To Catch an Intruder: Part A—Uncluttered Scenario

Michael P. Atkinson, Moshe Kress, Roberto Szechtman

Operations Research Department, Naval Postgraduate School, Monterey, California 93943

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Abstract: We analyze an interdiction scenario where an interceptor attempts to catch an intruder as the intruder moves through the area of interest. A motivating example is the detection and interdiction of drug smuggling vessels in the Eastern Pacific and Caribbean. We study two models in this article. The first considers a nonstrategic target that moves through the area without taking evasive action to avoid the interdictor. We determine the optimal location the interceptor should position itself to best respond when a target arrives. The second model analyzes the strategic interaction between the interceptor and intruder using a Blotto approach. The intruder chooses a route to travel on and the interceptor chooses a route to patrol. We model the interaction as a two-player game with a bilinear payoff function. We compute the optimal strategy for both players and examine several extensions. 2017 Wiley Periodicals, Inc. Naval Research Logistics 64: 29–40, 2017

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1. INTRODUCTION

We study a situation where a defender attempts to intercept a moving intruder. This situation is motivated by the continuous effort of law-enforcement agencies to interdict drug-carrying vessels, which are sent from South America, in the Caribbean Sea and Eastern Pacific [1].

We consider two basic scenarios. In the first scenario, the defender executes a detection and interdiction operation. A search asset (e.g., sonobuoys or a surveillance aircraft) collects information in the area where the intruder may travel (e.g., certain sea-lanes in the ocean) and the *interceptor* (e.g., a surface ship) attempts to physically capture the intruder based on the information obtained from the search asset. We assume, in this scenario, that the intruder is nonstrategic—it neither affects the interceptor's actions nor responds to them. It simply schedules and routes its travel based on its own objectives and constraints. The defender only has some probabilistic information, based on past experience or exogenous sources of intelligence, regarding the schedule and route of the intruder's travel in the area of interest. This information is used for deploying the interceptor. The interceptor's assets, typically over the horizon boats or helicopters, are fast. However, these assets have limited endurance and are based on relatively slow ships such as Coast Guard Cutters or Navy Frigates. Therefore, properly staging the interceptor in the area crucially affects how long it takes to successfully interdict the intruder. In the first scenario, we seek to determine the optimal staging point for the interceptor.

In the second scenario, the intruder is strategic and it tries to "outsmart" the defender. To simplify the modeling and analysis—without sacrificing too much realism—we discretize the area and assume that the intruder moves in one of a finite number of routes. Similarly to scenario one, the interceptor has to determine where to locate its interceptor, but this time absent a detector. Both the intruder and the interceptor are moving and therefore it is not enough that the two routes just intersect—timing is crucial and thus interception could occur only if the intruder and interceptor select the same route. In that case, there is a positive, route-dependent, probability that the interceptor interdicts the intruder. Otherwise the intruder traverses the route undetected.

The search, detection and tracking problem of a moving target has been treated quite extensively in the literature [2–5]. In particular, the general problem of searching for a target that follows a conditionally deterministic target motion—a situation similar to the one considered in the first scenario—is discussed in detail in [6]. However, the focus of the modeling and analysis in the aforementioned references is on search and detection—not on the physical interdiction of the target. A detection-interdiction situation involving a moving target is studied in [7, 8]. In these papers, all intruders head toward the same known location, whereas in our first model a target traverses through the area of interest with an unknown

Correspondence to: Roberto Szechtman (rszechtm@nps.edu)

destination. Another detection-interdiction model appears in [9] where multiple targets, blended among benign objects, are searched and then interdicted. A crucial simplifying assumption in that model was that once detected, the target becomes stationary until it is interdicted. Time and space were discretized, and a stochastic optimization model dynamically determined which cell to search next. The exact model turned out to be intractable for practical scenarios and therefore a heuristic algorithm was proposed. The main changes in the first scenario studied here, compared to [9], are: (a) time and space are continuous, (b) the scenario is continuously dynamic, (c) the environmental setting is such that the detection and interdiction occur in a constrained region called a *channel* and the detection occurs at a horizontal position in the channel. Our primary contribution appears in Proposition 4, which specifies the optimal location to stage the interceptor so it can best-respond to an intruder.

The second, strategic, scenario is a variant of the well-known Blotto Game (see e.g., [10]). However, unlike the original Blotto model, routes are associated with different rewards to the intruder and different interception capabilities of the defender. The base model for the second scenario is a type of logistics game, which was first introduced in [12] and also appears in [11]. Our main result for this scenario is Proposition 5, which presents the optimal strategies for both players and the value of the game.

The rest of the article is organized as follows: in Section 2, we consider the nonstrategic case and in Section 3, we analyze the 1-on-1 game between the intruder and the defender. We present concluding remarks in Section 4. Proofs of propositions appear in the Appendix. In the rest of the article, we refer to the intruder as the *target*.

2. SCENARIO 1: NONSTRATEGIC TARGET

Consider the two-dimensional Euclidean space $\mathbb{R}^2$. A target, traveling on a straight line in the general positive direction of y, is detected at time 0 at location $(x, 0)$. The detection location, along with the direction of the movement, is passed on to an *interceptor*, which is tasked to physically meet the target and interdict it. We do not model the detection mechanism; it could occur, for example, via deployed sonobuoys or a patrol by a surveillance aircraft. The target travels at a constant heading α away from the y axis where $\alpha \in (-90°, +90°)$. The interceptor positions itself at a location (b, D), waiting for detection information. Upon receiving information regarding the location $(x, 0)$ and heading α of the target, the interceptor immediately begins its pursuit of the target. We assume that the target, unaware of being detected, does not take evasive actions but continues along the same heading. See Fig. 1 for the illustration of the scenario.

While the interceptor starts the pursuit knowing $(x, 0)$ and α, the decision regarding its staging location (b, D) must be made in advance, before knowing this information. The objective of the interceptor is to minimize the expected time until interdiction. The target travels at speed u and the interceptor travels at speed v. For simplicity, we assume that the interceptor travels faster than the target ($v > u$), which is realistic in many real-life law-enforcement applications (e.g., drug trafficking).

To determine when the interceptor will capture the target, we solve the following quadratic equation (see Fig. 1).

$$\sqrt{(x + ut\sin\alpha - b)^2 + (D - ut\cos\alpha)^2} = vt,$$

which has solution

$$t = \frac{-u(D\cos\alpha - (x-b)\sin\alpha) + \sqrt{u^2(D\cos\alpha - (x-b)\sin\alpha)^2 + (D^2 + (x-b)^2)(v^2 - u^2)}}{v^2 - u^2}.$$

If α and x are independent random variables having uniform distributions on $[-\overline{\alpha}, +\overline{\alpha}]$ and $[-W, +W]$, respectively, then the optimal horizontal location of the interceptor is uniquely determined, as shown in the following proposition.

PROPOSITION 1: If the target's detection location and heading are independent uniform random variables, each symmetrically distributed around 0, then the optimal horizontal location is at $b = 0$.

This result follows from the symmetry of the problem and its formal proof appears in Appendix A.1. The optimal vertical position D is determined by the shortest expected time to interception. This expected time $g(D)$ is given by

$$g(D) = \frac{1}{4W\overline{\alpha}} \int_{-W}^{W} \int_{-\overline{\alpha}}^{\overline{\alpha}} \frac{-u(D\cos\alpha - x\sin\alpha) + \sqrt{u^2(D\cos\alpha - x\sin\alpha)^2 + (D^2 + x^2)(v^2 - u^2)}}{v^2 - u^2} d\alpha\, dx, \quad (1)$$

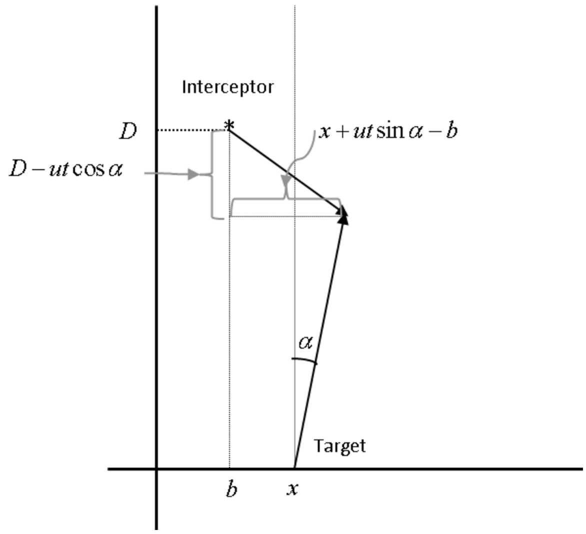

Figure 1. Interceptor located at position (b, D) pursues target who leaves from position $(x, 0)$ on heading α.

and thus the optimal vertical position is $D^* = \text{argmin}_D g(D)$. We assume that $W > 0$ because otherwise the interceptor knows with certainty the target location at detection and thus $D^* = 0$. Solving for D^* analytically is not possible, but a numerical solution is straightforward as $g(D)$ is convex, as stated in the next proposition, which is proved in Appendix A.2.

PROPOSITION 2: If the target's detection location and heading are independent uniform random variables, each symmetrically distributed around 0, then the expected interception time, $g(D)$, is convex in D.

Before turning to analyze the behavior of D^*, we present two propositions. The first states that the optimal position is never on the x-axis. The second proposition states that D^* scales linearly with W. We prove these in Appendices A.3 and A.4, respectively.

PROPOSITION 3: If the target's detection location and heading are independent uniform random variables, each symmetrically distributed around 0, then the optimal vertical position $D^* > 0$.

PROPOSITION 4: If the target's detection location and heading are independent uniform random variables, each symmetrically distributed around 0, then the optimal interceptor location is a linear function of W: $D^*(W, v, u, \overline{\alpha}) = \beta(v, u, \overline{\alpha})W$.

The exact form of $\beta(v, u, \overline{\alpha})$ in Proposition 4 is quite complicated; it is an implicit function and involves integration over α. It can be shown however that the optimal D^* depends on the relative speed. For details, see Appendix A.4.

Figure 2 plots the optimal position D^* as a (linear) function of W for various values of $\overline{\alpha}$ and v/u. The first general conclusion from observing the six graphs that comprise Fig. 2 is that the optimal staging position D^* gets closer to the detection point as the relative velocity of the interceptor increases compared to the target. This general trend is true for all values of W and $\overline{\alpha}$. However, it is interesting to note that $\overline{\alpha}$ has a significant impact on this relationship, as observed from the changes in the ordering of the curves for different values of $\overline{\alpha}$. For a relatively slow interceptor, the interceptor positions itself closer to the detection point for small $\overline{\alpha}$, and farther away for large $\overline{\alpha}$. This tendency is reversed for higher interceptor velocities. This non-monotone behavior is further demonstrated in Figs. 3 and 4. Note that the curves in Fig. 3 intersect for low velocity ratio v/u but then they uniformly merge for larger values of v/u. This non-monotonicity is well demonstrated in Fig. 4 where D^* is plotted against $\overline{\alpha}$ for various values of v/u.

To explain this counterintuitive non-monotonic behavior, it would be helpful to roughly categorize the interdiction process as either *approach* or *chase*. Figure 5 illustrates the approach and chase concepts. From a mathematical perspective, an approach scenario corresponds to a negative inner product between the target heading and interceptor heading, and the chase scenario corresponds to a positive inner product. In an approach scenario, the target and interceptor are generally heading toward each other and the time until interdiction scales roughly according to $D/(v+u)$. For the chase scenarios, the target heads away from the interceptor and the time until interdiction scales roughly according to $D/(v-u)$. The interceptor obviously prefers an approach scenario to a chase one. In order to enhance the probability of a desired approach scenario, the staging position D must be relatively far from the detection point. Thus, the interceptor faces a tradeoff; on the one hand it prefers a smaller D so that the initial distance to the target is smaller. Conversely, it would like to choose a large value of D to enhance the likelihood of an approach scenario. As $\overline{\alpha}$ increases from 0, the target may take more extreme headings, which increases the likelihood of chase scenarios. For a fast interceptor ($v \gg u$), the interdiction process will be quick, even when chasing, therefore a small D is preferred to minimize distance. For a smaller value of v, the chase situations can significantly increase the expected interdiction time (recall time scales $\sim D/(v-u)$). As $\overline{\alpha}$ increases, D^* must also increase to ensure most combinations of (x, α) lead to approach scenarios. Hence, we see that D^* rises rapidly for $v = 1.01$ in Fig. 4. However, the curve eventually reaches a maximum and decreases as $\overline{\alpha}$ approaches $\pi/2$. For targets with large value of α, the interdiction is likely to be a chase scenario regardless of the value of D as the target heads in nearly a horizontal heading. Therefore, the

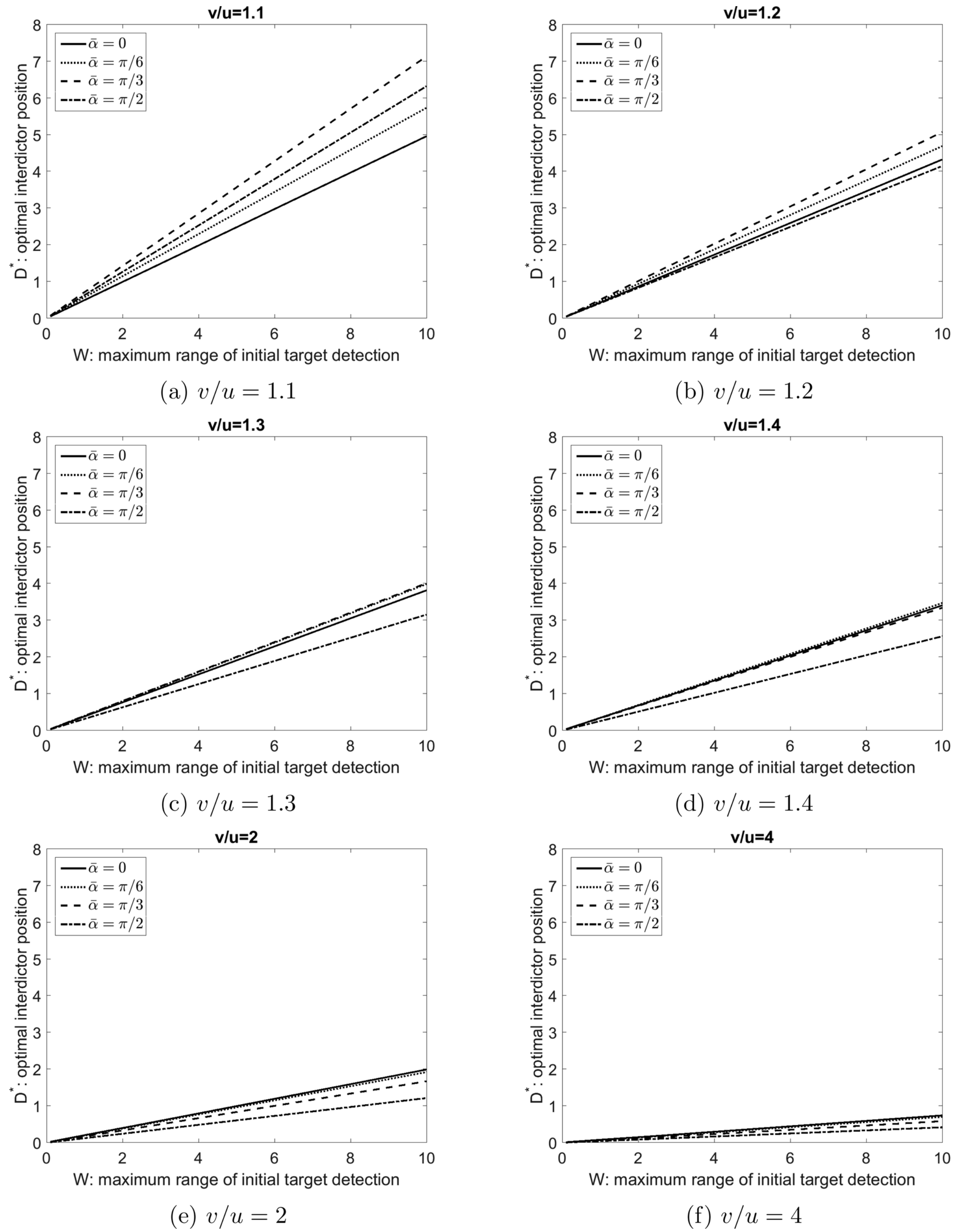

Figure 2. Optimal interdiction location as a function of the target support W and target heading $\overline{\alpha}$.

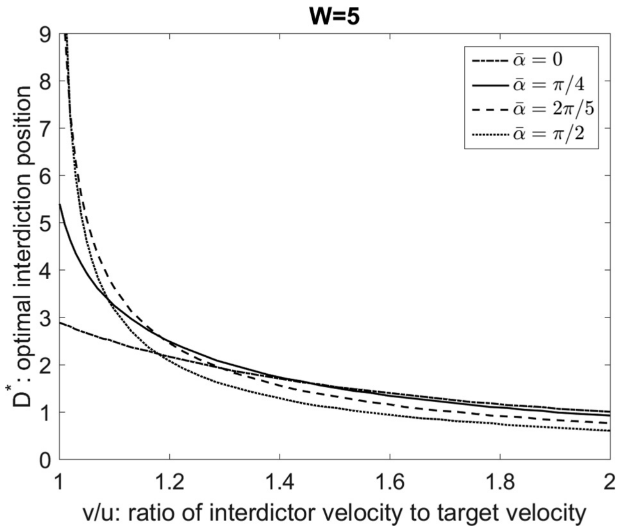

Figure 3. Optimal interdiction location as a function v/u for various target headings $\overline{\alpha}$ and $W = 5$.

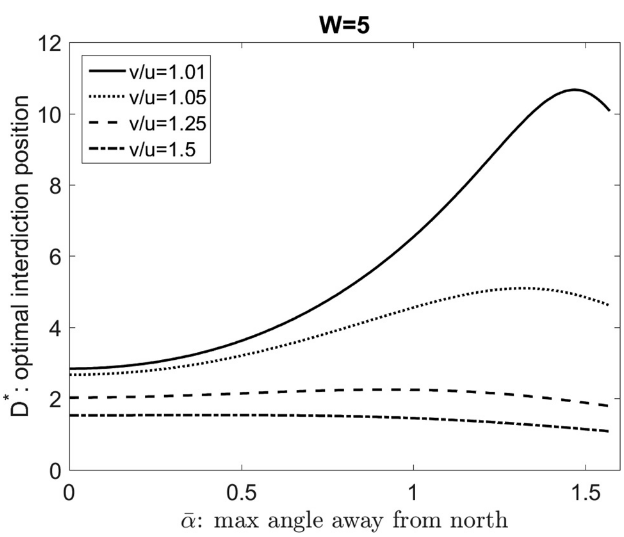

Figure 4. Optimal interdiction location as a function of $\overline{\alpha}$ for various interdictor velocities v and $W = 5$.

optimal location D^* decreases slightly for $\overline{\alpha}$ close to $\pi/2$ so the interceptor can better react to those types of targets.

3. SCENARIO 2: STRATEGIC TARGET

In this section, we consider a simple, Blotto-type, game-theoretic model. In a Blotto game, two players distribute their forces simultaneously across n sectors of a battlefield. In the simplest variant, each sector is won by the player that allocates the largest force level to the sector, and the overall payoff is the fraction of sectors won by a player. For more details on Blotto games, see [10]. The target and interceptor

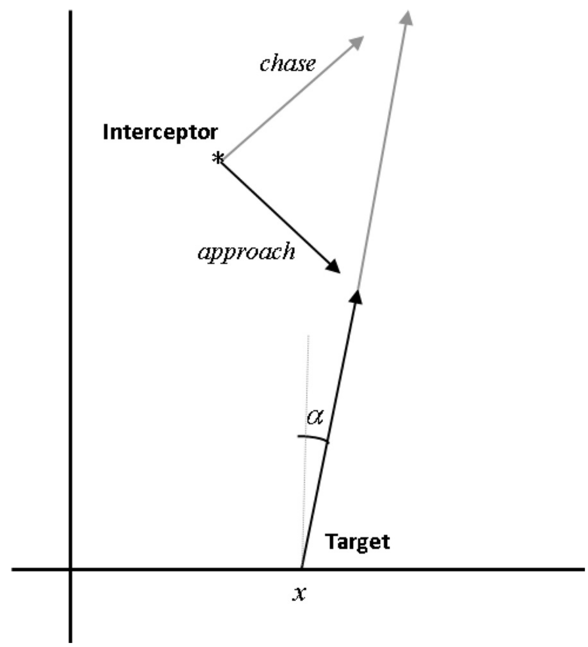

Figure 5. The interceptor *approaches* the target if the interceptor and target are roughly traveling toward each other, otherwise the interceptor *chases* the target.

are strategic and both attempt to optimize their objectives; the target wishes to safely complete its travel while the interceptor attempts to catch the target before it completes its travel. To simplify the modeling and analysis—without sacrificing too much realism—we discretize the area and assume that the target travels along one of a finite number of *routes* (say, sea lanes) known to the interceptor.

3.1. Basic Model

There are I routes available to the target. Each route i is associated with a reward r_i for the target, $i = 1, 2, \ldots, I$. The reward is the difference between the expected revenue at the destination and the costs of travel on that route—costs that are determined by the length of the route, its physical characteristics and its transportation requirements. The target chooses to travel on route i with probability p_i. The interceptor selects patrol route i with probability q_i and intercepts the target with probability c_i if the latter travels on the same route as the interceptor. The probabilities p_i and q_i, $i = 1, 2, \ldots, I$, are the decision variables, or the *strategies*, for the target and interceptor, respectively, $\sum_{i=1}^{I} p_i = \sum_{i=1}^{I} q_i = 1$.

The target wishes to maximize its expected reward, while the interceptor wishes to minimize it. The expected reward for the target, if it chooses route i, is $r_i(1 - c_i q_i)$.

We assume that both the intruder and the interceptor have information about past interceptions and therefore they know the values of c_i and $r_i, i = 1, 2, \ldots, I$. Given the strategy profile $\boldsymbol{q} = (q_1, q_2, \ldots, q_I)$ of the interceptor and $\boldsymbol{p} = (p_1, p_2, \ldots, p_I)$ of the target, the expected reward for the target is:

$$V(\boldsymbol{p}, \boldsymbol{q}) = \sum_{i=1}^{I} r_i p_i (1 - c_i q_i). \tag{2}$$

We assume a simultaneous interaction between the interceptor and target. While, as in any two-player zero-sum game, the problem may be formulated as primal and dual linear optimization problems, we adopt here a more direct approach that results in a closed-form analytic solution. The optimal strategies $\boldsymbol{p^*}$ and $\boldsymbol{q^*}$ satisfy the standard Nash equilibrium condition:

$$V(\boldsymbol{p^*}, \boldsymbol{q^*}) \geq V(\hat{\boldsymbol{p}}, \boldsymbol{q^*}), \quad \forall \hat{\boldsymbol{p}} \quad \text{s.t. } \hat{p}_i \geq 0, \text{ for } i = 1, 2, \ldots, I \text{ and } \sum_{i=1}^{I} \hat{p}_i = 1,$$

$$V(\boldsymbol{p^*}, \boldsymbol{q^*}) \leq V(\boldsymbol{p^*}, \hat{\boldsymbol{q}}), \quad \forall \hat{\boldsymbol{q}} \quad \text{s.t. } \hat{q}_i \geq 0, \text{ for } i = 1, 2, \ldots, I \text{ and } \sum_{i=1}^{I} \hat{q}_i = 1.$$

To solve for $\boldsymbol{p^*}$ and $\boldsymbol{q^*}$, we note the interaction can be modeled as a concave-convex game (see in Section 5 in [11]) as $V(\cdot, \cdot)$ in Eq. (2) is a concave function of $\boldsymbol{p}$ for every fixed $\boldsymbol{q}$, and a convex function of $\boldsymbol{q}$ for every fixed $\boldsymbol{p}$. Concave-convex games always have a saddle point [11] so we can determine the simultaneous game solution by considering the two sequential game variants.

As with many games, our solution has both sides choosing a strategy such that the opponent is indifferent among its own options. Before stating the formal solution, we label the routes in descending order of r_i and introduce new variables θ and k_l. Suppose $r_1(1 - c_1) \geq r_2$ then obviously both the target and interceptor will only focus on route 1 because this is the best expected reward for the target and best hedging for the interceptor. In this case, $k_1 = r_1(1 - c_1)$ and $\theta = 1$. Otherwise, the interceptor chooses q_1 and q_2, $q_1 + q_2 = 1$, such that $r_1(1 - q_1 c_1) = r_2(1 - q_2 c_2) \equiv k_2$. If $k_2 \geq r_3$, then $\theta = 2$, $q_i^* = q_i$, $i = 1, 2$, and $q_i^* = 0$, $i = 3, 4, \ldots, I$. If $k_2 < r_3$, the interceptor chooses q_1, q_2 and q_3, $q_1 + q_2 + q_3 = 1$, and so on. Thus, the parameter k_l is the expected reward to the target along routes 1 through l when the interceptor chooses $\boldsymbol{q}$ according to this equalization procedure across the first l routes. The parameter θ is an index specifying the set of routes $1, 2, \ldots, \theta$ the target and interceptor should assign positive probability to in the optimal solution. In general, we have:

$$k_l = \frac{-1 + \sum_{i=1}^{l} c_i^{-1}}{\sum_{i=1}^{l} (c_i r_i)^{-1}}, \quad l = 1, 2, \ldots, I.$$

$$\theta = \begin{cases} I & \text{if } k_{I-1} < r_I \\ \min\{i \,:\, k_i \geq r_{i+1}\} & \text{otherwise,} \end{cases}$$

We now state the main result of this section, which specifies the optimal strategies for both players and the value of the game. The proof appears in Appendix B.1.

PROPOSITION 5: The interceptor's optimal strategy $\boldsymbol{q}^*$ is given by

$$q_i^* = \begin{cases} \frac{1}{c_i}\left(1 - \frac{k_\theta}{r_i}\right) & \text{if } i = 1, 2, \ldots, \theta \\ 0 & \text{if } i = \theta + 1, \theta + 2, \ldots, I. \end{cases} \tag{3}$$

The target's optimal strategy $\boldsymbol{p}^*$ is given by

$$p_i^* = \begin{cases} \frac{(r_i c_i)^{-1}}{\sum_{j=1}^{\theta} (r_j c_j)^{-1}} & \text{if } i = 1, 2, \ldots, \theta \\ 0 & \text{if } i = \theta + 1, \theta + 2, \ldots, I. \end{cases} \tag{4}$$

The expected reward for the target, and value of the game, equals k_θ.

Routes that do not yield a (relatively) large reward to the target receive no effort from the interceptor or target. Large changes in r_i can change θ and hence the final strategies substantially. If we only consider small changes in r_i and c_i so that θ remains fixed, then Eqs. (3) and (4) provide insight into the solution. As the probability of interdiction c_i increases, the target decreases its probability of taking that route. As a result the interceptor also decreases its probability of using the route, even though the interceptor is more effective on the route. An increase in the reward to the target r_i causes the interceptor to increase its probability of selecting that route to deny the target the larger reward, which causes the target to decrease its probability of using the more lucrative route.

We specialize the above result along three lines. First, we find that for r_i roughly constant across $i = 1, 2, \ldots, I$, both the interceptor and target are more likely to choose routes with small values of c_i, which leads to a significant advantage for the target. In the case where $r_i = 1$ for all i, the interceptor and target have the same optimal strategy: $p_i^* = q_i^* = c_i^{-1} / \sum_{j=1}^{I} c_j^{-1}$ for $i = 1, 2, \ldots, I$. The value of the game is easily seen to be

$$k_I = \left(1 - \frac{1}{\sum_{j=1}^{I} c_j^{-1}}\right), \tag{5}$$

which equals the expected reward for the target.

Second, if instead the c_i are roughly constant, then the interceptor and target have differing strategies. The target puts greater weight on routes with lower rewards, and the interceptor puts more weight on routes with higher rewards. In these situations, the interceptor effectively concedes the lower reward routes to the target and focuses its efforts on

Table 1. Expected reward for target: 2.26

	1	2	3	4
r_i	2.5	2.5	2.5	2.5
c_i	0.2	0.4	0.6	0.8
p_i^*	0.48	0.24	0.16	0.12
q_i^*	0.48	0.24	0.16	0.12

Table 2. Expected reward for target: 2.57

	1	2	3	4
r_i	4	3	2	1
c_i	0.5	0.5	0.5	0.5
p_i^*	0.43	0.57	0	0
q_i^*	0.71	0.29	0	0

Table 3. Expected reward for target: 3.2

	1	2	3	4
r_i	4	3	2	1
c_i	0.2	0.4	0.6	0.8
p_i^*	1	0	0	0
q_i^*	1	0	0	0

protecting the higher reward routes to ensure the target has little incentive to use one of the higher reward routes. In the case where we assume $c_i = c$, with $0 \leq c \leq 1$ for all i, the likelihood the interceptor patrols a patrolled route is proportional to the relative decrease in the reward for the target: $(r_i - k_\theta)/r_i$.

Finally using similar logic, if route j has $c_j \approx 0$, then the target will use this route with high probability and the expected reward will be $k_\theta \approx r_j$. The interceptor will patrol route j with very low probability, both because it has little chance of interdicting the target on route j and because it instead patrols higher reward routes to ensure the target has no incentive to use those routes and potentially generate larger rewards. In the special case where $r_i = 1$ and $c_i = 1$ for all i, the interdictor and target both choose routes uniformly and the expected reward is simply $(1 - 1/I)$, which is the probability that the target and interceptor choose different routes.

We finish with a few numerical illustrations. In all examples, we use four routes and set $\sum_{i=1}^{4} r_i = 10$ and $\sum_{i=1}^{4} c_i = 2$ to provide some basis for comparison. In the case that $r_i = 2.5$ and $c_i = 0.5$ for all i, both the target and interceptor choose a route uniformly at random ($p_i^* = q_i^* = 0.25$ for all i) and the expected reward for the target is $r_i(1 - c_i/I) = 2.5(1 - 0.5/4) = 2.1875$. Tables 1–3 present the solution for three other examples. In Table 1, the routes are homogeneous with respect to the reward r_i, but it is much easier for the target to avoid detection on the lower indexed routes. Consequently, both players use the same mixed strategy of putting more weight on the routes with smaller interdiction probabilities. The rewards vary across the routes in Tables 2 and 3, which lead the interceptor and target to ignore certain routes. In the extreme case where the interception probability is low for the high reward route, both sides choose route 1 with certainty (Table 3). Table 2 illustrates a constant c_i scenario described above, which leads to asymmetric strategies: the target puts higher weight on the lower reward route and the interceptor puts higher weight on the higher reward route.

3.2. Extensions

We present two possible extensions to the basic one-on-one game. The first extension addresses the situation where the defender operates more than one interceptor. The second extension analyzes the case where both the target and the interceptor can affect the interception probabilities values $c_i, i = 1, 2, \ldots, I$ by investing resources in stealth and sensors, respectively. In this section, we set the rewards obtained by the target equal to 1 for all routes (i.e., $r_i = 1$ for all i).

3.2.1. Multiple Interceptors with Time-Dependent Effectiveness

Suppose that the defender can deploy several independent and homogeneous interceptors in such a way that it can control the length of time each route is patrolled by an interceptor. The probability of interception depends on these times. Let t_i denote the total patrol time in route i and $\sum_{i=1}^{I} t_i \leq T$.

We adopt the principle of a *random search* model where interception occurs, conditioned on the target choosing route i, with probability $1 - \exp(-\gamma_i t_i)$. The parameter γ_i is a measure of the interception effectiveness on route i. The random search model is a common model used in search theory both for its conservative assumptions (i.e., a real searcher should be able to perform better than randomly searching the area) and its analytic tractability. The exponential structure of the probability comes from assuming a Bernoulli detection event in any time interval, and then sending the interval and corresponding Bernoulli probability to 0. For more information on the random search model, see chapter 2 of [5].

The interceptor chooses $\boldsymbol{t} = (t_1, t_2, \ldots, t_I)$ and the target chooses to travel on route i with probability p_i, and we represent it using the same notation as in the base model $\boldsymbol{p} = (p_1, p_2, \ldots, p_I)$. This leads to the following reward function for the target:

$$V(\boldsymbol{p}, \boldsymbol{t}) = \sum_{i=1}^{I} p_i \exp(-\gamma_i t_i),$$

This formulation is a variant of a logistics game introduced in [12] and examined in chapter 5.3 of [11]. As in the base model, we have a concave-convex game and thus we can determine the optimal strategies by examining sequential interactions. In the min-max game, the interceptor sets $t_i^* = T\gamma_i^{-1}/\sum_{j=1}^{I}\gamma_j^{-1}$, to ensure $\gamma_k t_k = \gamma_j t_j$ for all $1 \le j,k \le I$. The min-max value is $\exp(-T/\sum_{j=1}^{I}\gamma_j^{-1})$. For the max-min game, given a $\boldsymbol{p}$, the interceptor chooses t_i to equalize the derivatives of the summands: $p_k\gamma_k\exp(-\gamma_k t_k) = p_j\gamma_j\exp(-\gamma_j t_j)$ for all *j, k*. Inspection reveals that setting $p_i^* = \gamma_i^{-1}/\sum_{j=1}^{I}\gamma_j^{-1}$ will produce a response by the interceptor of t_i^* and yield a payoff equal to the min-max value, which implies that this allocation must be the optimal strategy for the target. Consequently, we get the same type of solution as for the basic model.

3.2.2. *Affecting Interception Probabilities*

The target and interceptor may affect the probabilities c_i by investing resources in the routes. The target invests in protection and camouflage, and the interceptor invests in detection and interception capabilities such as sensors and surveillance platforms. Let $c_i(x_i, y_i)$ denote the interception probability if the target and interceptor invest x_i and y_i in route *i*, respectively, where $\sum_{i=1}^{I} x_i \le x_{\max}$, $\sum_{i=1}^{I} y_i \le y_{\max}$. The parameters $x_{\max}$, $y_{\max}$ are the total resources available for the intruder and interceptor, respectively. We assume that $c_i(\cdot,\cdot) \in [0,1]$ is continuously differentiable, decreasing function of x_i and increasing function of y_i.

Both the target and interceptor are faced with a two-stage decision: (a) a strategic decision where to invest their resources, and (b) an operational decision where to act once resources are in place. For example, the interceptor might invest in better sensors and faster interdiction vessels in the first stage, while the target could invest in quieter engines, lower profile vessels, and enhanced communication technology. We assume that each side observes the strategic resource allocation of the other in the first stage, and then the second stage proceeds in the same simultaneous fashion as in the basic model of Section 3.1. If we denote the target's first-stage allocation $\boldsymbol{x} = (x_1, x_2, \ldots, x_I)$ and the interceptor's first-stage allocation $\boldsymbol{y} = (y_1, y_2, \ldots, y_I)$, then the value function for the target in this two stage game follows from Eq. (5),

$$W(\boldsymbol{x},\boldsymbol{y}) = 1 - \frac{1}{\sum_{k=1}^{I}\frac{1}{c_k(x_k,y_k)}}.$$

The interceptor chooses its strategic allocation $\boldsymbol{y}$ first. The target observes it and decides on its own first-stage allocation $\boldsymbol{x}$, which is then observed by the interceptor. With this perfect information regarding the first-stage strategic investments, both sides simultaneously choose their routes in the second stage. It is reasonable to assume that the interceptor's first-stage strategic plan will be known by the target before the target makes its first-stage allocation as the interceptor's plan involves more effort and resources to implement than the target's corresponding plan, which is typically less elaborate. It is also the worst case scenario for the interceptor. The assumption that the interceptor knows the target's first-stage allocation before the interceptor makes its operational second-stage decision is also reasonable. For example, it will not take the interceptor long to determine that the target is using stealthier or faster vessels along certain routes. To compute the first-stage resource allocation, we solve $\min_{\boldsymbol{y}}\max_{\boldsymbol{x}} W(\boldsymbol{x},\boldsymbol{y})$, which is equivalent to

$$\begin{aligned} \min_{y}\ \max_{x}\quad & \sum_{k=1}^{I}\frac{1}{c_k(x_k,y_k)} \\ \text{s.t.}\quad & \sum_{k=1}^{I} x_k \le x_{\max} \\ & \sum_{k=1}^{I} y_k \le y_{\max} \\ & x_k, y_k \ge 0 \quad \forall k = 1,2,\ldots,I \end{aligned}$$

Proposition 6 specifies the targets' optimal allocation strategy under certain restrictions on the structure of $c_i(\cdot,\cdot)$. We show that the target takes an all-or-nothing approach by allocating all $x_{\max}$ to one route.

PROPOSITION 6: If $1/c_i(\cdot, y_i)$ is increasing and convex for all y_i then the optimal strategy for the target is to invest all its resources in one route. Specifically, the target's optimal response to allocation $\boldsymbol{y}$ of the interceptor is to allocate all $x_{\max}$ resources to route $j^*(\boldsymbol{y})$, where

$$j^*(\boldsymbol{y}) = \arg\max_j U_j(\boldsymbol{y}),$$

and

$$U_j(\boldsymbol{y}) = \frac{1}{c_j(x_{\max}, y_j)} + \sum_{k\ne j}\frac{1}{c_k(0,y_k)}. \tag{6}$$

The proof of Proposition 6 appears in Appendix B.2. The assumption in Proposition 6 that $1/c_i(\cdot, y_i)$ is convex allows for many reasonable possibilities for $c_i(\cdot,\cdot)$. One example is $c_i(x_i, y_i) = \alpha_i(y_i)e^{-\beta_i(x_i,y_i)}$ for $0 \le \alpha_i(\cdot) \le 1$ and $\beta_i(\cdot, y_i)$ a non-negative and convex function for all y_i.

To determine the interceptor's allocation, we solve

$$\boldsymbol{y}^* = \min_{y}\left(\max_j U_j(\boldsymbol{y})\right).$$

While the target's optimal allocation has a nice all-or-nothing form, there is in general no intuitive characterization of the interceptor's allocation.

4. CONCLUSION AND FUTURE RESEARCH

In this article, we present two models, which differ in the way the defender views the intruder. In the first model, the intruder is assumed to be nonstrategic and the defender has to decide where to deploy its intercepting force in a continuous area of interest, given some knowledge regarding the tactical boundaries of the intruder. In the second model, the intruder is strategic and the defender has the additional task of "outsmarting" the intruder. The defender's best practices are identified and analyzed for both cases, including some extensions.

This paper is the first part of a two-part paper addressing an intruder-defender problem. In this article, we assume a "sterile" environment where the only participants in the scenario are the intruder (target) and the defender (interceptor). In some situations, such as insurgencies or drug trafficking, the intruder may attempt to reduce its signature by blending in a crowd of similarly looking objects (e.g., people, vehicles, boats). The potential effect of blending affects the strategies of both the intruder and defender, as will be discussed in a later companion paper.

APPENDIX A: PROOFS FOR SECTION 2 PROPOSITIONS

The Appendix contains the proofs of the propositions and other supplementary information. Appendix A contains material related to Section 2.

A.1. Interceptor Position at the Center of the Area

Given the symmetry of the target's movement, it is intuitive that the optimal horizontal position lies at $b=0$. We now prove this rigorously and thus show that the defender needs to optimize only one parameter—the vertical location D. We show that, for any value of D, if the interceptor is located at $(b, D), b > 0$ then it can reduce the expected interdiction time by moving to $(0, D)$.

The expected time to capture the target when the interceptor starts at position (b, D) is a constant multiplication (with respect to b) of

$$\int_{-W}^{W}\int_{-\overline{\alpha}}^{\overline{\alpha}} -(D\cos\alpha - (x-b)\sin\alpha) + \sqrt{(D\cos\alpha - (x-b)\sin\alpha)^2 + (D^2 + (x-b)^2)(v^2/u^2 - 1)}\,d\alpha\, dx.$$

The terms outside the square root do not affect the optimal value of b because the sin term integrates to 0. This leaves us to consider

$$\int_{-W}^{W}\int_{-\overline{\alpha}}^{\overline{\alpha}} \times \sqrt{(D\cos\alpha - (x-b)\sin\alpha)^2 + (D^2 + (x-b)^2)(v^2/u^2 - 1)}\,d\alpha\, dx. \tag{7}$$

We next define two functions related to the term under the square root in Eq. (7):

$$f_1(x,\alpha,b,D) = (D\cos\alpha - (x+b)\sin\alpha)^2 + (D^2 + (x+b)^2)(v^2/u^2 - 1)$$

$$f_2(x,\alpha,b,D) = (D\cos\alpha + (x+b)\sin\alpha)^2 + (D^2 + (x+b)^2)(v^2/u^2 - 1).$$

We can express the term under the square root in Eq. (7) as $f_1(x, \alpha, -b, D)$. To complete the proof, we will show that

$$\int_{-W}^{W}\int_{-\overline{\alpha}}^{\overline{\alpha}} \sqrt{f_1(x,\alpha,-b,D)}\,d\alpha\, dx \geq \int_{-W}^{W}\int_{-\overline{\alpha}}^{\overline{\alpha}} \sqrt{f_1(x,\alpha,0,D)},\, d\alpha\, dx, \tag{8}$$

which implies that the expected time to capture the target is smaller starting at position $(0, D)$ than it is from position (b, D).

Breaking up the left-hand-side of Eq. (8) into the sum of four integrals over separate quadrants of $[-W, W] \times [-\overline{\alpha}, \overline{\alpha}]$, and then changing variables such that the limits of integration are all $[0, W] \times [0, \overline{\alpha}]$ yields

$$\begin{aligned}\int_{-W}^{W}\int_{-\overline{\alpha}}^{\overline{\alpha}} \sqrt{f_1(x,\alpha,-b,D)}\,d\alpha\, dx &= \int_{0}^{W}\int_{0}^{\overline{\alpha}} \sqrt{f_1(x,\alpha,-b,D)}\,d\alpha\, dx + \int_{0}^{W}\int_{0}^{\overline{\alpha}} \sqrt{f_2(x,\alpha,-b,D)}\,d\alpha\, dx \\ &+ \int_{0}^{W}\int_{0}^{\overline{\alpha}} \sqrt{f_2(x,\alpha,b,D)}\,d\alpha\, dx + \int_{0}^{W}\int_{0}^{\overline{\alpha}} \sqrt{f_1(x,\alpha,b,D)}\,d\alpha\, dx.\end{aligned} \tag{9}$$

We now group the f_1 terms and f_2 terms, respectively:

$$\begin{aligned}\int_{-W}^{W}\int_{-\overline{\alpha}}^{\overline{\alpha}} \sqrt{f_1(x,\alpha,-b,D)}\,d\alpha\, dx &= \int_{0}^{W}\int_{0}^{\overline{\alpha}} \sqrt{f_1(x,\alpha,-b,D)} + \sqrt{f_1(x,\alpha,b,D)}\,d\alpha\, dx \\ &+ \int_{0}^{W}\int_{0}^{\overline{\alpha}} \sqrt{f_2(x,\alpha,-b,D)} + \sqrt{f_2(x,\alpha,b,D)}\,d\alpha\, dx.\end{aligned}$$

To finish the proof, we show that $\sqrt{f_1}$ and $\sqrt{f_2}$ are convex in b. This convexity implies that

$$\begin{aligned}\int_{-W}^{W}\int_{-\overline{\alpha}}^{\overline{\alpha}} \sqrt{f_1(x,\alpha,-b,D)}\,d\alpha\, dx &\geq \int_{0}^{W}\int_{0}^{\overline{\alpha}} 2\sqrt{f_1(x,\alpha,0,D)}\,d\alpha\, dx \\ &+ \int_{0}^{W}\int_{0}^{\overline{\alpha}} 2\sqrt{f_2(x,\alpha,0,D)}\,d\alpha\, dx\end{aligned} \tag{10}$$

By a similar change of variable manipulation used to produce Eq. (9), we can show that the right-hand-side of Eq. (10) is equivalent to the right-hand-side of Eq. (8). This implies that the condition in Eq. (8) holds, which completes the proof.

First, we show that the second derivative of $\sqrt{f_1}$ is positive. Performing the differentiation, we see that the second derivative is positive if $2f_1 > \frac{(f_1')^2}{f_1''}$. Below are the first and second derivatives of f_1 with respect to b

$$f_1'(x,\alpha,b,D) = -2\sin\alpha(D\cos\alpha - (x+b)\sin\alpha) + 2(x+b)(v^2/u^2 - 1)$$

$$f_1''(x,\alpha,b,D) = 2\sin^2\alpha + 2(v^2/u^2 - 1)$$

Thus, to show convexity, we must show that

$$(D\cos\alpha - (x+b)\sin\alpha)^2 + (D^2 + (x+b)^2)(v^2/u^2 - 1) > \frac{(-\sin\alpha(D\cos\alpha - (x+b)\sin\alpha) + (x+b)(v^2/u^2 - 1))^2}{\sin^2\alpha + (v^2/u^2 - 1)}. \tag{11}$$

We sketch out the details below for why this inequality holds. First

$$D^2\cos^2\alpha + D^2(\sin^2\alpha + (v^2/u^2 - 1)) > 0$$

We next add $-2(x+b)(D\cos\alpha - (x+b)\sin\alpha)\sin\alpha$ to both sides and then multiply both sides by $(v^2/u^2 - 1)$. This produces the following inequality

$$
\begin{aligned}
&(D\cos\alpha-(x+b)\sin\alpha)^2(v^2/u^2-1)+(x+b)^2(v^2/u^2-1)\sin^2\alpha\\
&\quad+D^2(v^2/u^2-1)(\sin^2\alpha+(v^2/u^2-1))\\
&\quad>-2(x+b)(v^2/u^2-1)(D\cos\alpha-(x+b)\sin\alpha)\sin\alpha
\end{aligned}
$$

Next add $(D\cos\alpha-(x+b)\sin\alpha)^2\sin^2\alpha+(x+b)^2(v^2/u^2-1)^2$ to both sides

$$
\begin{aligned}
&((D\cos\alpha-(x+b)\sin\alpha)^2\\
&\quad+(D^2+(x+b)^2)(v^2/u^2-1))(\sin^2\alpha+(v^2/u^2-1))\\
&\quad>(-\sin\alpha(D\cos\alpha-(x+b)\sin\alpha)+(x+b)(v^2/u^2-1))^2
\end{aligned}
$$

Dividing both sides by $\sin^2\alpha+(v^2/u^2-1)$ yields Eq. (11). A slight modification of the above steps reveals that $\sqrt{f_2}$ is also convex with respect to b.

A.2. Expected Interdiction Time is Convex in *D*

The steps to prove convexity are nearly identical to the convexity arguments in Appendix A.1. We thus only provide a roadmap to the proof here and omit most of the details, which are very similar to the steps in in Appendix A.1. Define:

$$
\tilde{g}(D)=\int_{-W}^{W}\int_{-\overline{\alpha}}^{\overline{\alpha}}\sqrt{(D\cos\alpha-x\sin\alpha)^2+(D^2+x^2)(v^2/u^2-1)}d\alpha\,dx. \tag{12}
$$

It is easily seen that if $\tilde{g}(D)$ is convex then so is $g(D)$. Also note that $\tilde{g}(D)$ depends on v/u. As in Appendix A.1, we define two functions related to the term under the square root

$$
f_1(x,\alpha,D)=(D\cos\alpha-x\sin\alpha)^2+(D^2+x^2)(v^2/u^2-1) \tag{13}
$$

$$
f_2(x,\alpha,D)=(D\cos\alpha+x\sin\alpha)^2+(D^2+x^2)(v^2/u^2-1). \tag{14}
$$

We now write $\tilde{g}(D)$ in terms of $f_1(x,\alpha,D)$ and $f_2(x,\alpha,D)$ so that the limits are all $[0,W]\times[0,\overline{\alpha}]$

$$
\tilde{g}(D)=2\int_0^W\int_0^{\overline{\alpha}}\sqrt{f_1(x,\alpha,D)}+\sqrt{f_2(x,\alpha,D)}d\alpha\,dx. \tag{15}
$$

To show that $\tilde{g}(D)$ is convex, it suffices to show that both $\sqrt{f_1(x,\alpha,D)}$ and $\sqrt{f_2(x,\alpha,D)}$ are convex in D. As in Appendix A.1, one can go through the brute force differentiation to show that the 2nd derivatives of $\sqrt{f_1}$ and $\sqrt{f_2}$ are positive. We omit the details here.

A.3. Proof that $D^* > 0$

To prove that the optimal D^* is positive, we will show that $g'(0)<0$. We first write $g(D)$ in terms of $\tilde{g}(D)$ introduced in Eq. (12) of Appendix A.2:

$$
g(D)=-\frac{uD\sin\overline{\alpha}}{\overline{\alpha}(v^2-u^2)}+\frac{u}{4W\overline{\alpha}(v^2-u^2)}\tilde{g}(D). \tag{16}
$$

We next show $\tilde{g}'(0)=0$, which implies $g'(0)=-\frac{u\sin\overline{\alpha}}{\overline{\alpha}(v^2-u^2)}<0$ and completes the proof. In Appendix A.2, we write $\tilde{g}(D)$ in terms of two auxiliary functions f_1 and f_2. See Eqs. (13)–(15) for the relationship between $\tilde{g}(D)$, f_1, and f_2.

To differentiate $\tilde{g}(D)$, we interchange derivatives and integrals because $\sqrt{f_1(x,\alpha,D)}$ and $\sqrt{f_2(x,\alpha,D)}$ are continuous as are their partial derivatives with respect to D. Differentiating inside the integral:

$$
\begin{aligned}
&\frac{d}{dD}(\sqrt{f_1(x,\alpha,D)}+\sqrt{f_2(x,\alpha,D)})\\
&\quad=\frac{\cos\alpha(D\cos\alpha-x\sin\alpha)+D(v^2/u^2-1)}{\sqrt{f_1(x,\alpha,D)}}\\
&\qquad+\frac{\cos\alpha(D\cos\alpha+x\sin\alpha)+D(v^2/u^2-1)}{\sqrt{f_2(x,\alpha,D)}}
\end{aligned}
$$

Evaluating the right hand side at $D=0$ yields

$$
\frac{-x\cos\alpha\sin\alpha}{\sqrt{f_1(x,\alpha,0)}}+\frac{x\cos\alpha\sin\alpha}{\sqrt{f_2(x,\alpha,0)}}
$$

However, $f_1(x,\alpha,0)=f_2(x,\alpha,0)$ and the above simplifies to 0. Consequently, $\tilde{g}'(0)=0$, $g'(0)<0$, and $D^*>0$.

A.4. Optimal Interceptor Location is a Linear Function of *W*

By Propositions 2 and 3, we know there is a unique minimizer D^* of $g(D)$, such that $D^*>0$. To solve for D^*, it suffices to find the D^* such that $g'(D^*)=0$. Using the representation from Eq. (16) with $\tilde{g}(D)$ (see Eq. (12)), we write $g'(D)$

$$
g'(D)=-\frac{u\sin\overline{\alpha}}{\overline{\alpha}(v^2-u^2)}+\frac{u}{4W\overline{\alpha}(v^2-u^2)}\tilde{g}'(D).
$$

Thus, D^* satisfies

$$
\tilde{g}'(D^*)=4W\sin\overline{\alpha}.
$$

As discussed in Appendix A.3, we can interchange differentiation and integration to compute $\tilde{g}'(D)$

$$
\tilde{g}'(D)=\int_{-\overline{\alpha}}^{\overline{\alpha}}\int_{-W}^{W}\frac{(D\cos\alpha-x\sin\alpha)\cos\alpha+D(v^2/u^2-1)}{\sqrt{(D\cos\alpha-x\sin\alpha)^2+(D^2+x^2)(v^2/u^2-1)}}dxd\alpha.
$$

With the help of *Wolfram's Mathematica Integrator* [13] we integrate out the inner integral with respect to x:

$$
\begin{aligned}
\tilde{g}'(D)=&\int_{-\alpha}^{\alpha}\frac{(v^2/u^2-1)(v^2/u^2)D}{((v^2/u^2-1)+\sin^2\alpha)^{3/2}}\\
&\times\log\left(\frac{W(v^2/u^2-1)+W\sin^2\alpha-D\cos\alpha\sin\alpha+\sqrt{((v^2/u^2-1)+\sin^2\alpha)((D\cos\alpha-W\sin\alpha)^2+(v^2/u^2-1)(D^2+W^2))}}{-W(v^2/u^2-1)-W\sin^2\alpha-D\cos\alpha\sin\alpha+\sqrt{((v^2/u^2-1)+W\sin^2\alpha)((D\cos\alpha+\sin\alpha)^2+(v^2/u^2-1)(D^2+W^2))}}\right)\\
&+\frac{\cos\alpha\sin\alpha}{(v^2/u^2-1)+\sin^2\alpha}\times\left(\sqrt{(D\cos\alpha+W\sin\alpha)^2+(v^2/u^2-1)(D^2+W^2)}-\sqrt{(D\cos\alpha-W\sin\alpha)^2+(v^2/u^2-1)(D^2+W^2)}\right)d\alpha.
\end{aligned} \tag{17}
$$

Next, we write $D^* = \beta(W, v, u, \overline{\alpha})W$. When we substitute this form of D^* into Eq. (17) to solve $\tilde{g}'(D^*) = 4W \sin\overline{\alpha}$, a factor of W cancels out. This leaves us with the following implicit function for β (we have dropped the functional form for β for notational convenience):

$$4\sin(\overline{\alpha}) = \int_{-\alpha}^{\alpha} \frac{(v^2/u^2-1)(v^2/u^2)\beta}{((v^2/u^2-1)+\sin^2\alpha)^{3/2}} \times \log\left(\frac{(v^2/u^2-1)+\sin^2\alpha-\beta\cos\alpha\sin\alpha+\sqrt{((v^2/u^2-1)+\sin^2\alpha)((\beta\cos\alpha-\sin\alpha)^2+(v^2/u^2-1)(\beta^2+1))}}{-(v^2/u^2-1)-\sin^2\alpha-\beta\cos\alpha\sin\alpha+\sqrt{((v^2/u^2-1)+\sin^2\alpha)((\beta\cos\alpha+\sin\alpha)^2+(v^2/u^2-1)(\beta^2+1))}}\right) + \frac{\cos\alpha\sin\alpha}{(v^2/u^2-1)+\sin^2\alpha} \times \left(\sqrt{(\beta\cos\alpha+\sin\alpha)^2+(v^2/u^2-1)(\beta^2+1)} - \sqrt{(\beta\cos\alpha-\sin\alpha)^2+(v^2/u^2-1)(\beta^2+1)}\right) d\alpha. \quad (18)$$

W does not appear in Eq. (18) and thus β does not depend upon it. Consequently, we can write $D^* = \beta(v, u, \overline{\alpha})W$ as a linear function of W, where $\beta(v, u, \overline{\alpha})$ is the value of β that solves Eq. (18).

APPENDIX B: PROOFS FOR SECTION 3 PROPOSITIONS

Appendix B contains material related to Section 3.

B.1. Solution to Strategic Model from Section 3.1

As our game has a saddle point, to determine the solution we consider the min-max and max-min sequential games. First, we examine the min-max game where the interceptor chooses $\boldsymbol{q}$ first and the target observes the decision and responds. Thus, for a given $\boldsymbol{q}$, the target chooses a pure strategy where $p_i = 1$ for the route i with largest value of $r_i(1-c_iq_i)$. The interceptor, being in a disadvantageous position, selects a strategy that hedges against the worst case scenario, that is, minimizes (over $\boldsymbol{q}$) the maximum (over $\boldsymbol{p}$) value of $V(\boldsymbol{p}, \boldsymbol{q}) = \sum_{i=1}^{I} r_i p_i (1 - c_i q_i)$.

Obviously, if $r_1(1-c_1) \equiv k_1 \geq r_2$, then $q_1^* = 1$ because this is the best hedging for the interceptor. Otherwise, the interceptor chooses q_1 and $q_2, q_1+q_2 = 1$, such that $r_1(1-q_1c_1) = r_2(1-q_2c_2) \equiv k_2$. If $k_2 \geq r_3$, then $q_i^* = q_i, i = 1, 2$ and $q_i^* = 0, i = 3, 4, \ldots, I$. If $k_2 < r_3$, we choose q_1, q_2, and q_3, $q_1+q_2+q_3 = 1$, such that $r_1(1-q_1c_1) = r_2(1-q_2c_2) = r_3(1-q_3c_3) \equiv k_3$. We check if $k_3 \geq r_4$ to decide whether to stop, in which case $q_i^* = q_i$, $i = 1, 2, 3$ and $q_i^* = 0$, $i = 4, 5, \ldots, I$, or proceed. The process continues in this fashion. Straightforward calculations reveal that

$$k_l = \frac{-1 + \sum_{i=1}^{l} c_i^{-1}}{\sum_{i=1}^{l} (c_i r_i)^{-1}}, \quad l = 1, 2, \ldots, I.$$

This process of optimally allocating $\boldsymbol{q}$ to the routes ends at route θ such that $k_\theta \geq r_{\theta+1}$ for the first time; continuing on to the next route will not reduce the maximum reward for the target and therefore the interceptor will gain nothing. Thus,

$$r_1(1-c_1q_1) = r_2(1-c_2q_2) = \cdots = r_\theta(1-c_\theta q_\theta) = k_\theta, \quad (19)$$

for some $\theta \in \{1, 2, \ldots I\}$.

There are two possible scenarios. First, if $k_{I-1} < r_I$ then, following the explanation above, $\theta = I$, and from Eq. (19) it follows that $q_i^* = \frac{1}{c_i}(1 - \frac{k_I}{r_i})$, $i = 1, 2, \ldots, I$. We observe that $k_{I-1} < r_I$ if and only if

$$\sum_{j=1}^{I-1} \frac{1}{c_j}\left(1 - \frac{r_I}{r_j}\right) < 1. \quad (20)$$

The second possible scenario is when $k_\theta \geq r_{\theta+1}$ for the first time, in which case we stop the process at route $\theta \in \{1, 2, \ldots I-1\}$. That is,

$$\theta = \min\left\{ i : \sum_{j=1}^{i} \frac{1}{c_j}(1 - \frac{r_{i+1}}{r_j}) \geq 1 \right\} \quad (21)$$

is the largest index of the route that is patrolled with positive probability.

It follows from Eqs. (19)–(21) that Eq. (3) is true. The target can achieve an expected reward no greater than k_θ by construction and will achieve that expected reward if the target chooses any route 1 through θ with certainty. Consequently, the min-max value is k_θ.

We next turn to the max-min problem, where the target has to choose its strategy $\boldsymbol{p}$ before the interceptor. To complete the proof, it suffices to show that the strategy $\boldsymbol{p}^*$ in Eq. (4) will produce a value of k_θ. For $\boldsymbol{p}^*$, the expected reward to the target as a function of $\boldsymbol{q}$ is

$$\begin{aligned}
\sum_{i=1}^{\theta} r_i p_i^*(1-c_iq_i) &= \sum_{i=1}^{\theta} r_i p_i^* - \sum_{i=1}^{\theta} c_i r_i p_i^* q_i \\
&= \frac{\sum_{i=1}^{\theta} r_i (r_i c_i)^{-1}}{\sum_{j=1}^{\theta} (r_j c_j)^{-1}} - \frac{\sum_{i=1}^{\theta} q_i c_i r_i (r_i c_i)^{-1}}{\sum_{j=1}^{\theta} (r_j c_j)^{-1}} \\
&= \frac{\sum_{i=1}^{\theta} c_i^{-1}}{\sum_{j=1}^{\theta} (r_j c_j)^{-1}} - \frac{\sum_{i=1}^{\theta} q_i}{\sum_{j=1}^{\theta} (r_j c_j)^{-1}} \\
&\geq \frac{-1 + \sum_{i=1}^{\theta} (c_i)^{-1}}{\sum_{j=1}^{\theta} (r_j c_j)^{-1}} \\
&= k_\theta
\end{aligned}$$

The interceptor can achieve a value of k_θ for $\boldsymbol{p}^*$ by choosing any route 1 through θ with certainty (i.e., $q_i = 1$ for any $i \in 1, 2, \ldots \theta$.) Since we have found a max-min strategy that generates the min-max value, $\boldsymbol{p}^*$ is an optimal strategy for the max-min game and hence of our simultaneous game. Finally, k_θ is the value of the simultaneous game.

B.2. Optimal First-Stage Allocation for Target

For a fixed interceptor strategy $\boldsymbol{y}$, the target solves the following problem to determine its best response

$$\max_{\boldsymbol{x}} \sum_{k=1}^{I} \frac{1}{c_k(x_k, y_k)}.$$

By our assumptions $\frac{1}{c_j(\cdot, y_j)}$ is monotone increasing and convex for all y_j and j, and hence the target wants to maximize a linear combination of convex functions. As the only constraint is that the allocation must sum to x_{max}, the expression is maximized if the target allocates all x_{max} to one of the summands. Consequently, the interceptor only need to consider the I alternatives where the target allocates all x_{max} to one route. If the target allocates x_{max} to route j, the expected reward is $U_j(\mathbf{y})$ in Eq. (6), and the proof is complete.

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