Statistics and Probability Letters 100 (2015) 85–92

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Multidimensional hitting time results for Brownian bridges with moving hyperplanar boundaries

Michael P. Atkinson *, Dashi I. Singham

Operations Research Department, Naval Postgraduate School, United States

ARTICLE INFO

Article history:
Received 4 April 2014
Received in revised form 15 May 2014
Accepted 6 February 2015
Available online 14 February 2015

Keywords:
Brownian bridge
First-passage time
Hyperplane boundaries

ABSTRACT

We calculate several hitting time probabilities for a correlated multidimensional Brownian bridge process, where the boundaries are hyperplanes that move linearly with time. We compute the probability that a Brownian bridge will cross a moving hyperplane if the end-points of the bridge lie on the same side of the hyperplane at the starting and ending times, and we derive the distribution of the hitting time if the endpoints lie on opposite sides of the moving hyperplane. Our third result calculates the probability that this process remains between two parallel hyperplanes, and we extend this result in the independent case to a hyperrectangle with moving faces. To derive these quantities, we rotate the coordinate axes to transform the problem into a one-dimensional calculation.

1. Introduction

In this paper we generalize some of the one-dimensional first passage time results for conditional Brownian motion and extend them to the multidimensional realm. Conditioning on the value of a Brownian motion at a particular time $T > 0$ produces a Brownian bridge process on the interval $[0, T]$. Several well known first passage time results exist for the one-dimensional Brownian bridge (Beghin and Orsingher, 1999; Abundo, 2002), however few studies consider the multidimensional Brownian bridge process. We focus on hyperplane boundaries that move at a constant speed. In the multidimensional case, we do not assume the components behave independently; the underlying Brownian motion can evolve according to an arbitrary correlation structure.

The papers by Beghin and Orsingher (1999) and Abundo (2002) resemble our work most closely. Beghin and Orsingher (1999) contains one-dimensional results for the Brownian bridge for stationary boundaries. Both Beghin and Orsingher (1999) and Abundo (2002) examine a two-sided boundary crossing probability result with symmetric boundaries. Our first result generalizes this one-dimensional, two-sided boundary result to the asymmetric case. Scheike (1992) calculates the probability that the standard Brownian bridge crosses a linear boundary, as well as other Brownian motion crossing results. He uses the time inversion property of Brownian motion to prove several of his results. Abundo (2002) follows the same approach, as will we. Abundo (2002) also extends the work of Beghin and Orsingher (1999) to moving boundaries and provides the foundation upon which we will derive the multidimensional results.

We present three results for correlated multidimensional Brownian bridge processes. We first compute the probability that such a process hits a linear moving hyperplane, given that the initial and final locations of the Brownian bridge lie on the same side of the hyperplane at the initial and final times, respectively. When the initial and final locations of the Brownian bridge lie on opposite sides of the moving hyperplane, we derive the distribution of the hitting time. Finally, we

* Corresponding author.
E-mail address: mpatkins@nps.edu (M.P. Atkinson).

http://dx.doi.org/10.1016/j.spl.2015.02.006
0167-7152/Published by Elsevier B.V.

calculate the probability that the correlated Brownian bridge will remain between two moving parallel hyperplanes. When the components of the Brownian bridge evolve independently, we extend this last result to compute the probability that the process remains within a hyperrectangle with moving faces. We derive the multidimensional quantities of interest in this paper by rotating the coordinate system so that the first component lies perpendicular to the hyperplanes; this procedure reduces the problem to a single dimension. While we leverage the one-dimensional results to derive the multidimensional results, the extension is not trivial. Furthermore, moving from the one-dimensional setting to a multidimensional correlated process with multidimensional boundaries represents a significant generalization.

Several results exist for hitting times in multiple dimensions for Brownian motion. Iyengar (1985) and Metzler (2010) examine planar correlated Brownian motion and derive results for the first time either component reaches a fixed level. Wendel (1980), Yin (1999) and Betz and Gzyl (1994) derive first passage time results for multidimensional Brownian motion with spherical boundaries. Di Crescenzo et al. (1991) presents hitting time results for general diffusion problems and a method for transforming a multidimensional problem to a one-dimensional problem. We take a similar approach in this paper. To the best of our knowledge, the only multidimensional result for the Brownian bridge appears in Buchmann and Petersen (2006). However, they only examine the independent case with one stationary boundary. Wu (2012) contains a good overview of the literature on first passage time problems related to Brownian motion.

The hitting time calculations presented in this paper may prove useful beyond the theoretical realm. We may know the location of an entity at certain time points and Brownian motion may appropriately describe the behavior of the entity between those points. For example, recent work by ecology researchers models the movement patterns of animals using Brownian bridges (Bullard, 1999; Horne et al., 2007; Kranstauber et al., 2012; Fischer et al., 2013). If Brownian bridge dynamics adequately represent an animal's migration pattern, then the results from this paper could aid in determining the probability an animal will stray outside the boundary of a wildlife refuge, computing the expected time until a herd reaches a particular grazing region during its migration, or choosing boundaries of a park to protect a certain species.

We define notation and review the one-dimensional results in Section 2. In Section 3 we generalize a symmetric boundary result to the asymmetric case for a one-dimensional Brownian bridge. In Section 4 we present the main results of the paper for the correlated multidimensional Brownian bridge. Section 5 concludes.

2. Notation and preliminaries

We define B_t as an n-dimensional standard Brownian motion with independent components and W_t as an n-dimensional Brownian motion with zero mean and a positive-definite covariance matrix Σ. If L_Σ is the lower triangular Cholesky factor of Σ, then $W_t \overset{D}{=} L_\Sigma B_t$, where $\overset{D}{=}$ denotes equivalence in distribution. We define X_t as the n-dimensional Brownian bridge of interest with the same covariance matrix Σ as the underlying Brownian motion W_t. Without loss of generality, we assume that the Brownian bridge starts at time $t = 0$ at position $X_0 = x_0 \in \mathbb{R}^n$ and finishes at time $t = T$ at position $X_T = x_T \in \mathbb{R}^n$. We can write the Brownian bridge as

$$X_t \overset{D}{=} x_0 + (x_T - x_0)\frac{t}{T} + W_t - \frac{t}{T}W_T. \tag{2.1}$$

We consider hyperplane boundaries that move at a constant speed. We define the hyperplane according to its normal vector, $a \in \mathbb{R}^n$, and thus at time t the boundary takes the form $\{x \in \mathbb{R}^n \mid a'x = b + ct\}$, with $b, c \in \mathbb{R}$ and a' denoting the transpose of a.

We summarize the three main results we will present for the multidimensional case from Section 4. We first calculate the probability the Brownian bridge hits the hyperplane if both x_0 and x_T lie on the same side of the moving hyperplane at times 0 and T:

$$\mathbf{P}\left[\bigcup_{t\in[0,T]} ((a'x_0 - b)(a'X_t - b - ct) \le 0)\right]. \tag{2.2}$$

We will also analyze the case where the hyperplane separates x_0 and x_T. Formally this occurs when $(a'x_0 - b)(a'x_T - b - cT) < 0$. In this case, we define the hitting time $\tau := \inf\{s > 0 : (a'x_0 - b)(a'X_s - b - cs) \le 0\}$ and compute its distribution:

$$\mathbf{P}[\tau \le t] = \mathbf{P}\left[\bigcup_{s\in[0,t]} ((a'x_0 - b)(a'X_s - b - cs) \le 0)\right], \quad \text{for } 0 \le t \le T. \tag{2.3}$$

Finally, we will compute the probability that the Brownian bridge remains between two moving parallel hyperplanes: $a'x = b_1 + c_1 t$ and $a'x = b_2 + c_2 t$,

$$\mathbf{P}\left[\bigcap_{t\in[0,T]} ((a'X_t - b_1 - c_1 t)(a'X_t - b_2 - c_2 t) \le 0)\right]. \tag{2.4}$$

We will use (2.4) to derive the probability a Brownian bridge with i.i.d. components stays within a hyperrectangle with moving faces.

We next review prior results for the one-dimensional case that we will use in the multidimensional setting after performing an appropriate rotation of the coordinate axes. Without loss of generality, we will assume the vector a has unit length. Thus in the one-dimensional case, the boundary takes the form $x = b + ct$ and x_0 and x_T are scalars. We denote by σ^2 the variance parameter of the process, so that $W_t \overset{D}{=} \sigma B_t$ and X_t follows from Eq. (2.1). Result 2.1 presents the probability that a one-dimensional Brownian bridge will cross a linearly moving boundary, given that the initial and final locations lie on the same side of the boundary (e.g., $x_0 < b$ and $x_T < b + cT$). This appears in Equations (2.2) of Abundo (2002) and (6) of Scheike (1992) in slightly different notation.

Result 2.1. If $(x_0 - b)(x_T - b - cT) > 0$ then

$$\mathbf{P}\left[\bigcup_{t\in[0,T]} ((x_0 - b)(X_t - b - ct) \le 0)\right] = \exp\left(-2\frac{|x_0 - b|\,|x_T - b - cT|}{\sigma^2 T}\right). \tag{2.5}$$

We next present the hitting time distribution when the initial and final locations lie on opposite sides of the boundary (e.g., $x_0 < b$ and $x_T > b + cT$). If the linear boundary separates the starting and ending locations, then with probability one the Brownian bridge will cross the boundary on the interval $[0, T]$. We denote by τ the first hitting time of the boundary. The following result specifies the cumulative distribution function for τ and is a special case of (2.8) in Abundo (2002).

Result 2.2. If $(x_0 - b)(x_T - b - cT) < 0$ then

$$\begin{aligned}\mathbf{P}[\tau \le t] &= \mathbf{P}\left[\bigcup_{s\in[0,t]} ((x_0 - b)(X_s - b - cs) \le 0)\right]\\ &= \exp\left(2\frac{|x_0 - b|\,|x_T - b - cT|}{\sigma^2 T}\right)\Phi\left(-\frac{|x_0 - b|(T - t) + t|x_T - b - cT|}{\sigma\sqrt{tT(T - t)}}\right)\\ &\quad + \left(1 - \Phi\left(\frac{|x_0 - b|(T - t) - t|x_T - b - cT|}{\sigma\sqrt{tT(T - t)}}\right)\right), \quad \text{for } 0 \le t \le T.\end{aligned} \tag{2.6}$$

3. Asymmetric two-sided boundary extension

We next examine the probability that a one-dimensional Brownian bridge will stay between two asymmetric linear boundaries: $b_1 + c_1 t$ and $b_2 + c_2 t$. Both Abundo (2002) and Beghin and Orsingher (1999) only consider the symmetric case where $b_1 = -b_2$ and $c_1 = -c_2$. The following result from (3.5) of Abundo (2002) gives the probability of staying within the linear boundaries in the symmetric case:

Result 3.1. If $b > 0,\ c \ge 0,\ x_0 = 0$, and $|x_T| < b + cT$ then

$$\mathbf{P}\left[\bigcap_{t\in[0,T]} ((X_t - b - ct)(X_t + b + ct) \le 0)\right] = \sum_{j=-\infty}^{j=\infty} (-1)^j \exp\left(-2bj\frac{(b + cT)j - x_T}{\sigma^2 T}\right). \tag{3.1}$$

We will analyze the more general case by assuming only that both x_0 and x_T lie between two linear boundaries that do not cross during times $t \in [0, T]$.

Proposition 3.2. *If the following three conditions hold*

- $(x_0 - b_1)(x_0 - b_2) < 0$ *(initial point lies between boundaries)*
- $(x_T - b_1 - c_1 T)(x_T - b_2 - c_2 T) < 0$ *(final point lies between boundaries)*
- $(b_1 - b_2)(b_1 + c_1 T - b_2 - c_2 T) > 0$ *(boundaries do not cross for $t \in [0, T]$)*

then the probability that the Brownian bridge stays within the two boundaries is

$$\begin{aligned}&\mathbf{P}\left[\bigcap_{t\in[0,T]} ((X_t - b_1 - c_1 t)(X_t - b_2 - c_2 t) \le 0)\right]\\ &= G\left(\frac{|x_0 - b_1|}{\sigma}, \frac{|x_T - b_1 - c_1 T|}{\sigma T}, \frac{|x_0 - b_2|}{\sigma}, \frac{|x_T - b_2 - c_2 T|}{\sigma T}\right),\end{aligned} \tag{3.2}$$

where

$$G(\alpha, \beta, \gamma, \delta) = 1 - \sum_{k=1}^{\infty} (e^{-2A_k} + e^{-2B_k} - e^{-2C_k} - e^{-2D_k}), \tag{3.3}$$

with

$$\begin{aligned} A_k &= k^2\gamma\delta + (k-1)^2\alpha\beta + k(k-1)(\gamma\beta + \delta\alpha) \\ B_k &= (k-1)^2\gamma\delta + k^2\alpha\beta + k(k-1)(\gamma\beta + \delta\alpha) \\ C_k &= k^2(\gamma\delta + \alpha\beta) + k(k-1)\gamma\beta + k(k+1)\delta\alpha \\ D_k &= k^2(\gamma\delta + \alpha\beta) + k(k+1)\gamma\beta + k(k-1)\delta\alpha. \end{aligned} \tag{3.4}$$

Proof. Abundo (2002) presents the symmetric case where $b_1 = -b_2$ and $c_1 = -c_2$. The proof for the more general case proceeds in a similar fashion to the symmetric case, so we closely mimic the steps used by Abundo (2002). We leverage a result from Doob (1949) regarding the probability a one-dimensional Brownian motion stays between two lines for all time. Doob's result also appears as Theorem 3.1 in Abundo (2002). For $\alpha, \beta, \gamma, \delta > 0$ Doob (1949) proved

$$\mathbf{P}[-(\alpha t + \beta) \le B_t \le \gamma t + \delta \text{ for all } t \ge 0] = G(\alpha, \beta, \gamma, \delta) \tag{3.5}$$

where (3.3) and (3.4) define $G(\cdot, \cdot, \cdot, \cdot)$. To prove our result we first rewrite (3.5) as

$$\mathbf{P}\left[\bigcap_{t\ge 0}((B_t + \alpha t + \beta)(B_t - \gamma t - \delta) \le 0)\right] = G(\alpha, \beta, \gamma, \delta). \tag{3.6}$$

We now proceed using the time inversion technique of Scheike (1992) and Abundo (2002). That is, we use the fact that $\tilde{B}_t = tB_{\frac{1}{t}}$ is also a standard Brownian motion. In the following steps we will explicitly write out X_t as a conditional Brownian motion:

$$\begin{aligned} &\mathbf{P}\left[\bigcap_{t\in[0,T]}((X_t - b_1 - c_1 t)(X_t - b_2 - c_2 t) \le 0)\right] \\ &= \mathbf{P}\left[\bigcap_{t\in[0,T]}((x_0 + \sigma B_t - b_1 - c_1 t)(x_0 + \sigma B_t - b_2 - c_2 t) \le 0) \mid x_0 + \sigma B_T = x_T\right] \\ &= \mathbf{P}\left[\bigcap_{s\ge\frac{1}{T}}\left(\left(sB_{\frac{1}{s}} + \frac{x_0 - b_1}{\sigma}s - \frac{c_1}{\sigma}\right)\left(sB_{\frac{1}{s}} + \frac{x_0 - b_2}{\sigma}s - \frac{c_2}{\sigma}\right) \le 0\right) \,\middle|\, B_{\frac{1}{T}} = \frac{x_T - x_0}{\sigma T}\right] \\ &= \mathbf{P}\left[\bigcap_{s\ge\frac{1}{T}}\left(\left(B_s + \frac{x_0 - b_1}{\sigma}s - \frac{c_1}{\sigma}\right)\left(B_s + \frac{x_0 - b_2}{\sigma}s - \frac{c_2}{\sigma}\right) \le 0\right) \,\middle|\, B_{\frac{1}{T}} = \frac{x_T - x_0}{\sigma T}\right] \\ &= \mathbf{P}\left[\bigcap_{s\ge\frac{1}{T}}\left(\left(B_s - B_{\frac{1}{T}} + \frac{x_0 - b_1}{\sigma}s + \frac{x_T - x_0 - c_1 T}{\sigma T}\right)\left(B_s - B_{\frac{1}{T}} + \frac{x_0 - b_2}{\sigma}s + \frac{x_T - x_0 - c_2 T}{\sigma T}\right) \le 0\right)\right] \\ &= \mathbf{P}\left[\bigcap_{t\ge 0}\left(\left(B_t + \frac{x_0 - b_1}{\sigma}t + \frac{x_T - b_1 - c_1 T}{\sigma T}\right)\left(B_t + \frac{x_0 - b_2}{\sigma}t + \frac{x_T - b_2 - c_2 T}{\sigma T}\right) \le 0\right)\right]. \end{aligned} \tag{3.7}$$

The expression in (3.7) has the same form as the left-hand-side of (3.6). By assumption, one of $\frac{x_0 - b_1}{\sigma}$ and $\frac{x_0 - b_2}{\sigma}$ is positive and the other is negative. We can define α as the positive value and $-\gamma$ as negative quantity. The corresponding value $\frac{x_T - b_i - c_i T}{\sigma T}$ will have the same sign as $\frac{x_0 - b_i}{\sigma}$ and we can define β and δ accordingly. However, by inspection of (3.3) and (3.4) we observe that $G(\alpha, \beta, \gamma, \delta) = G(\gamma, \delta, \alpha, \beta)$, and therefore we can arbitrarily choose how to distinguish between (α, β) and (γ, δ). Choosing

$$\begin{aligned} \alpha &= \frac{|x_0 - b_1|}{\sigma} \\ \beta &= \frac{|x_T - b_1 - c_1 T|}{\sigma T} \\ \gamma &= \frac{|x_0 - b_2|}{\sigma} \\ \delta &= \frac{|x_T - b_2 - c_2 T|}{\sigma T} \end{aligned} \tag{3.8}$$

produces the desired result. In the symmetric case where $b_1 = -b_2$ and $c_1 = -c_2$ and $x_0 = 0$, we define $b = |b_1|$ and $c = |c_1|$ and the expressions in (3.2)–(3.4) simplify greatly to produce result (3.1). □

Remark 3.3. The results in Sections 2 and 3 depend upon the initial and final locations and the boundary only through the absolute difference between the initial location and the boundary at time 0 (e.g., $|x_0-b|$) and the final location and the boundary at time T (e.g,. $|x_T-b-cT|$). When we extend these results to the multidimensional case in Section 4, we will replace the absolute values with the perpendicular distance between the initial and final locations and the hyperplane boundaries.

4. Multidimensional results

We start by stating several standard linear algebra results that we use to transform the multidimensional problem to a one-dimensional calculation. First we present the well known formula for the perpendicular distance between a point and a hyperplane. This appears in any standard linear algebra text (see for example page 450 of Cheney and Kincaid (2010)).

Result 4.1. If we have a point $\hat{x}$ in n-dimensional space, and a hyperplane defined by $\{x \in \mathbb{R}^n \mid a'x = b\}$, then the perpendicular distance from $\hat{x}$ to the hyperplane is

$$D = \frac{|a'\hat{x} - b|}{\sqrt{a'a}}. \tag{4.1}$$

Our analysis will require the Cholesky decomposition of the covariance matrix of the underlying multidimensional Brownian motion, Σ. For a positive definite symmetric matrix S, its Cholesky factor L_S is a lower triangular matrix such that $S = L_S L_S'$. See Chapter 4.2 of Golub and Van Loan (1996) for more details on the Cholesky decomposition and a proof of the following result.

Result 4.2. If S is a positive definite symmetric matrix with Cholesky factor L_S then $L_S(1,1) = \sqrt{S(1,1)}$, where (p, q) corresponds to the element of the matrix in the pth row and qth column.

Result 4.2 will prove beneficial because we will need element $(1, 1)$ of the Cholesky factor of the covariance matrix to compute the variance parameter of a Brownian motion after rotating our coordinate axes. We will use the following result when performing calculations with Cholesky factors. This result appears in Chapter 3.1.8 of Golub and Van Loan (1996).

Result 4.3. If L is an invertible lower triangular matrix, then its inverse L^{-1} is also lower triangular and $L^{-1}(1,1) = \frac{1}{L(1,1)}$.

Our next result will allow us to write the distribution of the product of an $n \times n$ matrix A with Brownian motion, AB_t, as an equivalent distribution of a matrix product using a lower triangular matrix.

Result 4.4. If A is an invertible and square matrix, and we define L_A as the lower Cholesky factor for AA', then $AB_t \stackrel{D}{=} L_A B_t$.

Proof. First we define $W_t = AB_t$ and $\tilde{W}_t = L_A B_t$. Both W_t and $\tilde{W}_t$ are multidimensional zero-mean Gaussian processes. Furthermore, they have the following covariance structures:

$$\begin{aligned} &\mathrm{Cov}(W_t, W_s) = \mathrm{Cov}(W_s, W_s) = \mathrm{Cov}(AB_s, AB_s) = AA's \quad \text{for } s \le t \\ &\mathrm{Cov}(\tilde{W}_t, \tilde{W}_s) = L_A L_A' s \quad \text{for } s \le t. \end{aligned} \tag{4.2}$$

By construction $L_A L_A' = AA'$, and consequently W_t and $\tilde{W}_t$ have the same covariance structure and evolve according to the same distribution. □

Remark 4.5. The Brownian bridge and underlying Brownian motion evolve with respect to the standard coordinate axes. Part of our analysis will rotate the axes and define a related Brownian motion by multiplying our original Brownian motion by a suitably chosen rotation matrix A. We will construct the rotated coordinate system such that the first component lies perpendicular to the hyperplane boundary. To accomplish this, we set the first column of A to the normalized version of a. Result 4.4 will allow us to perform analysis in terms of lower triangular Cholesky factors, rather than the original matrix A. This will lead to great simplifications in the calculations.

We now present the hitting time results from (2.2)–(2.4) in the multidimensional setting. Recall that we define the moving hyperplanes as $a'x = b + ct$, $t \in [0, T]$. Without loss of generality, we only consider unit length vectors a such that $a'a = 1$. We will first generalize Result 2.1 to multidimensional correlated Brownian bridges in Proposition 4.6, where we calculate the probability X_t will hit a moving hyperplane. As with Result 2.1, we assume the initial and final locations lie on the same side of the hyperplane.

Proposition 4.6. *If* $(a'x_0 - b)(a'x_T - b - cT) > 0$ *then*

$$\mathbf{P}\left[\bigcup_{t\in[0,T]} ((a'x_0 - b)(a'X_t - b - ct) \le 0)\right] = \exp\left(-2\frac{|a'x_0 - b||a'x_T - b - cT|}{a'\Sigma a T}\right). \tag{4.3}$$

Proof. We start in a similar fashion to the proof of Proposition 3.2 by rewriting (4.3) in terms of the underlying Brownian motion

$$\mathbf{P}\left[\bigcup_{t\in[0,T]}((a'x_0-b)(a'X_t-b-ct)\le 0)\right]$$

$$=\mathbf{P}\left[\bigcup_{t\in[0,T]}((a'x_0-b)(a'(x_0+L_\Sigma B_t)-b-ct)\le 0)\mid x_0+L_\Sigma B_T=x_T\right], \tag{4.4}$$

where L_Σ is the lower Cholesky factor of the covariance matrix Σ. We next construct an orthonormal matrix A by setting the vector a as the first column. We will not need the remaining $n-1$ columns of A so we do not specify their exact form. However, one could easily construct A from a using the Gram–Schmidt algorithm (see page 230 of Golub and Van Loan (1996)). The matrix A defines a rotated coordinate system where the first component lies perpendicular to the hyperplane boundary. We next modify (4.4) by multiplying certain components by the orthonormal matrices A and A'. By doing this, we effectively rotate the coordinate system, which will allow us to focus only on the evolution of the process perpendicular to the boundary. We rewrite (4.4) as

$$\mathbf{P}\left[\bigcup_{t\in[0,T]}((a'x_0-b)((a'A)(A'x_0+A'L_\Sigma B_t)-b-ct)\le 0)\mid A'x_0+A'L_\Sigma B_T=A'x_T\right]. \tag{4.5}$$

We next define L as the lower Cholesky factor of $A'\Sigma A$. By Result 4.4, $A'L_\Sigma B_t \overset{D}{=} LB_t$, and hence we can rewrite (4.5) as

$$\mathbf{P}\left[\bigcup_{t\in[0,T]}((a'x_0-b)((a'A)(A'x_0+LB_t)-b-ct)\le 0)\mid A'x_0+LB_T=A'x_T\right]. \tag{4.6}$$

We next turn our attention to $(a'A)(A'x_0+LB_t)$. By construction, $(a'A)=e_1'$, where e_1 denotes the standard unit vector in the first component ($e_1'=(1,0,0\ldots,0)$). In the following calculation we define A_i as the ith column of the matrix A, and recall that $A_1=a$. We now simplify $(a'A)(A'x_0+LB_t)$ to

$$e_1'\left(\begin{pmatrix} a'x_0\\ A_2'x_0\\ \vdots\\ A_n'x_0\end{pmatrix}+\begin{pmatrix} L(1,1)B_t(1)\\ L(2,1)B_t(1)+L(2,2)B_t(2)\\ \vdots\\ \sum_{i=1}^{n}L(n,i)B_t(i)\end{pmatrix}\right)=a'x_0+L(1,1)B_t(1) \tag{4.7}$$

where $B_t(i)$ is the Brownian motion in the ith component. Substituting (4.7) into (4.6) greatly simplifies the problem. We have transformed the event so that we only need to consider the evolution of a one-dimensional Brownian motion $B_t(1)$ and write (4.6) as

$$\mathbf{P}\left[\bigcup_{t\in[0,T]}((a'x_0-b)(a'x_0+L(1,1)B_t(1)-b-ct)\le 0)\mid A'x_0+LB_T=A'x_T\right]. \tag{4.8}$$

We can rewrite the conditioning event as $B_T=L^{-1}A'(x_T-x_0)$. By definition, $B_t(1)$ is independent of $B_T(j)$ for all $j>1$. Consequently, we only need to condition on the first component of the Brownian motion at time T: $B_T(1)$. To determine the first element of $L^{-1}A'(x_T-x_0)$, we note that the first row of $L^{-1}A'$ is $a'/L(1,1)$. This follows from Result 4.3 because L is a lower triangular Cholesky factor. Hence, $B_T(1)=a'(x_T-x_0)/L(1,1)$. Rearranging yields

$$\mathbf{P}\left[\bigcup_{t\in[0,T]}((a'x_0-b)(a'x_0+L(1,1)B_t(1)-b-ct)\le 0)\mid a'x_0+L(1,1)B_T(1)=a'x_T\right]. \tag{4.9}$$

Inspection of (4.9) reveals we have completed the transformation to a calculation involving only a one-dimensional process. The Brownian bridge starts at initial point $a'x_0$ and finishes at time T at point $a'x_T$, and the underlying Brownian motion evolves with $\sigma=L(1,1)$. By Result 4.2 we know $L(1,1)$ corresponds to the square root of element $(1,1)$ of $A'\Sigma A$. This element equals $a'\Sigma a$, and thus $L(1,1)=\sqrt{a'\Sigma a}$. Substituting these quantities into the one-dimensional Eq. (2.5) produces the multidimensional result in (4.3). □

The same technique of transforming a multidimensional problem into a one-dimensional calculation will allow us to extend Result 2.2 and Proposition 3.2 into their multidimensional analogs. We replace σ with $\sqrt{a'\Sigma a}$, and then using Result 4.1 we replace $|x_0-b|$ with $|a'x_0-b|$ and $|x_T-b-cT|$ with $|a'x_T-b-cT|$. Proposition 4.7 extends Result 2.2 to the probability distribution for the time until X_t will hit the hyperplane $a^T x = b + ct$, given the hyperplane separates the initial and final locations.

Proposition 4.7. *If* $(a^T x_0 - b)(a^T x_T - b - cT) < 0$ *then*

$$\mathbf{P}\left[\bigcup_{s\in[0,t]} ((a' x_0 - b)(a^T X_s - b - cs) \le 0)\right] \tag{4.10}$$

$$= \exp\left(2\frac{|a'x_0 - b|\,|a'x_T - b - cT|}{a'\Sigma a T}\right)\Phi\left(-\frac{|a'x_0 - b|(T-t) + t|a'x_T - b - cT|}{\sqrt{a'\Sigma a t T(T-t)}}\right)$$
$$+\left(1 - \Phi\left(\frac{|a'x_0 - b|(T-t) - t|a'x_T - b - cT|}{\sqrt{a'\Sigma a t T(T-t)}}\right)\right), \quad \text{for } 0 \le t \le T. \tag{4.11}$$

Finally we extend Proposition 3.2 to the probability that X_t will remain between two parallel hyperplanes given the initial and final points lie between both hyperplanes. We define the two hyperplanes by $a'x = b_1 + c_1 t$ and $a'x = b_2 + c_2 t$. Our methodology does not apply for non-parallel hyperplanes. However, the two planes can move at different rates: $c_1 \ne c_2$.

Proposition 4.8. *If*

- $(a'x_0 - b_1)(a'x_0 - b_2) < 0$ *(initial point between boundaries)*
- $(a'x_T - b_1 - c_1 T)(a'x_T - b_2 - c_2 T) < 0$ *(final point between boundaries)*
- $(b_1 - b_2)(b_1 + c_1 T - b_2 - c_2 T) > 0$ *(boundaries do not cross)*

then

$$\mathbf{P}\left[\bigcap_{t\in[0,T]} ((a'X_t - b_1 - c_1 t)(a'X_t - b_2 - c_2 t) \le 0)\right]$$
$$= G\left(\frac{|a'x_0 - b_1|}{\sqrt{a'\Sigma a}}, \frac{|a'x_T - b_1 - c_1 T|}{\sqrt{a^T \Sigma a T}}, \frac{|a'x_0 - b_2|}{\sqrt{a'\Sigma a}}, \frac{|a'x_T - b_2 - c_2 T|}{\sqrt{a'\Sigma a T}}\right) \tag{4.12}$$

where $G(\cdot,\cdot,\cdot,\cdot)$ *is defined in* (3.3).

We can extend the result in Proposition 4.8 if the components of the underlying Brownian motion evolve in an i.i.d. fashion (i.e., $\Sigma = \sigma^2 I$). In this case we compute the probability that the Brownian bridge remains within a hyperrectangle with moving faces in Corollary 4.9. We denote the ith parallel faces to the hyperrectangle by $a_i'x = b_{i1} + c_{i1} t$ and $a_i'x = b_{i2} + c_{i2} t$ for $1 \le i \le n$. We also require that a_i and a_j be orthogonal for $i \ne j$. Thus if a_i corresponds to the ith column of A, then A forms an orthonormal basis of $\mathbb{R}^n$ that defines the orientation of the hyperrectangle.

Corollary 4.9. *If*

- *A is an orthonormal matrix with columns a_i*
- *X_t is Brownian bridge with covariance matrix $\Sigma = \sigma^2 I$*
- $(a_i'x_0 - b_{i1})(a_i'x_0 - b_{i2}) < 0$ *for all* $i = 1, 2, \ldots, n$ *(initial point in hyperrectangle)*
- $(a_i'x_T - b_{i1} - c_{i1} T)(a_i'x_T - b_{i2} - c_{i2} T) < 0$ *for all* $i = 1, 2, \ldots, n$ *(final point in hyperrectangle)*
- $(b_{i1} - b_{i2})(b_{i1} + c_{i1} T - b_{i1} - c_{i1} T) > 0$ *for all* $i = 1, 2, \ldots, n$ *(parallel faces do not cross)*

then

$$\mathbf{P}\left[\bigcap_{i=1}^{n}\bigcap_{t\in[0,T]} ((a_i'X_t - b_{i1} - c_{i1} t)(a_i'X_t - b_{i2} - c_{i2} t) \le 0)\right]$$
$$= \prod_{i=1}^{n} G\left(\frac{|a_i'x_0 - b_{i1}|}{\sigma}, \frac{|a_i'x_T - b_{i1} - c_{i1} T|}{\sigma T}, \frac{|a_i'x_0 - b_{i2}|}{\sigma}, \frac{|a_i'x_T - b_{i2} - c_{i2} T|}{\sigma T}\right) \tag{4.13}$$

where $G(\cdot,\cdot,\cdot,\cdot)$ *is defined in* (3.3).

5. Conclusion

In this paper we extend the results of Scheike (1992), Beghin and Orsingher (1999), and Abundo (2002) to compute hitting time quantities of correlated multidimensional Brownian bridge processes to moving hyperplane boundaries. Our technique of rotating the coordinate axes provides an approach that could be used to generalize other hitting time results related to Brownian motion to the multidimensional setting. For example, Abundo (2002) considers piecewise linear boundaries in one-dimension. In the multidimensional case this would correspond to the hyperplane boundary changing from $a'x = b + ct$ when $t \le t^*$ to $a'x = b^* + c^* t$ when $t \ge t^*$. Iyengar (1985) and Metzler (2010) examine the first time planar Brownian motion hits either a horizontal line or a vertical line. Using our approach, one could generalize this problem to analyze the first time planar Brownian motion hits one of two perpendicular lines.

References

Abundo, M., 2002. Some conditional crossing results of Brownian motion over a piecewise-linear boundary. Statist. Probab. Lett. 58 (2), 131–145.
Beghin, L., Orsingher, E., 1999. On the maximum of the generalized Brownian bridge. Lith. Math. J. 39 (2), 157–167.
Betz, C., Gzyl, H., 1994. Hitting spheres from the exterior. Ann. Probab. 22 (1), 177–179.
Buchmann, F.M., Petersen, W.P., 2006. An exit probability approach to solving high dimensional Dirichlet problems. SIAM J. Sci. Comput. 28 (3), 1153–1166.
Bullard, F., 1999. Estimating the home range of an animal: a Brownian bridge approach (Master's thesis), University of North Carolina, Chapel Hill.
Cheney, W., Kincaid, D., 2010. Linear Algebra: Theory and Applications. Jones & Bartlett Learning.
Di Crescenzo, A., Giorno, V., Nobile, A., Ricciardi, L., 1991. On the reduction to one dimensional of first-passage-time problems for diffusion processes. J. Math. Phys. Sci. 25, 599–611.
Doob, J.L., 1949. Heuristic approach to the Kolmogorov–Smirnov theorems. Ann. Math. Statist. 20 (3), 393–403.
Fischer, J.W., Walter, W.D., Avery, M.L., 2013. Brownian bridge movement models to characterize birds' home ranges. Condor 115 (2), 298–305.
Golub, G.H., Van Loan, C.F., 1996. Matrix Computations. In: Johns Hopkins Studies in the Mathematical Sciences, Johns Hopkins University Press.
Horne, J.S., Garton, E.O., Krone, S.M., Lewis, J.S., 2007. Analyzing animal movements using Brownian bridges. Ecology 88 (9), 2354–2363.
Iyengar, S., 1985. Hitting lines with two-dimensional Brownian motion. SIAM J. Appl. Math. 45 (6), 983–989.
Kranstauber, B., Kays, R., LaPoint, S.D., Wikelski, M., Safi, K., 2012. A dynamic Brownian bridge movement model to estimate utilization distributions for heterogeneous animal movement. J. Anim. Ecol. 81 (4), 738–746.
Metzler, A., 2010. On the first passage problem for correlated Brownian motion. Statist. Probab. Lett. 80 (5), 277–284.
Scheike, T.H., 1992. A boundary-crossing result for Brownian motion. J. Appl. Probab. 448–453.
Wendel, J.G., 1980. Hitting spheres with Brownian motion. Ann. Probab. 8 (1), 164–169.
Wu, T.-L., 2012. Linear and non-linear boundary crossing probabilities for Brownian motion and related processes (Ph.D. thesis), University of Manitoba.
Yin, C., 1999. The joint distribution of the hitting time and place to a sphere or spherical shell for Brownian motion with drift. Statist. Probab. Lett. 42 (4), 367–373.