Online Discrete Convex Optimization

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Online Combinatorial Optimization

Consider a matroid \((N, I)\), where the ground set \(N = [n] := \{1, \ldots, n\}\) is finite and \(I \subseteq 2^N\) is the associated family of independent sets.

### Online Setting

At each time \(t \in [T] := \{1, \ldots, T\}\):

- Select an independent set \(S_t \in I\) to play.
- The cost incurred by \(S_t\) is evaluated according to an adversarially chosen function \(f_t : I \rightarrow [0, 1]\).

We will focus on the full-information setting.

- i.e., \(f_t\)'s are revealed via a value oracle.

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### Objective: Approximate Regret Minimization

Do as well, over the time horizon \(T\), as some constant \(\alpha \in [0, 1]\) times the best fixed \(S \in I\) in hindsight, i.e. minimize the \(\alpha\)-regret

\[
\text{Regret}_\alpha(T) := \sum_{t=1}^{T} f_t(S_t) - \alpha \min_{S \in I} \sum_{t=1}^{T} f_t(S)
\]

### Examples of applications include online versions of:

- Routing on Networks
- Selecting Products to Produce
- Selecting Items to Bid On
- Selecting Locations of Facilities, Sensors, . . .
Online Submodular Optimization

**Assumption**

The $f_t$'s are **submodular**, i.e.,

$$f_t(S) + f_t(S') \geq f_t(S \cup S') + f_t(S \cap S')$$

for all $S, S' \in 2^N$ and $t \in [T]$

**Special Case:** The $f_t$'s are **modular** (aka linear) if

$$f_t(S) = \sum_{a \in S} c_t(a) \text{ for some } c_t : N \rightarrow [0, 1]$$

Submodularity models diminishing marginal returns (e.g., profits, coverage).

There are algorithms whose incurred $\alpha$-regret grows at most sub-linearly in $T$ for, e.g.,

- cardinality-constrained modular optimization for $\alpha = 1$ [Audibert Bubeck & Lugosi, 2014]
- combinatorially-constrained monotone submodular minimization [Jegelka & Bilmes, 2010]
- cardinality-constrained monotone submodular maximization for $\alpha = 1 - \frac{1}{e}$ [Streeter & Golovin, 2008], [Harvey Liaw & Soma, 2020]
- matroid-constrained monotone submodular maximization for $\alpha = 1 - \frac{1}{e} - \epsilon$ [Golovin Krause & Streeter 2014], [Harvey Liaw & Soma, 2020]
**Motivation:** Online Convex Optimization

For optimization problems of the form

$$\min_{x \in \mathbb{R}^n} f(x), \quad f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\},$$

*convexity* is a useful way to distinguish between “easy” and “hard” problems [Rockafellar, 1970]

There is a well-developed theory for online minimization when the decisions are vectors in $\mathbb{R}^n$ and the functions $f_t$ are convex [Hazan, 2022].

**Examples:**

- 1-regret bounds that grow with $\sqrt{T}$
- 1-regret lower bounds that scale with $\sqrt{T}$

**Question**

Is there a useful notion of convexity in combinatorial optimization?

From p. 309 in [Rockafellar & Wets, 2009]
There is more than one way to usefully define convexity for functions on $\mathbb{Z}^n$ [Murota, 2003].

- **Desiderata:** duality between convexity and concavity, separation, …

- Not enough to assume that the discrete function (on $\mathbb{Z}^n$) can be extended to a convex function on $\mathbb{R}^n$ [Murota, 2003].

There are two notions of discrete convexity that are useful for modeling certain discrete optimization problems [Murota, 2003]:

- **L♮-Convexity** ("Lattice Convexity")
- **M♮-Convexity** ("Matroid Convexity")
Online $L^\triangledown$-Convex Minimization

Proposition (p. 145 in [Murota, 2003])

A function $f : 2^N \rightarrow \mathbb{R} \cup \{+\infty\}$ is submodular if and only if its indicator function on $\{0, 1\}^n$ is $L^\triangledown$-convex.

▶ $L^\triangledown$-convex minimization is equivalent to submodular minimization

Projected Subgradient Descent (PSD)

Given a step size $\eta$ and an initial $x_1 \in [0, 1]^n$, for $t = 1, \ldots, T$ do the following:

1. Round $x_t$ to an indicator vector for a subset $S_t$.
2. Play $S_t$.
3. Compute a subgradient $p_t$ of the Lovász extension of $f_t$ at $x_t$.
4. Take a projected gradient step in the direction of $-p_t$ from $x_t$ to $x_{t+1}$

Theorem [Jegelka & Bilmes 2010]

If the rounding procedure is $\alpha$-approximate, then the expected $\alpha$-regret of PSD is bounded by $\sqrt{nT}$.
Online $M^\natural$-Concave Maximization

Proposition (p. 179 in [Murota, 2003])

If $g : \{0,1\}^n \to \mathbb{R} \cup \{+\infty\}$ is $M^\natural$-concave, then its associated set function

$$f : 2^N \to \mathbb{R} \cup \{+\infty\}$$

is submodular.

$M^\natural$-concave maximization is a special case of submodular maximization.

Projected Supergradient Ascent (PSA)

Given a step size $\eta$ and initial $x_1 \in [0,1]^n$, for $t = 1, \ldots, T$ do the following:

1. For each $i \in N$, draw a threshold $\tau_i$ uniformly at random.
2. Play the subset $S_t = \{i : x_t(i) > \tau_i\}$
3. Compute a “supergradient” $p_t$ of $f_t$ at $S_t$, i.e., a $p$ that minimizes

$$p \cdot x_t - f^0(p), \quad p \in \mathbb{Z}^n, \|p\|_\infty \leq 2n$$

4. Take a projected gradient step in the direction of $p_t$ from $x_t$ to get $x_{t+1}$

Theorem [Chen, H. Kawase & Soma]

For $\eta = 1/(2n\sqrt{T})$, the expected $(1 - \frac{1}{e})$-regret of PSA is bounded by $2n^2\sqrt{T}$.
Ongoing Work

Current Work:

▸ Derive a parameter-free version of the online $M^h$-concave maximization algorithm

▸ Study FTPL-type algorithms [Jegelka & Bilmes, 2010]

▸ Hardness conjecture for online $M^h$-concave maximization.

Future Work: Extension to online optimization of functions on $\mathbb{Z}^n$

▸ Inventory Control [Chen & Li, 2021]

▸ Bike Sharing [Freund Henderson & Shmoys 2022], [Shioura 2022]

▸ Games [Fujishige Goemans Harks Peis & Zenklusen, 2015]