

APPLYING DATA ENVELOPMENT ANALYSIS PRINCIPLE IN ORDINAL MULTI CRITERIA DECISION ANALYSIS

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Abstract. We consider a multicriteria decision analysis (MCDA) problem where importance of criteria, and evaluations of alternatives with respect to the criteria, are expressed on a qualitative ordinal scale. Using the extreme-point principle of Data Envelopment Analysis (DEA), we develop a two-parameter method for obtaining overall ratings of the alternatives when preferences and evaluations are made on an ordinal scale. We assume no parametric setup other than the two parameters that reflect minimum intensities of discriminating among rank positions: one parameter for the alternatives' ranking and one for the criteria ranking. These parameters are bounded by the ordinal input data, and they imply a universal tie among the alternatives when both parameters are selected to be zero. We describe the model, discuss its theoretical underpinning, and demonstrate its application.

Keywords: Multiple criteria analysis, ordinal scale, DEA, extreme-point, discriminating factor.

1. Introduction

We consider an ordinal multi-criteria decision problem where all inputs – weights of criteria and evaluations of alternatives with respect to the criteria – are given in terms of qualitative ordinal rankings or ordinal scores such as the Likert scale [17], [2]. Ordinal elicitations of assessments and preferences in an MCDA setting are less cognitively demanding than cardinal ones [16]. In a decision environment that may be uncertain and nebulous, a simple ordering is more natural to express preferences than cardinal values [8], [15]. Therefore, it may be easier to implement an ordinal preference setting, rather than cardinal, in complex multi-criteria problems. The question is how to aggregate the ordinal input into an overall evaluation that can prioritize the alternatives in a reasonable and consistent way.

The main thrust of this note is to propose a useful, transparent and easily-implementable method for assessing multicriteria linear value-functions in a purely ordinal setting. The method has the following two main features: (1) It is founded on the solid theoretical ground of the well-established Data Envelopment Analysis (DEA) methodology [5], (2) Besides two parameters, controlled by the decision-maker and functioning as “discriminating rheostat”, the method is fully ordinal and does not rely on any cardinal input, including no criteria weights.

There have been several attempts to address a variant of this problem where the objective is to elicit cardinal weights of criteria from their ordinal ranking. Methods range from procedures [1] and models [12], [13] converting qualitative (linguistic) terms into cardinal

or fuzzy settings, which are essentially equivalent to quantitative approaches, to theoretical approaches based on partial preorder of reference alternatives whose values are known, as well as from other partial preorders [18], to a transformation, called global outranking method, from the two dimensional (alternative, criterion) space into a single dimensional ranking of the alternatives [21]. Other approaches for dealing with ordinal data in an MCDA setting are reported, among others, in [4], [11], [14], [19], [23], and [24]. Recent contributions to the topic address non-uniform scales, which are treated using the concept of ordinal proximity measure [9], [10]. Non-uniform scales are implicitly present in our approach in this paper, as shown later on.

The main thrust of this note is to propose a useful, transparent and easily-implementable method for assessing multicriteria linear value-functions in a purely ordinal setting. The method has the following two main features: (1) It is founded on the solid theoretical ground of the well-established Data Envelopment Analysis (DEA) methodology [5], (2) Besides two parameters, controlled by the decision-maker and functioning as “discriminating rheostat”, the method is fully ordinal and does not rely on any cardinal input, including no criteria weights.

The method is based on an extreme-point principle and models developed in [6] and [7]. The models are MCDA applications of DEA [5], which is a widely-accepted method for measuring efficiencies of decision making units in a most equitable way. The multiple parameters in [6] and the non-linearity in [7] are reduced here to a two-parameter linear optimization model, which is transparent to and easily understandable by decision makers. The model is also easily implementable for real-world problems on a spreadsheet. An important feature highlighted here is the sensitivity of the decision outcome to the two parameters called *discriminating factors*, which represent the decision-maker’s general strength of opinion or preferences.

Section 2 describes the model and Section 3 presents an example and short analysis. Concluding remarks are given in Section 4.

2. Model

In many legacy MCDA models (e.g., the Analytic Hierarchy Process (AHP) [22]) the setup is rather simple: each of m criteria is assigned a weight w_i , $i = 1, \dots, m$, and each alternative j , $j = 1, \dots, n$, is evaluated with respect to each criterion i and is given a value v_{ij} . The overall evaluation V_i of alternative i is the weighted sum of its values, that is, $V_j = \sum_{i=1}^m V_{ij}$, where $V_{ij} = w_i v_{ij}$. In the purely ordinal setting this type of evaluations are not applicable anymore. If we only know that, say, $w_1 > w_2 > \dots > w_m$ and that, for some criterion i , $v_{i1} > v_{i2} > \dots > v_{in}$, what can we say about the V_{ij} s values and ultimately about the alternatives’ final ratings V_j , $j = 1, \dots, n$?

Suppose that on each criterion, the n alternatives are evaluated on an ordinal scale having r rank positions. For example, if $r = 5$ then we have the Likert scale (Likert, 1932; Allen and Seaman, 2007). Similarly, the importance or weight of each of the m criteria is determined on an ordinal scale having s rank positions. The choices of the ordinal scales – the parameters

r and s – are completely arbitrary. For example, the alternatives and criteria could be strictly ordered on n and m rank positions, respectively. Higher rank (lower index number) indicates higher value for an alternative and higher importance (weight) for a criterion. Let x_{ij} denote the “value” of being ranked in the j –*th* rank position with respect to a criterion ranked in the i –*th* rank position. We make no specific requirements regarding the values x_{ij} other than the following natural assumptions:

- (1) The values are nonnegative, $x_{ij} \geq 0$.
- (2) A higher rank of an alternative on a certain criterion has a larger value than a lower rank on that criterion,

$$x_{ij} - x_{i,j+1} > 0, \quad i = 1, \dots, s, \quad j = 1, \dots, r-1. \quad (1)$$

The third assumption relates to the rank positions of the criteria. The effect of the ordering of the criteria leads to two possible interpretations: a strict one and a relaxed one. In the strict interpretation we require that an alternative’s rank position on a higher ranked criterion has a larger value x_{ij} than the same rank position on a lower ranked criterion. That is,

$$x_{ij} - x_{i+1,j} > 0, \quad i = 1, \dots, s-1, \quad j = 1, \dots, r. \quad (2)$$

The logic behind these relations is that being, say, best on a more important criterion carries more value (larger V_{ij} value) than rated best on a less important criterion. A more relaxed assumption is simply to require that the total value assigned to a higher ranked criterion is larger than the total value of a lower ranked criterion, that is

$$\sum_{j=1}^r (x_{ij} - x_{i+1,j}) > 0, \quad i = 1, \dots, s-1. \quad (2a)$$

We will analyze and compare the two aforementioned interpretations of criteria rankings.

Next, we define,

$$a_j^k(l) = \begin{cases} 1 & \text{if alternative } k \text{ is ranked } j\text{-th on criterion } l \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

$$c_i^l = \begin{cases} 1 & \text{if criterion } l \text{ is ranked in the } i\text{-th position} \\ 0 & \text{otherwise} \end{cases} \quad (4)$$

If the x_{ij} values were known, then the overall evaluation, or rating, of alternative k would be:

$$V_k = \sum_{l=1}^m \sum_{i=1}^s \sum_{j=1}^r a_j^k(l) c_i^l x_{ij}, \quad k = 1, \dots, n. \quad (5)$$

Similarly to many other MCDM models (e.g., [22]) the alternative with the highest V value is the most preferred. The question is how to determine the x_{ij} values in a reasonable and “fair” way, consistent with equations (1) and (2) (or, alternatively, (2a)).

To do so, we propose an extreme-point principle, which is well established in the theory of Data Envelopment Analysis (DEA) [5]. The idea is to give each alternative its “best shot” for presenting itself. In other words, each alternative can select its own x_{ij} values such that it would get the highest possible rating subject to the constraints (1) and (2) (or (2a)), and the requirement that when the selected x_{ij} values are applied to the other alternatives, no alternative gets a rating higher than a certain bound which is normalized to be equal to 1. This is an “equitable” concept in the sense that the rating of each alternative is evaluated in exactly the same way as the others’.

To put this idea into practice, we define two *discriminating factors*. Similarly to the idea in (Punkka and Salo 2013) the discriminating factors express how clearly a decision-maker can distinguish among rank positions of the alternatives and the criteria. These factors are related to the concept of proximity measure described in [9] and [10], and are somewhat similar to the concept of value-difference in [20]. Define ε as the discriminating factor for the alternatives rank position. This is the minimum gap between two consecutive rank positions of alternatives. Formally,

$$x_{ij} - x_{i,j+1} \geq \varepsilon, \quad i = 1, \dots, s, \quad j = 1, \dots, r-1. \quad (6)$$

Similarly, we define δ as the discriminating factor for the criteria. That is,

$$x_{ij} - x_{i+1,j} \geq \delta, \quad i = 1, \dots, s-1, \quad j = 1, \dots, r \quad (7)$$

for the strict case, and

$$\sum_{j=1}^r (x_{ij} - x_{i+1,j}) > \delta, \quad i = 1, \dots, s-1 \quad (7a)$$

for the relaxed case.

As in standard DEA (Primal) model, we solve n linear programming problems – one for each alternative. As a convention for simplifying notation, an alternative for which the rating V is computed, and therefore appears in the objective function, we assign the index 0. Thus, we solve the following n LP problems:

$$Max V_0 = Max \sum_{l=1}^m \sum_{i=1}^s \sum_{j=1}^r a_j^0(l) c_i^l x_{ij} \quad (8)$$

s.t.

$$\sum_{l=1}^m \sum_{i=1}^s \sum_{j=1}^r a_j^k(l) c_i^l x_{ij} \leq 1, \quad k = 1, \dots, n \quad (9)$$

$$\text{Constraints (6) and (7) (or (7a))} \quad (10)$$

$$x_{ij} \geq 0. \quad (11)$$

In the DEA context, alternatives for which the objective in (8) reaches the value 1 are said to be *efficient*. In our MCDA context we call such alternative *non-dominated*. Alternatives for which the corresponding functional in (8) is less than 1 are *dominated* by those alternatives for which the functional is equal 1. Based on the ordinal evaluations, the objective of the decision maker is to narrow down the set of non-dominated alternatives and thus identify the desired alternatives. If $\varepsilon = 0$ then all the alternatives are trivially non-dominated as they all can reach the highest rating of 1. In that case each alternative can set $x_{ij} = \frac{1}{m}$, $i = 1, \dots, s$, $j = 1, \dots, r$, and all ratings V_k , $k = 1, \dots, n$, become the same at 1. As the values of ε and δ increase from 0, alternatives begin to separate in their values; the better alternatives maintain the rating 1 but the other alternatives start to get values V lower than 1. This process ends when either ε or δ become too large so that problem (8) – (11) becomes infeasible. For example, if $\varepsilon = \delta$ then the maximum possible value of ε is obtained when it is taken as a decision variable in (8) – (11), and we add the term $M\varepsilon$ to the objective function, where M is a large constant. Formally,

$$Max \sum_{l=1}^m \sum_{i=1}^s \sum_{j=1}^r a_j^0(l) c_i^l x_{ij} + M\varepsilon \quad (12)$$

s.t.

$$\sum_{l=1}^m \sum_{i=1}^s \sum_{j=1}^r a_j^k(l) c_i^l x_{ij} \leq 1, \quad k = 1, \dots, n \quad (13)$$

$$\text{Constraints (6) and (7) (or 7a)} \quad (14)$$

$$\varepsilon, x_{ij} \geq 0. \quad (15)$$

3. Example and Analysis

To demonstrate the model and its features, and analyze its outcomes, consider the following example with $n = 6$ alternatives and $m = 4$ criteria. The value of the alternatives with respect to the criteria, and the importance of the criteria are evaluated on a Likert scale, that is, $r = s = 5$. To be consistent with the notation thus far, we reverse the conventional definition of the scale and assign the following rank values for an alternative and a criterion, respectively:

- 1 – Most Valuable/Important,
- 2 – Very Valuable/Important,
- 3 – Valuable/ Important,
- 4 – Rather Valuable/ Important,
- 5 – Marginally Valuable/Important.

Table 1 presents the Likert scores for the criteria.

Table 1. Likert Scores for Criteria Importance

Criterion	Likert Score
C1	2
C2	1
C3	4
C4	5

Table 2 presents the Likert scores for the six alternatives, labeled A1, ..., A6, with respect to the four criteria. Notice that no alternative is trivially dominated by another, that is, no alternative is ranked lower than another alternative on all criteria. Therefore, there is no obvious inferior alternative that we could discard from further consideration.

Table 2. Likert Scores for Alternatives' Values with Respect to the Four Criteria.

	A1	A2	A3	A4	A5	A6
C1	2	4	3	2	2	3
C2	3	2	5	3	2	1
C3	3	2	1	5	1	4
C4	2	2	2	1	3	5

First, we consider the strict case where we use constraints (7) in problem (8) – (11). Table 3 presents the non-dominated alternatives whose functional (8) reaches the value 1 and therefore they are most preferred. We can see that as the discriminating factors ε and δ increase, fewer alternatives stay non-dominated. Also notice that, as discussed above, when $\varepsilon = 0$, which practically means that the decision maker has no clear way of distinguishing among the alternatives – all alternatives are equally valuable. Table 3 also displays the range of discriminating factors for which problem (8) – (11) (with constraints (7)) is feasible.

Table 3: Non-Dominated Alternatives for the Case of Strict Criteria Discrimination

		ε				
		0	.02	.04	.06	.08
δ	0	A1,...,A6	A1,...,A6	A1,...,A6	A5,A6	Infeasible
	.02	A1,...,A6	A1,...,A6	A1,...,A6	A5,A6	Infeasible
	.04	A1,...,A6	A1,...,A6	A5,A6	Infeasible	Infeasible
	.06	A1,...,A6	A1,...,A6	A5,A6	Infeasible	Infeasible
	.08	A1,...,A6	A1,A2,A5,A6	Infeasible	Infeasible	Infeasible
	.1	A1,...,A6	Infeasible	Infeasible	Infeasible	Infeasible

Alternatives A5 and A6 remain non-dominated throughout. No feasible combination of the discriminating factors can bring any of these two alternatives to an objective value smaller than 1. The conclusion is that, based on the Likert scores provided for the importance of the alternatives, and the values of the alternatives with respect to the criteria, the two alternatives, A5 and A6, are equally desired at the top.

Table 4 presents the V values for the six alternatives when we assume that $\varepsilon = \delta$. In that case, the maximum feasible value of the common discriminating parameter is 0.05. We can see now in detail the general picture presented in Table 3. First, if the common discriminating

factor is less than or equal 0.03, all six alternatives are non-dominated; the decision maker cannot identify a smaller set of desired alternatives. For discriminating factor greater than or equal .04 we see that the six alternatives are divided into 3 subsets: the non-dominated alternatives A5 and A6, the second tier of alternatives: A1 and A2, and the inferior alternatives: A3 and A4. These observations are consistent with the results in Table 3.

Table 4: V Scores for Alternatives for Equal Discriminating Factors

$\varepsilon = \delta$	Alternatives					
	A1	A2	A3	A4	A5	A6
0	1	1	1	1	1	1
.01	1	1	1	1	1	1
.02	1	1	1	1	1	1
.03	1	1	1	1	1	1
.04	.99	.99	.95	.95	1	1
.05	.90	.90	.85	.85	1	1

Table 5, which is similar to Table 3, presents the non-dominated alternatives for the relaxed case where constraints (7a) replace constraints (7) in problem (8) – (11). Notice that in this case the significant values of δ are order of magnitude larger than those values in the strict case, which is reasonable considering we compare now sums of rank values.

Table 5: Non-Dominated Alternatives for the Case of Relaxed Criteria Discriminating

	ε					
	0	.02	.04	.06	.08	
δ	0	A1,...,A6	A1,...,A6	A1,...,A6	A1,...,A6	A5,A6
	.1	A1,...,A6	A1,...,A6	A1,...,A6	A1,...,A6	A5,A6
	.2	A1,...,A6	A1,...,A6	A1,...,A6	A1,A2,A4,A5,A6	Infeasible
	.3	A1,...,A6	A1,...,A6	A1,...,A6	A5,A6	Infeasible
	.4	A1,...,A6	A1,...,A6	A1,...,A6	Infeasible	Infeasible
	.5	A1,...,A6	A1,...,A6	A1,A2,A4,A5,A6	Infeasible	Infeasible
	.55	A1,...,A6	A1,...,A6	A5,A6	Infeasible	Infeasible
	.6	A1,...,A6	A1,...,A6	Infeasible	Infeasible	Infeasible
	.7	A1,...,A6	Infeasible	Infeasible	Infeasible	Infeasible
.8	Infeasible	Infeasible	Infeasible	Infeasible	Infeasible	

As one can see, the results of the relaxed model are consistent with those of the strict model presented in Table 3. If we assume that is $\varepsilon = 0.1\delta$ then the maximum feasible discriminating factor is $\delta = .476$, in which case both alternatives A5 and A6 are still at the top with $V_{A5} = V_{A6} = 1$. For the other four alternatives, we have, $V_{A1} = V_{A2} = .91$ and $V_{A3} = V_{A4} = .86$, which is consistent with the results in Table 4, for the strict discrimination case. We conclude that for the two types of the criteria weights' discriminating factors – the strict and the relaxed – the results are consistent: alternatives A5 and A6 are indistinguishable

and ranked at the top. The set of non-dominated alternatives expands as the decision maker is less convinced about discriminating among rank positions.

4. Conclusions

We presented a simple, equitable and intuitive method for evaluating MCDA problems with ordinal inputs. The method has a theoretical foundation derived from extreme-point principle manifested in Data Envelopment Analysis. Two parameters – one associated with the criteria and one with the alternatives – called discriminating factors, play a role of lower bounds on gaps among rank positions. Higher lower bounds indicate stronger preferences among rank positions. Low values of the discriminating factors, which indicates weak preferences and/or opinions, result in more tied alternatives at the top rating, as one would expect. As the discriminating factors increase better alternatives are distinguished from the other. The methodology presented is flexible enough to accommodate variability among criteria where the decision maker may be more confident about his preferences on one criterion than on another. In our model such a situation would be manifested by a vector of ε values – one for each criterion – rather than a single factor. We also note that the proposed model can accommodate additional, not necessarily ordinal, requirements and assumptions regarding the criteria and the alternatives. For example, if the weight of criterion C2 in the example in Section 3 is evaluated to be at least two times the weight of C1, then problem (8) – (11) is

modified by adding the constraint
$$\sum_{j=1}^6 x_{2j} - 2 \sum_{j=1}^6 x_{1j} \geq 0.$$

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