel models the time until a shooter scores a successful hit follows an exponential distribution

[9],[10][11]. Our problem can be viewed as two sequential duels – one at each layer. Red, the attacker, wins if it successfully penetrates the two layers. Blue wins if it detects Red in one of the layers. As in the duel literature, we model Red's penetration time and Blue's detection time as exponential random variables.

Although we later on somewhat relax it, we assume that the two layers of defense require different attack and defense capabilities. For example, hacking layer 1 requires a much different set of skills than hacking layer 2. Therefore, both Red and Blue have to decide how to allocate their respective cyber-resources (money and manpower) between the two layers. Obviously, Red must allocate non-zero resources to each one of the two layers in order to have a non-zero probability to win.

If Blue and Red allocate x_i and y_i of their respective resources to attack and defend layer $i, i = 1, 2,$, respectively, then the expected time until Red penetrates layer i and Blue detects the attack on that layer, are $\frac{1}{\mu_i y_i}$ and $\frac{1}{\lambda_i x_i}$, respectively. We normalize resources to unitless parameters such that $0 \leq x_i, y_i \leq 1$, $i = 1, 2.$ and $x_1 + x_2 = y_1 + y_2 = 1$. The last condition simply says that not utilizing all of one's resources is a dominated strategy. Otherwise, Red (Blue) should simply allocate the remaining resources to either layer and the probability of successful penetration will increase (decrease).

The parameters λ_i and μ_i incorporate two factors. The first is Blue's (Red's) intrinsic, or "per-capita", effectiveness (e.g., cyber qualifications and experience of individual computer analysts) in layer i. As mentioned earlier, the characteristics of the two layers might be very different, and so Blue could be effective at defending one layer but not the other (e.g., $\lambda_1 \gg \lambda_2$). The second factor is the overall level of resources (e.g., number of computer analysts) at Blue's (Red's) disposal. Recall that we normalize resources to lie within [0,1] and so while $x_i = 0.5$ and $y_i = 0.5$ are equivalent from a relative standpoint, they might differ substantially from an absolute perspective. The units of λ_i and μ_i are 1/(time) since the resources x_i and y_i are unitless. $\frac{1}{\lambda_i}$ ($\frac{1}{\mu_i}$) is the expected amount of time for Blue (Red) to defend (penetrate) layer i when Blue (Red) utilizes all available resources in layer i. In this paper, we only consider linear functions of resources: $\lambda_i x_i$ and $\mu_i y_i$. We leave for future work analysis of non-linear relationships between resources and the rates.

Recall that the engagement is asymmetric: Red must successfully defeat both layers to achieve its objective, whereas Blue only needs to detect Red in one layer. Assuming the layers are independent, the probability Red wins is:

$$P[\text{Red wins}] = \frac{\mu_1 y_1}{\mu_1 y_1 + \lambda_1 x_1} \times \frac{\mu_2 y_2}{\mu_2 y_2 + \lambda_2 x_2} = \frac{\alpha_1 y_1}{\alpha_1 y_1 + x_1} \times \frac{\alpha_2 y_2}{\alpha_2 y_2 + x_2} \tag{1}$$

where $\alpha_i \equiv \frac{\mu_i}{\lambda_i}$ is the Red-Blue *effectiveness ratio* at layer $i, i = 1, 2$. Recall from the discussion above that the α_i ratio incorporates both the quality and quantity aspects of the two adversaries. Note also that if $y_i = 0$, in some layer, then $P[\text{Red wins}] = 0$ regardless of what Blue does.

Recall we assume the two layers of defense require different types of resources (e.g., cyber skills or tools). However, this may not always be the case; cyber personnel who successfully hack layer 1 may be able to also hack layer 2. Although we primarily focus on the situation where resources cannot be reused ($x_1 + x_2 = 1$, $y_1 + y_2 = 1$), we will show some numerical examples where one side, say Blue, can fully reuse its resources (e.g., $x_1 = x_2 = 1$)

We first consider one-sided situations in Sections 3–4 where we fix y_i (x_i) and optimize x_i (y_i) and then study a simultaneous game in Section 5. We conclude this section by presenting a table of model parameters.

Symbol	Range	Description
λ_i	$(0,\infty)$	Blue defensive effectiveness in layer i
μ_i	$(0,\infty)$	Red offensive effectiveness in layer i
α_i	$(0,\infty)$	$\alpha_i \equiv \frac{\mu_i}{\lambda_i}$: Red-Blue effectiveness ratio at layer i
C	$(0,\infty)$	$C \equiv \frac{\alpha_1}{\alpha_2}$: the effectiveness ratio in layer 1 relative to layer 2
D	$(0,1)$	Partial reward for Red when Red stops after layer 1 (Section 6 only)
q	$(0,1)$	Probability Red wins immediately after penetrating layer 1 (Section 7 only)
x_i	$[0,1]$	Blue's defensive resource allocation in layer i
y_i	$[0,1]$	Red's offensive resource allocation in layer i
x	$[0,1]$	When x appears without a subscript, it is Blue's allocation in layer 1. In this case Blue allocates $1-x$ to layer 2
y	$[0,1]$	Red's allocation in layer 1. In this case Red allocates $1-y$ to layer 2
$b(x;y_1,y_2)$	$[0,1]$	Red win-probability when Blue allocates x in layer 1 and $(1-x)$ in layer 2 and Red allocates y_i to layer i (Section 3 only)
$r(y;x_1,x_2)$	$[0,1]$	Red win-probability when Blue allocates x_i to layer i and Red allocates y in layer 1 and $(1-y)$ in layer 2 (Section 4 only)
$g(x,y)$	$[0,1]$	Red win-probability when Blue allocates x in layer 1 and $(1-x)$ in layer 2 and Red allocates y in layer 1 and $(1-y)$ in layer 2 (Section 5 only)

Table 1: Model Parameters

3 Blue's Defense Allocation

In this section we assume Red's allocation is fixed to y_1 and y_2, and Blue knows the values of $\mu_1 y_1$ and $\mu_2 y_2$. Blue optimizes the allocation x to layer 1, which determines the allocation $1-x$ to layer 2, such that its detection and threat-elimination rates are $\lambda_1 x$ and $\lambda_2(1-x)$ for layers 1 and 2, respectively. We first rewrite the Red win-probability in (1) to highlight the functional dependence on x:

$$P[\text{Red wins}] \equiv b(x;y_1,y_2) = \frac{\alpha_1 y_1}{\alpha_1 y_1 + x} \times \frac{\alpha_2 y_2}{\alpha_2 y_2 + (1-x)} \tag{2}$$

Blue wishes to minimize $b(x;y_1,y_2)$ subject to $x \in [0,1]$. This is equivalent to minimizing $\log b(x;y_1,y_2)$:

$$\log b(x;y_1,y_2) = \log \alpha_1 y_1 - \log(\alpha_1 y_1 + x) + \log \alpha_2 y_2 - \log(\alpha_2 y_2 + (1-x)). \tag{3}$$

It is easily seen that $\log b(x; y_1, y_2)$ is convex in x. Setting the derivative of $\log b(x)$ to 0 yields the unconstrained minimizer of $b(x; y_1, y_2)$:

$$\hat{x} = \frac{\alpha_2 y_2 - \alpha_1 y_1 + 1}{2}. \tag{4}$$

Note that there are boundary conditions for $\hat{x}$ that are affected by the effectiveness ratios $\alpha_i, i = 1, 2$. Intuitively, Blue should concentrate its resources where it has a better chance of detecting Red. Specifically, if $\alpha_1 y_1 \geq 1 + \alpha_2 y_2$, then Blue should invest all its resources in protecting layer 2. Conversely, if $\alpha_2 y_2 \geq 1 + \alpha_1 y_1$, then Blue should only focus on layer 1. When $\alpha_1 y_1$ and $\alpha_2 y_2$ are more similar ($-1 < \alpha_2 y_2 - \alpha_1 y_1 < 1$) the interior solution $\hat{x}$ given by (4) is optimal, and Blue allocates resources to both layers. We summarize Blue's optimal allocation in the following proposition:

Proposition 1. *Blue's optimal defense allocation for layer 1 is*

$$x^* = \min(\max(\hat{x}, 0), 1). \tag{5}$$

where $\hat{x}$ is defined by (4).

The constrained minimizer x^* in Proposition 1 follows by combining the unconstrained minimizer $\hat{x}$ with the convexity of $\log b(x; y_1, y_2)$.

We conclude this section by examining the worst case scenario for Blue, when Red is able to reuse all of its resources allocated to layer 1 in layer 2, that is, $y_1 = y_2 = 1$. As discussed in Section 2, this could occur if Red is able to use the same personnel or tools to hack both layers. We do not have data to estimate the parameters – they are typically classified – however, fortunately, we only need the relative quantities α_i, which should be easier to estimate compared to individual parameters. Arguably, $\alpha_1 \geq \alpha_2$; as Red penetrates deeper into the network, it becomes more vulnerable to Blue's detection capabilities. Figure 1 presents the optimal allocation x^* for Blue, as a function of α_1 for several values of $C \equiv \frac{\alpha_1}{\alpha_2}$. The parameter ranges we consider in Figure 1 and the rest of the paper correspond to moderate settings where Blue and Red have similar capabilities (i.e., α_i do not assume extreme values) and one layer is not significantly more difficult to penetrate than the other (i.e., α_1 and α_2 have the same magnitude). For small values of α_1 the optimal allocation x^* is the unconstrained minimizer $\hat{x}$, which by inspection of (4) is just a line with an intercept of $\frac{1}{2}$ and a slope of $\frac{1}{2}(\frac{1}{C} - 1)$. Notice, as trivially observed from (4), that if the two layers have equal effectiveness ratios ($C = 1$), Blue should equally split its resources between the two layers, regardless of the actual value of the effectiveness ratio α_1. As C increases (i.e., the effectiveness ratio in layer 1 increases compared to layer 2), the fraction of Blue's resources directed to layer 1 decreases. For a given $C > 1$, as α_1 increases (i.e., Red becomes more effective compared to Blue in layer 1) x^* decreases to the point where Blue should abandon layer 1 and put all of its resources in layer 2 (e.g., when $C = 5$ and $\alpha_1 \geq 1.25$).

As mentioned above, it is most likely that $C \geq 1$. The case $C < 1$ is presented in the plot just as a reference.

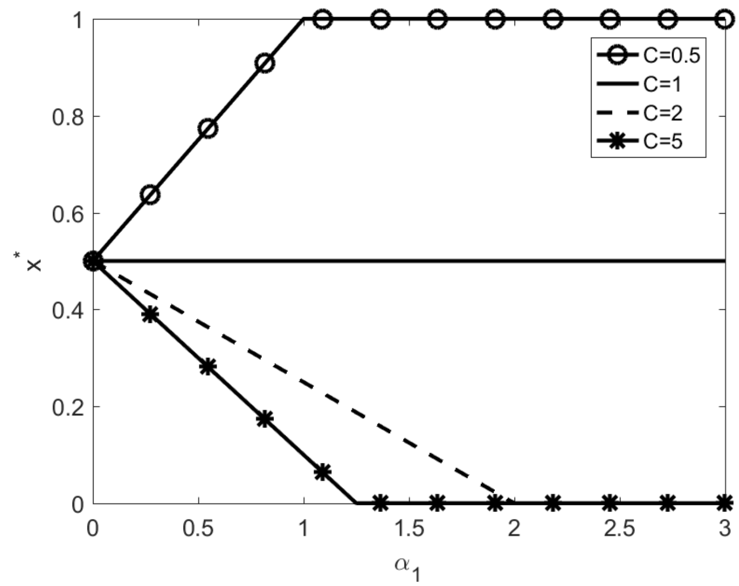


Figure 1: Blue's optimal allocation at layer 1, x^*, as a function of α_1 for several values of $C \equiv \frac{\alpha_1}{\alpha_2}$. Red is able to reuse all its resources from layer 1 in layer 2: $y_1 = y_2 = 1$

4 Red's Attack Allocation

We now assume that Blue's allocation is fixed at x_1 and x_2 and Red optimizes its resource allocation while knowing the values of $\lambda_1 x_1$ and $\lambda_2 x_2$. Red optimizes y to layer 1 and $1-y$ to layer 2. Thus, Red's problem is to choose y that maximizes

$$P[\text{Red wins}] \equiv r(y; x_1, x_2) = \frac{\alpha_1 y}{\alpha_1 y + x_1} \times \frac{\alpha_2(1-y)}{\alpha_2(1-y) + x_2}. \tag{6}$$

Equation (6) is a special case of (1). Note that $r(0; x_1, x_2) = r(1; x_1, x_2) = 0$ for any $x_1, x_2 \in [0, 1]$, whereas $r(y; x_1, x_2) > 0$ for any $0 < y < 1$. Hence unlike Blue, who might optimally concentrate all of its resources only in one layer (see Section 3), Red must allocate positive effort to each layer because otherwise $P[\text{Red wins}] = 0$. Thus, the optimal allocation must lie in the interior: $y^* \in (0, 1)$. The following proposition presents the optimal allocation.

Proposition 2. *Red's optimal attack allocation to layer 1 is*

$$y^* = \frac{-x_1(\alpha_2 + x_2) + \sqrt{x_1 x_2 (\alpha_1 + x_1)(\alpha_2 + x_2)}}{\alpha_1 x_2 - \alpha_2 x_1} \quad \textit{for } \frac{\alpha_1}{x_1} \neq \frac{\alpha_2}{x_2} \tag{7}$$

When the denominator of (7) is 0 ($\frac{\alpha_1}{x_1} = \frac{\alpha_2}{x_2}$), $y^* = 0.5$.

The proof for Proposition 2 proceeds in a similar fashion to the logic in Section 3 for Blue's defense allocation. We show that $r(y; x_1, x_2)$ is a concave function of y and y^* in (7) satisfies the first order condition. Full details of the proof for Proposition 2 appears in Appendix B.2.

As with Figure 1, Figure 2 displays the optimal resource allocation y^* for Red as a function of α_1 for six values of $C \equiv \frac{\alpha_1}{\alpha_2}$. Similarly to Figure 1, we assume a worst case for Red where Blue can fully reuse its resources in layer 2: $x_1 = x_2 = 1$. In the special case when the two layers are equal in terms of effectiveness ratios ($C = 1$), the optimal allocation is to equally split the resources between the two layers. Also, notice from Figure 2 that, unlike the case for Blue in Figure 1, Red's resource allocation is quite insensitive to both the effectiveness ratios α_i and the relative effectiveness between the two layers C. As observed above, Red has to engage in both layers to succeed, but Figure 2 shows that Red's level of engagement in the two layers is close to parity, unless both α_1 and C are very large.

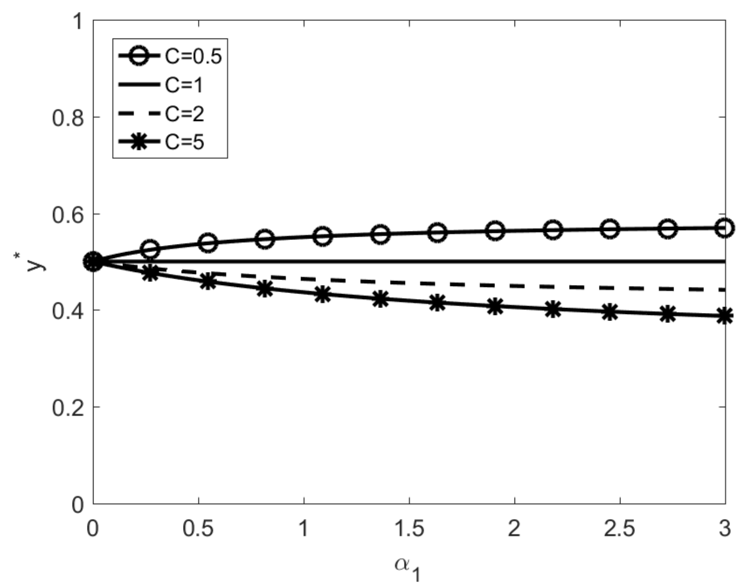


Figure 2: Red's optimal allocation at layer 1, y^*, as a function of α_1 for several values of $C \equiv \frac{\alpha_1}{\alpha_2}$. Blue is able to reuse all its resources from layer 1 in layer 2: $x_1 = x_2 = 1$

5 Simultaneous Allocation

In the previous two sections we assume that Blue (Red) allocates its finite resource against a fixed Red (Blue) allocation. Suppose now that both sides choose how to allocate their limited resources between the two layers simultaneously. As in Sections 3–4, we assume that resources in layer 1 cannot be reused in layer 2: $x_1 + x_2 = y_1 + y_2 = 1$. Hence Blue (Red) only needs to choose its allocation x (y) in layer 1, with the remaining $1 - x$ ($1 - y$) going to layer 2. Both Blue and Red know the effectiveness ratios α_1, α_2, but do not know the allocation of effort (y, x) in the opposite side. In this case, equation (1) can be written as

$$P[\text{Red wins}] \equiv g(x, y) = \frac{\alpha_1 y}{\alpha_1 y + x} \times \frac{\alpha_2(1-y)}{\alpha_2(1-y) + (1-x)}. \quad (8)$$

Red wishes to maximize $g(x, y)$ while Blue wants to minimize it. Examining the second derivative of $g(x, y)$ reveals that $g(x, y)$ is a strictly convex function of x for a fixed y, and strictly concave function of y for a fixed x. Therefore, we have a concave-convex game (see Section 5.2 of [39]), which implies $g(x, y)$ has a saddle point, which is the solution of the allocation game of the cyber resources.

Proposition 3. *The unique solution of the simultaneous zero-sum allocation game between Red and Blue is*

$$x^* = y^* = \frac{1}{1 + \frac{\alpha_1+1}{\alpha_2+1}} \tag{9}$$

The value of the game – the probability that Red wins – is

$$v^* = \frac{\alpha_1}{\alpha_1 + 1} \times \frac{\alpha_2}{\alpha_2 + 1}. \tag{10}$$

Because we have a concave-convex game, we just need to verify that (x^*, y^*) in (9) satisfies the first order conditions. The complete proof of Proposition 3 appears in Appendix B.3.

Figure 3 shows the layer 1 resource allocation for both Blue and Red. As in the one-sided cases, we see that if the effectiveness ratios are the same in both layers ($C = 1$) then the allocation is equal in the two layers, regardless of the actual value of the ratio α_1. When the effectiveness ratio tilts, as one would expect, toward Blue at the second layer (i.e., C increases), the allocation of resources also tilts towards layer 2, albeit in moderate manner, as shown in Figure 3. The minimum fraction of resources Red (and Blue) must put in layer 1 is $\frac{1}{1+C}$ (when $C > 1$). Even if $\alpha_1 >> \alpha_2$ Red must still allocate some resources to layer 1 to penetrate it.

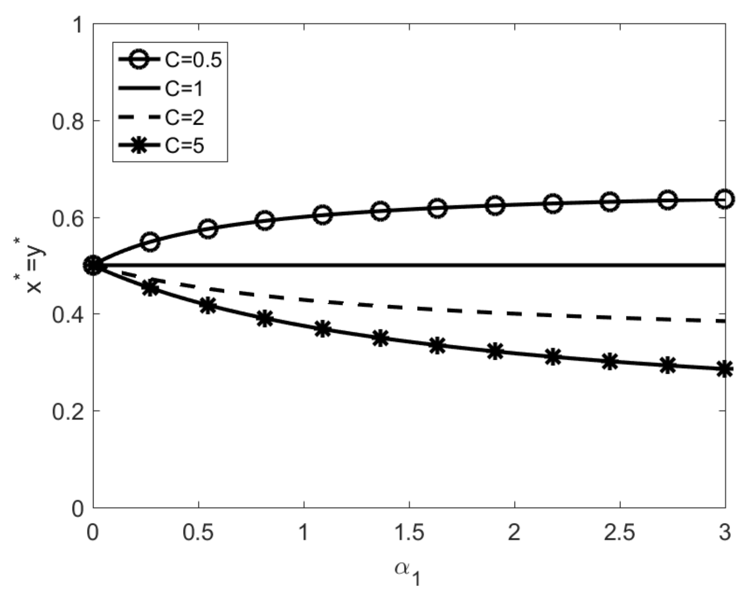


Figure 3: Blue and Red's optimal allocation at layer 1 as a function of α_1 for several values of $C \equiv \frac{\alpha_1}{\alpha_2}$.

6 Partial Reward for Red

Thus far we assume a binary situation: either Red successfully penetrates undetected the two defense layers of Blue, in which case Red is the winner, or Red is intercepted by Blue, either in layer 1 or layer 2, and Blue is the winner. Now suppose that Red can choose to stop after penetrating layer 1 and collect some partial reward $D < 1$. For example a hacker can stop after penetrating the DMZ and just download email messages. If Red decides to continue to layer 2 after successfully penetrating layer 1, Red will collect a reward of 1 if not intercepted by Blue in layer 2. Red receives a reward of 0 if Blue intercepts Red in either layer. That is, Red forfeits the D collected in layer 1 if Red continues to layer 2 and Blue intercepts Red at layer 2. So, the question here is regarding Red's stopping rule: shall Red stop after penetrating layer 1 or should Red continue to layer 2. Now in addition to Red and Blue choosing their resource allocations, Red must also choose whether to stop after layer 1. More specifically, we consider the simultaneous game situation where Blue decides on the value of x, and Red chooses both the value of y and whether to stop after layer 1 or proceed to layer 2. We use the nomenclature "choose layer 1" or "choose layer 2" to denote Red's options for its stop/continue decision.

Define $f_i(x, y)$ as the game payoff (Red expected reward) if Red chooses layer i with allocation y and Blue uses allocation x.

$$f_1(x, y) \equiv \frac{\alpha_1 y}{\alpha_1 y + x} D \tag{11}$$

$$f_2(x, y) \equiv \frac{\alpha_1 y}{\alpha_1 y + x} \frac{\alpha_2(1 - y)}{\alpha_2(1 - y) + (1 - x)} \tag{12}$$

For a small value of D, Red gains little benefit from stopping after layer 1 and thus Red chooses layer 2; therefore the solution is the same as in Proposition 3. For larger values of D, Red plays a mixed strategy; with some probability Red only attacks layer 1 and obviously puts all its resources in that layer. Otherwise, Red plans to attack layer 2 too and allocates resources to both layers.

Proposition 4. *If*

$$D < \frac{\alpha_2}{\alpha_2 + 1} \times \frac{\alpha_1 + \alpha_2 + 1}{\alpha_1 + \alpha_2 + 2} \tag{13}$$

then the solution of the game is the same as in Proposition 3. That is, Red chooses layer 2 and the resource allocation between the two layers will be the same for Red and Blue as in Equation (9). Otherwise, Red plays a mixed strategy across two options:

- *With probability p^* Red chooses to allocate all of its resources to layer 1 ($y = 1$).*
- *With probability $1 - p^*$ Red chooses layer 2 and only allocates a fraction y^* to layer 1.*

Blue uses a pure strategy and allocates x^* to layer 1. The triple (x^*, y^*, p^*) satisfies the following simultaneous equations

$$y = \frac{-x(\alpha_2 + (1-x)) + \sqrt{x(1-x)(\alpha_1 + x)(\alpha_2 + (1-x))}}{\alpha_1(1-x) - \alpha_2 x} \tag{14}$$

$$p = \frac{\alpha_2 y(1-y)(\alpha_1 + x)^2((\alpha_1 y + x) - (\alpha_2(1-y) + (1-x)))}{\alpha_2 y(1-y)(\alpha_1 + x)^2((\alpha_1 y + x) - (\alpha_2(1-y) + (1-x))) + D(\alpha_1 y + x)^2(\alpha_2(1-y) + (1-x))^2} \tag{15}$$

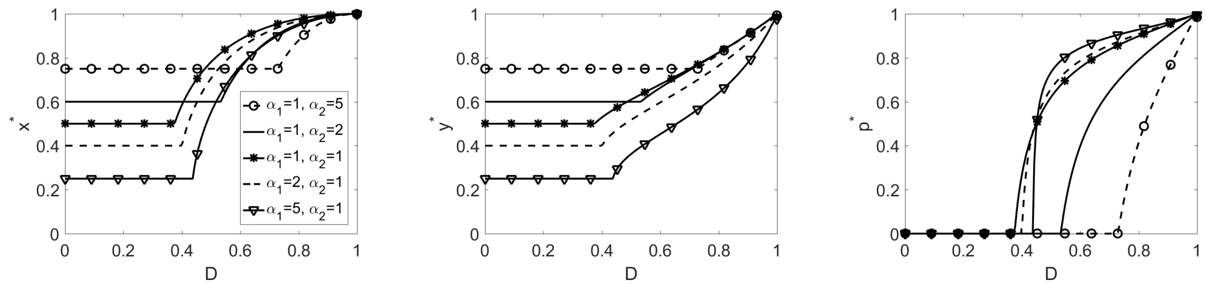
$$D(\alpha_1 y + x)(\alpha_2(1-y) + (1-x)) = \alpha_2 y(1-y)(\alpha_1 + x) \tag{16}$$

The proof of Proposition 4 appears in Appendix A. Equation (14) determines Red's best allocation y when choosing layer 2 if Blue allocates x. Equation (15) dictates Blue's best response to Red mixing with probability p and allocating y when Red chooses layer 2. Equation (16) equalizes the payoff between choosing layer 1 and layer 2 ($f_1(x^*, 1) = f_2(x^*, y^*)$), which allows for a Red mixed strategy to be optimal.

The threshold for D in (13) that determines whether Red solely chooses layer 2 is driven primarily by α_2. When Red is very effective in layer 2 (large α_2), then Red will attempt to penetrate layer 2 unless D is close to 1. For smaller values of α_2, Red is more likely to be satisfied with collecting D and stopping at layer 1.

While there is no closed form solution for (x^*, y^*, p^*) in (14)–(16), solving for these three parameters numerically is very straightforward as we only need to perform a grid-search over x, which fully determines the solutions for y and p via (14)–(15). We describe the grid-search approach in Appendix A.2.1. Figure 4 plots (x^*, y^*, p^*) from (14)–(16) vs. D for different values of α_1 and α_2. The curves are flat when D is less than the threshold in (13) and the solution is given by Proposition 3. As D increases, layer 1 becomes more enticing for Red as there is little marginal benefit to risking layer 2. However, as Proposition 4 reveals, Red never fully commits to layer 1 for $D < 1$. Figure 4 illustrates that, while theoretically Red never fully commits to layer 1 with certainty, practically, Red (and Blue) do put all the effort into layer 1 as $D \to 1$ since $x^*, y^*, p^* \to 1$.

x^* and y^* no longer equal each other once D increases beyond the threshold specified in (13). x^* more quickly increases to 1 than y^*. This occurs partly because x^* needs to account for Blue's response to Red choosing either layer 1 or layer 2, whereas y^* is the solution conditioned on Red choosing layer 2. Red also shifts its focus to layer 1 via its mixing probability p^* as D increases. x^* increases more quickly also because of the asymmetric nature of the engagement: Blue only needs to intercept in one layer, whereas Red needs to succeed in both, so Red cannot be as aggressive shifting toward layer 1.



(a) x^* vs. D (b) y^* vs. D (c) p^* vs. D

Figure 4: Blue (x^*) and Red's (y^*, p^*) optimal strategy as a function of D for several combinations of (α_1, α_2)

7 Early Victory

We consider here a similar situation to the one described in Section 6. Instead of partial reward D, we assume that there is a probability q that Red wins – it attains its attack goals – immediately after penetrating layer 1. In that case, Red does not need to proceed to layer 2, in which a successful attack guarantees a win. In the cyber DMZ scenario this could occur if a critical file, targeted by Red, is mistakenly moved by Blue into the DMZ. Such a situation could occur, for example, when an individual needs to work at home and emails themselves the critical file; once the file is on the email server, Red can gain access to it without penetrating layer 2. For the museum scenario, Red's target artifact has been moved to a less secure location in the museum for cleaning. This early victory setting represents Red getting lucky and only needing to exploit the outermost layer to win. For arbitrary Blue allocation (x_1, x_2) and Red allocation (y_1, y_2) the Red win-probability in (1) generalizes to:

$$P[\text{Red wins}] = \frac{\alpha_1 y_1}{\alpha_1 y_1 + x_1}\left(q + (1-q)\frac{\alpha_2 y_2}{\alpha_2 y_2 + x_2}\right) \tag{17}$$

The term outside the parentheses is the probability Red is successful in layer 1; Red still must penetrate layer 1 to win. If Red succeeds in layer 1, then with probability q Red wins, otherwise Red proceeds to layer 2 and must succeed in layer 2 to win. We assume that q is a fixed constant; future work could examine the situation where Red or Blue could modify q via resource allocation.

We extend the results from Section 3–5 in the following three subsections.

7.1 Blue's Defense Problem

Given fixed Red allocation (y_1, y_2), Blue's problem is to minimize

$$P[\text{Red wins}] = \frac{\alpha_1 y_1}{\alpha_1 y_1 + x}\left(q + (1-q)\frac{\alpha_2 y_2}{\alpha_2 y_2 + (1-x)}\right) \tag{18}$$

Define:

$$\tilde{x} = \frac{q + \alpha_2 y_2 - \sqrt{\alpha_2 y_2 (1-q)(q(1+\alpha_1 y_1) + \alpha_2 y_2)}}{q}. \tag{19}$$

Proposition 5. *Blue's optimal defense allocation for layer 1 is*

$$x^* = \min(\max(\tilde{x}, 0), 1). \tag{20}$$

where $\tilde{x}$ is defined by (19).

The proof of Proposition 5 appears in Appendix B.1.

7.2 Red's Attack Problem

Given fixed Blue allocation (x_1, x_2), Red's problem is to maximize

$$P[\text{Red wins}] = \frac{\alpha_1 y}{\alpha_1 y + x_1}\left(q + (1-q)\frac{\alpha_2(1-y)}{\alpha_2(1-y) + x_2}\right) \tag{21}$$

Define:

$$\tilde{y} = \frac{\alpha_2 x_1(\alpha_2 + x_2) - \sqrt{\alpha_2 x_1 x_2(\alpha_2 + x_2)(1-q)(\alpha_1 x_2 q + (\alpha_1 + x_1)\alpha_2)}}{\alpha_2(\alpha_2 x_1 - \alpha_1 x_2(1-q))}. \tag{22}$$

In the special case when the denominator of (22) equals 0, $\tilde{y}$ simplifies to

$$\tilde{y} = \frac{1}{2} + \frac{x_2 q}{2\alpha_2} \tag{23}$$

Proposition 6. *Red's optimal attack allocation for layer 1 is*

$$y^* = \min(\tilde{y}, 1). \tag{24}$$

where $\tilde{y}$ is defined by (22)–(23).

The proof of Proposition 6 appears in Appendix B.2. In the original formulation with $q = 0$ in Section 4, Red had to optimally allocate a positive amount to both layers. With $q > 0$, Red must still allocate a positive amount to layer 1. However, if q is large enough, Red might neglect layer 2 and allocate everything to layer 1 in the hope that the early victory occurs.

7.3 Simultaneous Allocation

When both Blue and Red optimally allocate their resources, then Red and Blue are engaged in a zero-sum game: Red wants to maximize and Blue minimize the following value

$$P[\text{Red wins}] = \frac{\alpha_1 y}{\alpha_1 y + x}\left(q + (1-q)\frac{\alpha_2(1-y)}{\alpha_2(1-y) + (1-x)}\right) \tag{25}$$

Proposition 7. *The unique solution of the simultaneous zero-sum game is*

$$x^* = y^* = \frac{q(\alpha_2+1)^2 + (1-q)\alpha_2(\alpha_2+1))}{q(\alpha_2+1)^2 + (1-q)\alpha_2(\alpha_1+\alpha_2+2)} \tag{26}$$

The game value (Red win-probability) is:

$$v^* = \frac{\alpha_1}{\alpha_1+1}\left(q + (1-q)\frac{\alpha_2}{\alpha_2+1}\right) \tag{27}$$

The proof of Proposition 7 appears in Appendix B.3. Figure 5 plots the relationship between x^*, y^* and q for different values of α_1 and α_2. x^* and y^* start at the solution given in Proposition 3 at $q = 0$ and increase toward 1 in a near linear fashion. In particular, linearity is attained in the case of parity, $\alpha_1 = \alpha_2 = 1$, where the allocation is $x^* = y^* = \frac{q+1}{2}$, and the probability Red wins is $v^* = \frac{q+1}{4}$. More generally, the relationship is linear whenever $\alpha_2 = \frac{1}{\alpha_1}$, in which case $x^* = y^* = q + (1-q)\frac{1}{\alpha_1+1}$.

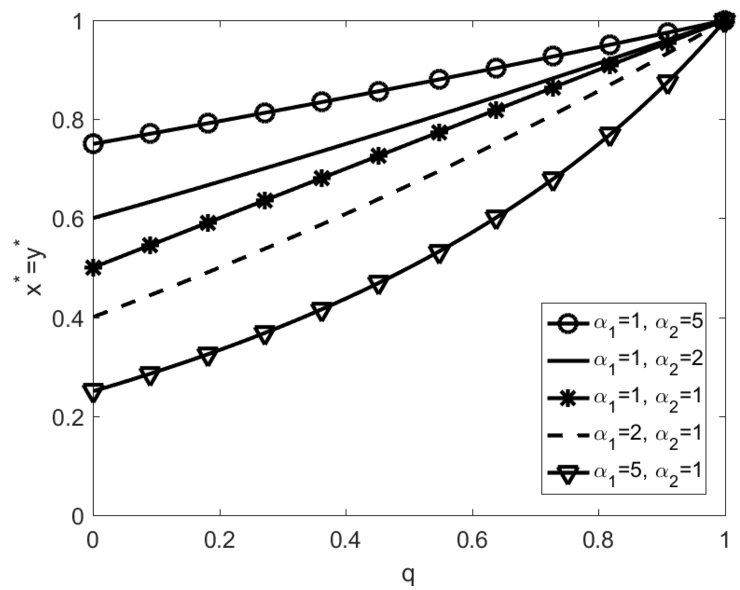


Figure 5: Blue and Red's optimal allocation at layer 1 as a function of q for several combinations of (α_1, α_2)

8 N-layer Simultaneous Game

In this section we extend the game theoretic model from Section 5 to N layers. Blue allocates $\mathbf{x} = (x_1, x_2, \ldots, x_N)$ to defend the layers and Red allocates $\mathbf{y} = (y_1, y_2, \ldots, y_N)$ to attack. Equation (1) generalizes to

$$g(\mathbf{x}, \mathbf{y}) \equiv P[\text{Red wins}] = \prod_{i=1}^{N} \frac{\alpha_i y_i}{\alpha_i y_i + x_i} \tag{28}$$

We assume resources cannot be reused across layers, that is, $\sum_{i=1}^{N} x_i = \sum_{i=1}^{N} y_i = 1, x_i, y_i \geq 0$. As in Section 5, the game payoff in (28) generates a concave-convex game and yields the following saddle point solution.

Proposition 8. *The unique optimal solution of the simultaneous zero-sum game is*

$$x_i^* = y_i^* = \frac{\frac{1}{\alpha_i+1}}{\sum_{j=1}^{N} \frac{1}{\alpha_j+1}} \tag{29}$$

The game value (Red win-probability) is:

$$v^* = \prod_{i=1}^{N} \frac{\alpha_i}{\alpha_i + 1} \tag{30}$$

The proof of Proposition 8 appears in Appendix C. Proposition 8 generalizes Proposition 5.

If α_i is large compared to α_j, $j \neq i$, then the optimal allocation x_i^* (y_i^*) is close to 0 for layer i. In this case Red is very effective relative to Blue in layer i, and so Blue essentially concedes layer i. If α_i is small compared to α_j, $j \neq i$, (i.e., Blue is very effective relative to Red in layer i), then the resource allocations (x_i^*, y_i^*) increase, but do not approach 1. Red has to successfully penetrate every layer, so cannot allocate too much to any one layer. For example, if $\alpha_i \approx 0$, and $\alpha_j \equiv \alpha$ are equal across the remaining layers $j \neq i$, equation (29) simplifies to

$$x_j^* = y_j^* = \begin{cases} \frac{\alpha+1}{N+\alpha} & \text{if } j = i \\ \frac{1}{N+\alpha} & \text{if } j \neq i \end{cases} \tag{31}$$

The resource allocation in layer i (where Red is ineffective) is $\alpha + 1$ times greater than the allocation in any of the other layers. For example, with $N = 7$ layers and $\alpha = 8$, Blue and Red allocate $x_i^* = y_i^* = 0.6$ to layer i, which leaves a substantial amount of resources for the other layers.

9 Conclusion

As in any contest, resource allocation in cyber warfare may determine the outcome of the confrontation. Specifically, when cyber resources, either offensive or defensive, are limited, actors engaged in cyber warfare must optimize the deployment of those resources and/or modify their tactics. In this paper we formulate a base model where Red needs to successfully penetrate both layers to achieve victory, whereas Blue only needs to detect Red in one of the layers. We also consider extensions where Red may achieve its objective without penetrating both layers. In the situation where Blue optimizes its allocation for a fixed Red allocation, Blue focuses all its resources on the layer where it has the advantage unless the relative effectiveness levels in the two layers are similar. This contrasts with the scenario when Red is the sole decision-maker against a fixed Blue allocation: Red always allocates resources to both layers in a nearly equal split that is fairly insensitive to the effectiveness of Red and Blue. In the game where both Red and Blue allocate resources, the allocation is symmetric and usually of moderate value; the allocation only approaches 0 or 1 when the two layers significantly differ in their effectiveness ratios. When Red can obtain rewards for just penetrating layer 1, both Red and Blue shift resources to layer 1. The models presented in this paper, combined with controlled Red Team/ Blue Team exercises and wargames, can guide cyber combat developers in determining where would be the highest "bang for the buck" in allocating resources in cyber attack or defense.

There are many avenues for future research. One could examine the notion of reusable resources more carefully. For example, the resources could be split into three bins: those that apply solely to layer 1, those that apply solely to layer 2, and those that apply to both layer 1 and layer 2. Presumably, the resources that specialize to only one layer are more effective than the general resources that can be used in both. Another possible extension is to generalize the fixed early-victory parameter q in Section 7 to account for resources that may affect its value. Red might be easier to detect when Red allocates more resources to a layer. Therefore, Blue's overall defensive rate may depend upon y_i in addition to x_i. Another related approach would have Red allocate its resources between a speed component and a stealth component of its attack plan. We assume complete information framework where both sides know all parameters. Future work could develop a Bayesian game for an incomplete information setting. If Red repeatedly attacks, a learning component could be incorporated where Blue and Red update their beliefs about their opponent's parameters after each round. Cyber data sets exist (e.g., [40]), however most are meant to be benchmarks for machine learning classifiers trying to detect cyber intrusions. Future work could perform an empirical exercise to examine our model by collecting data via experiments or cyber competitions.

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APPENDIX

A Partial Reward for Red

This section covers Proposition 4 from Section 6. A solution to the game exists. This follows because Blue and Red both have compact strategy spaces. See [42] and Theorem 2.7.1 and remark 2.7.2 of [43]. We first discuss pure-strategy equilibria and then cover mixed-strategies.

A.1 Pure Strategies

A pure strategy solution has Blue choose one x and Red chooses one layer and one corresponding y. There is no pure-strategy equilibrium where Red chooses layer 1. If there were such an equilibrium, then $x^* = y^* = 1$, otherwise Blue and Red would both have incentive to shift resources to layer 1 because the game would not proceed beyond layer 1. However, if $x^* = y^* = 1$, then Red would have incentive to choose layer 2 by shifting ϵ resources to layer 2 (i.e., $y^* = 1 - \epsilon$) and increase its reward from D to 1. This follows because layer 2 is unguarded by Blue ($x^* = 1$) and hence Red can effectively win layer 2 for free by shifting ϵ and increase its reward.

We now show that it is possible to have a pure-strategy equilibrium where Red chooses layer 2. If Red chooses layer 2, then the interaction is identical to Section 5 and hence x^* and y^* are defined in (9) of Proposition 3. Blue has no incentive to deviate from this solution by the same logic as in Proposition 3. However, Red may have incentive to shift to layer 1. If Red shifts to layer 1, it would also allocate all resources to layer 1 ($y = 1$). Hence, if Blue remains at x^* from (9), the best Red could do by shifting to layer 1 is

$$v_r \equiv \frac{\alpha_1}{\alpha_1 + x^*} D \tag{A.1}$$

If v_r in (A.1) exceeds the layer-2 solution, v^* from (10), then Red has incentive to deviate. Otherwise if $v_r < v^*$, Red is satisfied with the solution and Proposition 3 provides the equilibrium. Examining v_r in (A.1) and v^* from (10) yields the following condition

$$\frac{\alpha_1}{\alpha_1 + x^*} D < \frac{\alpha_1}{\alpha_1 + 1} \times \frac{\alpha_2}{\alpha_2 + 1} \tag{A.2}$$

Substituting in x^* from (9) and going through the algebra yields the threshold in (13) of Proposition 4.

A.2 Mixed Strategies

If (13) does not hold, then no pure-strategy solution exists. Recall the definition of $f_i(x, y)$ from (11)–(12): $f_i(x, y)$ is payoff if Red chooses layer i with allocation y and Blue chooses x.

We first examine Red's mixed strategy. Assume Red chooses layer 1 with probability p and layer 2 with probability $1 - p$. Since $f_1(x, y)$ in (11) increases in y for any fixed x, the Red strategy (layer 1, $y = 1$) dominates (layer 1, $y < 1$). Hence, any optimal mixed strategy must use allocation $y = 1$ whenever layer 1 is chosen. There is no dominance for layer 2, other than y must lie in the interior $y \in (0, 1)$. Therefore, we specify that Red allocates y in layer 2 according to the density $r_2(y)$. If Red uses the mixed strategy described in this paragraph against a fixed x, the payoff is

$$v(x) = pf_1(x, 1) + (1 - p) \int_0^1 r_2(y) f_2(x, y) dy \tag{A.3}$$

Turning to Blue's strategy, we note that $v(x)$ is a convex function over $x \in [0, 1]$ because both $f_1(x, y)$ and $f_2(x, y)$ are convex in x (see section 3.2.1 in [41]). Furthermore, because of the $f_1(x, 1)$ term in (A.3), $v(x)$ is a strongly convex function over $x \in [0, 1]$, and hence there is unique minimizer for Blue. Consequently, for any equilibrium with a Red mixed-strategy, Blue will use a pure strategy x that minimizes $v(x)$ in (A.3).

We have argued that an equilibrium must have the following properties: Blue plays pure strategy x and Red chooses the tuple $(p,\ r_2(y))$. However, $f_2(x, y)$ is concave in y for fixed x (we showed this in Section 4). Therefore if Blue plays pure strategy x at equilibrium, Red would not mix in layer 2 according to density $r_2(y)$, but would just use the one y that maximizes $f_2(x, y)$.

Putting the pieces together, a valid equilibrium must have: Blue plays x, Red chooses layer 1 with probability p (with all resources allocated to layer 1), and Red allocates y when choosing layer 2 (with probability $1 - p$). The game value in (A.3) can now be written as a function of the triple (x, y, p)

$$v(x, y, p) = pf_1(x, 1) + (1 - p)f_2(x, y) \tag{A.4}$$

To determine an (x^*, y^*, p^*) that generates an equilibrium, we have to show neither Red nor Blue will deviate. We start with Red. If Red mixes between layer 1 and layer 2, then Red must be indifferent between the two:

$$f_1(x^*, 1) = f_2(x^*, y^*) \tag{A.5}$$

Condition (A.5) is equivalent to (16) in Proposition 4.

In order for Red to not deviate from y^* when choosing layer 2, y^* must be the best response to Blue x^*. Section 4 examines the one-sided decision when Red chooses y for a fixed Blue allocation (x_1, x_2). If we substitute in $x_1 = x$ and $x_2 = (1 - x)$ in (7), we get Red's best response, which is a necessary condition for an equilibrium. This modification of (7) yields (14) in Proposition 4.

Turning to Blue, x^* must be the minimizer of $v(x, y^*, p^*)$. We first argue that neither $x = 0$ nor $x = 1$ can be the minimizer. Assume $x = 0$ minimizes $v(x, y^*, p^*)$. If $x = 0$ is part of an equilibrium then either ($x = 0, y = 1$, layer 1) is the equilibrium or ($x = 0, y = 0$, layer 2); it depends upon the relationship between D and $\frac{\alpha_2}{\alpha_2+1}$. However, neither ($x = 0, y = 1$, layer 1) nor ($x = 0, y = 0$, layer 2) is a valid equilibrium because in both cases Blue has incentive to increase x. Next assume $x = 1$ minimizes $v(x, y^*, p^*)$. If $x = 1$ is part of an equilibrium, then Red chooses layer 2 and sets $y = 1$ (or perhaps $1 - \epsilon$ to avoid 0/0 in layer 2). However ($x = 1, y = 1$, layer 2) cannot form an equilibrium because Blue would have incentive to shift δ to layer 2 to detect Red with certainty. Hence the minimizer of $v(x, y^*, p^*)$

lies in the interior of $x \in (0,1)$. Because $v(x, y^*, p^*)$ is strictly convex, the minimizer satisfies the first-order-condition. Setting $\frac{dv(x,y^*,p^*)}{dx} = 0$ and rearranging terms yields (15).

Equations (14)-(16) generate necessary conditions for an equilibrium. Because the game has a solution, there must exist a triple (x^*, y^*, p^*) that satisifes (14)-(16). This triple must also be unique. Assume there exists another triple $(\hat{x}, \hat{y}, \hat{p})$ that satisfies (14)-(16). By the same logic as above, $(\hat{x}, \hat{y}, \hat{p})$ would also be a solution to the game as neither Red nor Blue would deviate. However, zero-sum games have unique game values so $v(x^*, y^*, p^*) = v(\hat{x}, \hat{y}, \hat{p})$. The game value is

$$v(x^*, y^*, p^*) = f_1(x^*, 1) = \frac{\alpha_1}{\alpha_1 + x^*} \tag{A.6}$$

Because the game value only depends upon x^*, $\hat{x} = x^*$. However, x uniquely determines y by (14) and x and y jointly determine p by (14) . Therefore $(x^*, y^*, p^*) = (\hat{x}, \hat{y}, \hat{p})$ and the solution to (14)-(16) generates the unique equilibrium of the game.

A.2.1 Solving for Optimal (x^*, y^*, p^*)

Equations (14)–(16) uniquely determine the optimal triple (x^*, y^*, p^*). Unfortunately there is no closed form solution to (14)–(16). Solving for (x^*, y^*, p^*) numerically is straightforward as we only need to perform a grid-search over x. A given x determines y via (14), and x and y together determine p via (15). The grid search then determines which x (and corresponding y and p) satisfy (16). More formally we present the grid-search algorithm below

1. Define `xList` as a vector of equally spaced numbers between 0 and 1 with spacing `epsilon`
2. Initialize `yList` as a vector of y values corresponding to each x in `xList`
3. Initialize `pList` as a vector of p values corresponding to each x in `xList`
4. Initialize `sqDevList` as a vector measuring the difference between the left-hand side and right-hand side of equation (16) for each x in `xList`
5. `for i in length(xList)`
   (a) `x = xList[i]`
   (b) Compute `y` via equation (14)
   (c) `yList[i] = y`
   (d) Compute `p` via equation (15)
   (e) if `p < 0 or p > 1:  continue`
   (f) `pList[i] = p`
   (g) Compute `sqDev` by taking the squared difference of the left-hand side and right-hand side of equation (16)
   (h) `sqDevList[i] = sqDev`
6. `iBest = arg min sqDevList`
7. `return (xList[iBest], yList[iBest], pList[iBest])`

B Early Victory

This section covers the proofs for the results in Section 7. We examine the more general version in the Appendix with both q_1 and q_2, where q_i denotes the probability Red wins after layer i. The results in Section 7 correspond to the special case of $q_1 = q$ and $q_2 = 1$. The model in Sections 3–5 corresponds to the special case of $q_1 = 0$ and $q_2 = 1$, so the proofs in this section are valid for the results in Sections 3–5

B.1 Blue's Defensive Allocation

This section covers Proposition 1 and Proposition 5. We first show the objective is convex and then verify the optimality conditions. Blue desires to minimize the following objective:

$$b(x; y_1, y_2) \equiv P[\text{Red wins}] = \frac{\alpha_1 y_1}{\alpha_1 y_1 + x}\left(q_1 + (1-q_1)\frac{\alpha_2 y_2}{\alpha_2 y_2 + 1 - x} q_2\right) \tag{B.1}$$

For notational simplicity, we drop the dependence of $b(\cdot)$ on y_1 and y_2 for the remainder of this section and just write $b(x)$. We assume throughout that $y_1, y_2 > 0$. If $y_1 = 0$ the win-probability is 0 for all x and the problem is not interesting. If $y_2 = 0$ the optimal solution for Blue is trivial: $x^* = 1$.

We compute the first and second derivative of $b(x)$

$$\begin{aligned} b'(x) =& -q_1 \frac{\alpha_1 y_1}{(\alpha_1 y_1 + x)^2} - \frac{\alpha_1 y_1}{(\alpha_1 y_1 + x)^2}(1-q_1)\frac{\alpha_2 y_2}{\alpha_2 y_2 + 1 - x} q_2 \\ &+ \frac{\alpha_1 y_1}{\alpha_1 y_1 + x}(1-q_1)\frac{\alpha_2 y_2}{(\alpha_2 y_2 + 1 - x)^2} q_2 \end{aligned} \tag{B.2}$$

and

$$\begin{aligned} b''(x) =& 2q_1 \frac{\alpha_1 y_1}{(\alpha_1 y_1 + x)^3} \\ &+ 2\frac{\alpha_1 y_1}{(\alpha_1 y_1 + x)^3}(1-q_1)\frac{\alpha_2 y_2}{\alpha_2 y_2 + 1 - x} q_2 \\ &- 2\frac{\alpha_1 y_1}{(\alpha_1 y_1 + x)^2}(1-q_1)\frac{\alpha_2 y_2}{(\alpha_2 y_2 + 1 - x)^2} q_2 \\ &+ 2\frac{\alpha_1 y_1}{\alpha_1 y_1 + x}(1-q_1)\frac{\alpha_2 y_2}{(\alpha_2 y_2 + 1 - x)^3} q_2 \end{aligned} \tag{B.3}$$

We next show $b''(x) > 0$ on $x \in [0, 1]$ and thus $b(x)$ is strictly convex on that domain. The first line in (B.3) is non-negative (positive if $q_1 > 0$). The last 3 lines of (B.3) can be rewritten as

$$(1-q_1)q_2 w(x)v(x)((u(x) - z(x))^2 + u^2(x) + z^2(x)) > 0 \tag{B.4}$$

where

$$w(x,y) \equiv \frac{\alpha_1 y_1}{\alpha_1 y_1 + x} > 0 \tag{B.5}$$

$$v(x,y) \equiv \frac{\alpha_2 y_2}{\alpha_2 y_2 + 1 - x} > 0 \tag{B.6}$$

$$u(x,y) \equiv \frac{1}{\alpha_1 y_1 + x} > 0 \tag{B.7}$$

$$z(x,y) \equiv \frac{1}{\alpha_2 y_2 + 1 - x} > 0 \tag{B.8}$$

Consequently $b(x)$ is strictly convex on $x \in [0,1]$. Hence x^* is either 0, 1, or an interior point solution if $b'(x^*) = 0$ and $x^* \in (0,1)$. Let us examine the derivative $b'(x)$ from (B.2). First we multiply through by positive quantity $(\alpha_1 y_1 + x)^2(\alpha_2 y_2 + 1 - x)^2$ to remove the denominator; we will call this new expression $c(x)$. $c(x)$ has the same critical points as $b(x)$ and $sign(b(x)) = sign(c(x))$, so analyzing $c(x)$ is equivalent to $b(x)$ for our purposes.

$$\begin{aligned} b'(x) \propto c(x) \equiv & - q_1(\alpha_2 y_2 + 1 - x)^2 - (1 - q_1)\alpha_2 y_2(\alpha_2 y_2 + 1 - x)q_2 \\ & + (\alpha_1 y_1 + x)(1 - q_1)\alpha_2 y_2 q_2 \end{aligned} \tag{B.9}$$

Moving from (B.2) to (B.9), we also factor out the positive constant $\alpha_1 y_1$. Using the quadratic formula, we compute the two zeros of $c(x)$:

$$\tilde{x} = \frac{q_1(1 + \alpha_2 y_2) + \alpha_2 y_2 q_2(1 - q_1) - \sqrt{\alpha_2 y_2 q_2(1 - q_1)(q_1(1 + \alpha_1 y_1 + \alpha_2 y_2) + \alpha_2 y_2 q_2(1 - q_1))}}{q_1}. \tag{B.10}$$

$$\tilde{x}_2 = \frac{q_1(1 + \alpha_2 y_2) + \alpha_2 y_2 q_2(1 - q_1) + \sqrt{\alpha_2 y_2 q_2(1 - q_1)(q_1(1 + \alpha_1 y_1 + \alpha_2 y_2) + \alpha_2 y_2 q_2(1 - q_1))}}{q_1}. \tag{B.11}$$

Because $c(x)$ in (B.9) is a concave quadratic, we are interested in the smaller root $\tilde{x}$ in (B.10), as $b(x)$ increases as we increase past $\tilde{x}$. Furthermore, by inspection of (B.11) $\tilde{x}_2 > 1$ and so we focus on $\tilde{x}$ in (B.10), which corresponds to (19) in Section 7 (when $q_1 = q$ and $q_2 = 1$). Because $c(x)$ in (B.9) is a concave quadratic, if

- $\tilde{x} > 1$, then $b(x)$ is decreasing on $[0,1]$, and hence Blue's best option is $x^* = 1$.
- $\tilde{x} < 0$, then $b(x)$ is increasing on $[0,1]$ (since $\tilde{x}_2 > 1$), and hence Blue's best option is $x^* = 0$.
- $\tilde{x} \in (0,1)$, then $\tilde{x}$ is a local (and global) minimizer on $[0,1]$, and hence $x^* = \tilde{x}$.

Putting the three cases together yields the final expression in Proposition 5. Note Proposition 5 only presents the special case where $q_1 = q$ and $q_2 = 1$.

The analysis simplifies greatly for Proposition 1 in Section 3 when $q_1 = 0$ and $q_2 = 1$. The first order condition using (B.9) simplifies to

$$b'(x) \propto c(x) \equiv (\alpha_2 y_2 + 1 - x) + (\alpha_1 y_1 + x) = 0, \tag{B.12}$$

which yields $\hat{x}$ from (4).

B.2 Red's Offensive Allocation

This section covers Proposition 2 and Proposition 6. The proof proceeds in similar steps to Appendix B.1: show concavity and then solve for the first order conditions. Red desires to maximize the following objective:

$$r(y; x_1, x_2) \equiv P[\text{Red wins}] = \frac{\alpha_1 y}{\alpha_1 y + x_1} \left(q_1 + (1 - q_1) \frac{\alpha_2(1-y)}{\alpha_2(1-y) + x_2} q_2 \right) \tag{B.13}$$

We drop the dependence of $r(\cdot)$ on x_1 and x_2 for the remainder of this section and just write $r(y)$. We assume $\max(x_1, x_2) > 0$, otherwise the problem is trivial.

We compute the first and second derivative of $r(y)$

$$\begin{aligned} r'(y) =& q_1 \frac{\alpha_1 x_1}{(\alpha_1 y + x_1)^2} + (1 - q_1) q_2 \frac{\alpha_1 x_1}{(\alpha_1 y + x_1)^2} \frac{\alpha_2(1-y)}{\alpha_2(1-y) + x_2} \\ & - (1 - q_1) q_2 \frac{\alpha_1 y}{\alpha_1 y + x_1} \frac{\alpha_2 x_2}{(\alpha_2(1-y) + x_2)^2} \end{aligned} \tag{B.14}$$

and

$$\begin{aligned} r''(y) =& -2q_1 \frac{\alpha_1^2 x_1}{(\alpha_1 y + x_1)^3} \\ & - 2(1 - q_1) q_2 \frac{\alpha_1^2 x_1}{(\alpha_1 y + x_1)^3} \frac{\alpha_2(1-y)}{\alpha_2(1-y) + x_2} \\ & - 2(1 - q_1) q_2 \frac{\alpha_1 x_1}{(\alpha_1 y + x_1)^2} \frac{\alpha_2 x_2}{(\alpha_2(1-y) + x_2)^2} \\ & - 2(1 - q_1) q_2 \frac{\alpha_1 y}{\alpha_1 y + x_1} \frac{\alpha_2^2 x_2}{(\alpha_2(1-y) + x_2)^3} \end{aligned} \tag{B.15}$$

$r(y)$ is strictly concave on $y \in [0, 1]$ as $r''(y) < 0$ by inspection (B.15). To determine the optimal allocation y^*, let us examine the derivative $r'(y)$ from (B.14). First we multiply through by positive quantity $(\alpha_1 y + x_1)^2 (\alpha_2(1-y) + x_2)^2$ to remove the denominator; we will call this new expression $d(y)$.

$$\begin{aligned} r'(y) \propto d(y) \equiv & q_1 x_1 (\alpha_2(1-y) + x_2)^2 + (1 - q_1) q_2 x_1 \alpha_2 (1-y)(\alpha_2(1-y) + x_2) \\ & - (1 - q_1) q_2 \alpha_2 x_2 y (\alpha_1 y + x_1) \end{aligned} \tag{B.16}$$

Moving from (B.15) to (B.16), we also factor out the positive constant α_1. Rewriting (B.16) in standard quadratic form:

$$d(y) = ay^2 + by + c \tag{B.17}$$

where

$$a = \alpha_2(\alpha_2 x_1 (q_1 + q_2(1 - q_1)) - \alpha_1 x_2 q_2 (1 - q_1)) \tag{B.18}$$

$$b = -2\alpha_2 x_1 (\alpha_2 + x_2)(q_1 + q_2(1 - q_1)) \tag{B.19}$$

$$c = x_1 (\alpha_2 + x_2)(\alpha_2(q_1 + q_2(1 - q_1)) + q_1 x_2) \tag{B.20}$$

We consider the general case of $a \neq 0$ separately from the special case of $a = 0$.

B.2.1 $a \neq 0$

Using the quadratic formula we have two roots for $d(y)$:

$$\tilde{y} = \frac{\alpha_2 x_1(\alpha_2 + x_2)(q_1 + q_2(1-q_1)) - \sqrt{\alpha_2 x_1 x_2(\alpha_2 + x_2)(1-q_1)q_2(\alpha_1 x_2 q_1 + (\alpha_1 + x_1)\alpha_2(q_1 + q_2(1-q_1)))}}{\alpha_2(\alpha_2 x_1(q_1 + q_2(1-q_1)) - \alpha_1 x_2 q_2(1-q_1))} \tag{B.21}$$

$$\tilde{y}_2 = \frac{\alpha_2 x_1(\alpha_2 + x_2)(q_1 + q_2(1-q_1)) + \sqrt{\alpha_2 x_1 x_2(\alpha_2 + x_2)(1-q_1)q_2(\alpha_1 x_2 q_1 + (\alpha_1 + x_1)\alpha_2(q_1 + q_2(1-q_1)))}}{\alpha_2(\alpha_2 x_1(q_1 + q_2(1-q_1)) - \alpha_1 x_2 q_2(1-q_1))} \tag{B.22}$$

Root $\tilde{y}$ in (B.21), which corresponds to (22) in Section 7 (when $q_1 = q$ and $q_2 = 1$), is the relevant one for our analysis. We argue this by examining the $a > 0$ and $a < 0$ case separately, where a is defined in (B.18):

- When $a > 0$, $d(y)$ is a convex quadratic and hence we care about the smaller root because $d(y)$ is positive prior to the first root and therefore $r(y)$ is increasing on that domain. Clearly $\tilde{y} < \tilde{y}_2$ because the denominator of the roots is positive ($a > 0$ by assumption) and $\tilde{y}$ subtracts the square-root. $\tilde{y} > 0$ because $d(y)$ is a convex quadratic and $d(0) = c > 0$ by inspection of (B.20); the only other alternative for $\tilde{y} \leq 0$ would be if $y_2 \leq 0$ but inspection of (B.22) reveals $y_2 > 0$.

- When $a < 0$, $d(y)$ is a concave quadratic and hence we care about the larger root because $d(y)$ is positive between the first and second root and therefore $r(y)$ is increasing on that domain. $\tilde{y}_2 < 0$ since the numerator is positive and the denominator is negative. Because $d(0) = c > 0$, the second root must be positive because $d(y)$ is a concave quadratic. Therefore $\tilde{y}$ must be positive and thus $\tilde{y} > \tilde{y}_2$, and $\tilde{y}$ is the desired larger root.

We have shown that for $a \neq 0$, $\tilde{y}$ from (B.21) is the root that dictates Red's optimal allocation. We also showed that $\tilde{y} > 0$. If $\tilde{y} > 1$, then $d(y) > 0$ on $y \in [0, 1]$ and hence $r(y)$ increases on that domain and Red chooses $y^* = 1$. Otherwise if $\tilde{y} \in (0, 1)$, then $\tilde{y}$ is a local (and global) maximizer of $r(y)$ and $y^* = \tilde{y}$ from (B.21). Substituting in $q_1 = q$ and $q_2 = 1$ yields (22) in Proposition 6.

In the special case of Proposition 2 in Section 4, where $q_1 = 0$ and $q_2 = 1$, $\tilde{y}$ from (B.21) simplifies to y^* in (7). It can be easily verified that y^* in (7) lies in (0,1). First assume that $\alpha_1 x_2 > \alpha_2 x_1$. To verify $y^* > 0$, we just need to check the numerator in (7) is positive because the denominator is positive by $\alpha_1 x_2 > \alpha_2 x_1$:

$$y^* > 0 \tag{B.23}$$

$$\iff \sqrt{x_2(\alpha_1 + x_1)x_1(\alpha_2 + x_2)} > x_1(\alpha_2 + x_2) \tag{B.24}$$

Condition (B.24) holds when $\alpha_1 x_2 > \alpha_2 x_1$ because then $x_2(\alpha_1 + x_1) > x_1(\alpha_2 + x_2)$. To verify that $y^* < 1$, algebraic manipulation of (7) yields

$$y^* < 1 \tag{B.25}$$

$$\iff \sqrt{x_2(\alpha_1 + x_1)x_1(\alpha_2 + x_2)} < x_2(\alpha_1 + x_1) \tag{B.26}$$

Condition (B.26) holds when $\alpha_1 x_2 > \alpha_2 x_1$. We derive similar conditions to (B.23)–(B.26) when $\alpha_1 x_2 < \alpha_2 x_1$.

B.2.2 $a = 0$

In this case $d(y)$ is just a linear function

$$d(y) = by + c \tag{B.27}$$

Where b and c are defined in (B.19)–(B.20). The unique zero of $d(y)$ is :

$$\tilde{y} = \frac{c}{-b} = \frac{x_1(\alpha_2 + x_2)(\alpha_2(q_1 + q_2(1-q_1)) + q_1 x_2)}{2\alpha_2 x_1(\alpha_2 + x_2)(q_1 + q_2(1-q_1))} \tag{B.28}$$

$$= \frac{1}{2} + \frac{q_1 x_2}{2\alpha_2(q_1 + q_2(1-q_1))} \tag{B.29}$$

(B.29) corresponds to (23) in Section 7 (when $q_1 = q$ and $q_2 = 1$). Note $\tilde{y} \geq \frac{1}{2}$. The slope of the line in (B.27), b, is negative by (B.19). The intercept $d(0) = c$, is positive by (B.20). Hence $r(y)$ increases prior to $\tilde{y}$ and decreases after. If $\tilde{y} > 1$ then $y^* = 1$, otherwise $\tilde{y} \in [\frac{1}{2}, 1)$ is the local (and global) maximizer of $r(y)$ and $y^* = \tilde{y}$.

In the special case of Proposition 2, where $q_1 = 0$ and $q_2 = 1$, we have $\tilde{y} = \frac{1}{2}$ by inspection of (B.29). $a = 0$ in this case when $\frac{\alpha_1}{x_1} = \frac{\alpha_2}{x_2}$.

B.3 Game Theoretic Model

This section covers Proposition 3 and Proposition 7. The proof builds on the steps in Appendices B.1 and B.2. We show that we have a concave-convex game and verify the proposed solution is a saddle point by checking the KKT conditions (see Chapter 5.5 of [41]).

The game payoff is Red's win probability

$$g(x, y) \equiv P[\text{Red wins}] = \frac{\alpha_1 y}{\alpha_1 y + x}\left(q_1 + (1-q_1)\frac{\alpha_2(1-y)}{\alpha_2(1-y) + (1-x)}q_2\right) \tag{B.30}$$

$g(x, y)$ is a strictly convex function of x for fixed y. We showed this in Appendix B.1. Similarly, $g(x, y)$ is a strictly concave function of y for fixed x following the steps in Appendix B.2. We have a concave-convex game. We prove that solution (x^*, y^*) from (26) is a saddle point by verifying the KKT conditions. Since x and y are single variables and our proposed solution is in the interior, this simplifies to checking first order conditions: $g_x(x^*, y^*) = 0$ and $g_y(x^*, y^*) = 0$

We first examine $g_x(x^*, y^*)$. Appropriately modifying $b'(x)$ from (B.2)

$$\begin{aligned} g_x(x^*, y^*) = &-q_1\frac{\alpha_1 y^*}{(\alpha_1 y^* + x^*)^2} - \frac{\alpha_1 y^*}{(\alpha_1 y^* + x^*)^2}(1-q_1)\frac{\alpha_2(1-y^*)}{\alpha_2(1-y^*) + 1 - x^*}q_2 \\ &+ \frac{\alpha_1 y^*}{\alpha_1 y^* + x^*}(1-q_1)\frac{\alpha_2(1-y^*)}{(\alpha_2(1-y^*) + 1 - x^*)^2}q_2 \end{aligned} \tag{B.31}$$

Next we define β_i as the probability Red wins layer i given equal allocation for Red and Blue in layer i: $x_i = y_i$:

$$\beta_i = \frac{\alpha_i}{\alpha_i + 1} \tag{B.32}$$

Using β_i from (B.32) and noting that our proposed solution in Proposition 7 has $x^* = y^*$, we rewrite (B.31)

$$\begin{aligned} g_x(x^*, y^*) = -q_1\beta_1(1-\beta_1)\frac{1}{x^*} - (1-q_1)q_2\beta_1(1-\beta_1)\beta_2\frac{1}{x^*} \\ + (1-q_1)q_2\beta_1\beta_2(1-\beta_2)\frac{1}{1-x^*} \end{aligned} \tag{B.33}$$

Factoring out terms:

$$\begin{aligned} g_x(x^*, y^*) =& \beta_1\frac{1}{x^*(1-x^*)}\times \\ & (-q_1(1-\beta_1)(1-x^*) - (1-q_1)q_2(1-\beta_1)\beta_2(1-x^*) + (1-q_1)q_2\beta_2(1-\beta_2)x^*) \end{aligned} \tag{B.34}$$

x^* in (26) is defined such that second line in (B.34) is zero. To verify this, we set the second line in (B.34) to zero and solve for x^*

$$x^* = \frac{q_1(1-\beta_1) + (1-q_1)q_2(1-\beta_1)\beta_2}{q_1(1-\beta_1) + (1-q_1)q_2\beta_2(1-\beta_1+1-\beta_2)} \tag{B.35}$$

Substituting in the definition of β_i from (B.32) into (B.35) and simplifying terms yields (26) from Proposition 7 (when $q_1 = q$ and $q_2 = 1$).

We next need to verify that $g_y(x^*, y^*) = 0$. Going through the steps of modifying $r'(y)$ from (B.14), we end up with $g_y(x^*, y^*) = -g_x(x^*, y^*)$ because $x^* = y^*$. We omit the details here. Hence, $g_y(x^*, y^*) = g_x(x^*, y^*) = 0$ and (x^*, y^*) from (26) is a saddle point solution.

We show uniqueness by contradiction. We assume there exists some other solution $(\tilde{x}, \tilde{y}) \neq (x^*, y^*)$. The game value is unique and hence

$$g(\tilde{x}, \tilde{y}) = g(x^*, y^*) \tag{B.36}$$

Because $g(x, y)$ is strictly convex (concave) in x (y) for fixed y (x), $\tilde{x} \neq x^*$ and $\tilde{y} \neq y^*$. Consider the combination $(x^*, \tilde{y})$. Because $g(x, y)$ is strictly convex in x for fixed y, we have

$$g(\tilde{x}, \tilde{y}) < g(x^*, \tilde{y}) \tag{B.37}$$

By similar reasoning, since y^* is a best response to x^* we also have

$$g(x^*, \tilde{y}) < g(x^*, y^*) \tag{B.38}$$

Conditions (B.37)–(B.38) imply $g(\tilde{x}, \tilde{y}) < g(x^*, y^*)$, which contradicts (B.36). Similar logic reveals no mixed strategy solution exists because each side prefers playing its optimal pure strategy over mixing with other solutions that are inferior.

In the case of Proposition 3 of Section 5, where $q_1 = 0$ and $q_2 = 1$, (B.35) reduces to

$$x^* = \frac{1-\beta_1}{1-\beta_1+1-\beta_2} \tag{B.39}$$

which corresponds to (9) in Section 5 when we substitute for β_i from (B.32) into (B.39)

C N-layer Simultaneous Game

This section covers Proposition 8. The proof is similar to the proof in Appendix B.3: show that we have concave-convex game and verify the proposed solution is a saddle point by checking the KKT conditions.

The game payoff is Red's win probability

$$g(\mathbf{x}, \mathbf{y}) \equiv P[\text{Red wins}] = \prod_{i=1}^{N} \frac{\alpha_i y_i}{\alpha_i y_i + x_i} \tag{C.1}$$

Since the payoff is a product, we examine the log win probability

$$g_L(\mathbf{x}, \mathbf{y}) \equiv \log g(x, y) = \sum_{i=1}^{N} \log\left(\frac{\alpha_i y_i}{\alpha_i y_i + x_i}\right) \tag{C.2}$$

$g_L(\mathbf{x}, \mathbf{y})$ is a strictly convex function of $\mathbf{x}$ for fixed $\mathbf{y}$. This follows because $g_L(\mathbf{x}, \mathbf{y})$ is additive separable across the x_i and $g_L(\mathbf{x}, \mathbf{y})$ is strictly convex for each x_i:

$$\frac{\partial^2 g_L(\mathbf{x}, \mathbf{y})}{\partial x_i^2} = \frac{1}{(\alpha_i y_i + x_i)^2} > 0 \tag{C.3}$$

Similarly, $g_L(\mathbf{x}, \mathbf{y})$ is a concave function of $\mathbf{y}$ for fixed $\mathbf{x}$ as $g_L(\mathbf{x}, \mathbf{y})$ is concave for each y_i:

$$\frac{\partial^2 g_L(\mathbf{x}, \mathbf{y})}{\partial y_i^2} = -\frac{x_i(2\alpha_i y_i + x_i)}{y_i^2(\alpha_i y_i + x_i)^2} \leq 0 \tag{C.4}$$

We have a concave-convex game (see Section 5.2 of [39]), which implies the game has a saddle point. To prove that our solution $(\mathbf{x}^*, \mathbf{y}^*)$ from (29) is a saddle point we need to show that $\mathbf{x}^*$ is a minimizer of $g_L(\mathbf{x}, \mathbf{y}^*)$ and $\mathbf{y}^*$ is a maximizer of $g_L(\mathbf{x}^*, \mathbf{y})$. We proceed by verifying the KKT conditions (see Chapter 5.5 of [41]).

We start by showing $\mathbf{x}^*$ is a minimizer of $g_L(\mathbf{x}, \mathbf{y}^*)$. Since our proposed solution is in the interior, the KKT multipliers on the non-negative constraints are 0 by complementary slackness. The resource constraint ($\sum_{i=1}^{N} x_i \leq 1$) is tight, so the associated multiplier may be positive. We denote δ as the KKT multiplier associated with the resource constraint. Thus we need to verify that there exists a $\delta \geq 0$ that, in conjunction with our $(\mathbf{x}^*, \mathbf{y}^*)$ from (29), satisfies the first order KKT condition:

$$\left.\frac{\partial g_L(\mathbf{x}, \mathbf{y}^*)}{\partial x_i}\right|_{\mathbf{x}=\mathbf{x}^*} + \delta = 0 \quad \forall 1 \leq i \leq N \tag{C.5}$$

Substituting the derivative into (C.5) yields

$$-\frac{1}{\alpha_i y_i^* + x_i^*} + \delta = 0 \quad \forall 1 \leq i \leq N \tag{C.6}$$

Noting that $x_i^* = y_i^*$ transforms (C.6) into

$$\delta = \frac{1}{x_i^*}\frac{1}{\alpha_i + 1} \quad \forall 1 \leq i \leq N \tag{C.7}$$

Substituting in x_i^* from (29) into (C.7) yields the final expression for the KKT multiplier:

$$\delta = \sum_{j=1}^{N} \frac{1}{\alpha_j + 1} \tag{C.8}$$

Note that δ in (C.8) does not depend upon i, and $\delta \geq 0$. Hence the triple of $(\mathbf{x}^*, \mathbf{y}^*, \delta)$ defined by equations (29) and (C.8) satisfy the KKT conditions and therefore $\mathbf{x}^*$ is a best response to $\mathbf{y}^*$.

Similar logic shows that $\mathbf{y}^*$ is a maximizer of $g_L(\mathbf{x}^*, \mathbf{y})$. $(\mathbf{x}^*, \mathbf{y}^*)$ must satisfy the KKT condition:

$$-\left.\frac{\partial g_L(\mathbf{x}^*, \mathbf{y})}{\partial y_i}\right|_{\mathbf{y}=\mathbf{y}^*} + \gamma = 0 \quad \forall 1 \leq i \leq N \tag{C.9}$$

For some KKT multiplier $\gamma \geq 0$. Substituting the derivative into (C.9) yields

$$-\frac{\frac{x_i^*}{y_i^*}}{\alpha y_i^* + x_i^*} + \gamma = 0 \quad \forall 1 \leq i \leq N \tag{C.10}$$

Noting that $x_i^* = y_i^*$ simplifies (C.10)

$$\gamma = \frac{1}{x_i^*}\frac{1}{\alpha_i + 1} \quad \forall 1 \leq i \leq N \tag{C.11}$$

Expression (C.11) is the same as (C.7), and hence $\gamma = \delta$ and the triple $(\mathbf{x}^*, \mathbf{y}^*, \delta)$ defined by equations (29) and (C.8) satisfy the KKT conditions and hence $\mathbf{y}^*$ is a best response to $\mathbf{x}^*$. Therefore we have a saddle point solution: $(\mathbf{x}^*, \mathbf{y}^*)$.

Uniqueness follows by the same logic as in Appendix B.3