TECHNOMETRICS© VOL. 15, NO. 4 NOVEMBER 1973

The Efficiencies of Maximum Likelihood and Minimum Variance Unbiased Estimators of Fraction Defective in the Normal Case

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This paper compares two point estimators of fraction defective of a normal distribution when both population parameters are unknown; the minimum variance unbiased estimator, $\bar{F}(x)$, and the maximum likelihood estimator, $\hat{F}(x)$. Using minimum mean squared error as a criterion, it is shown that the choice of estimator depends upon the true value of $F(x)$, and the sample size. In the domain $.0005 \leq F(x) \leq .50$, the maximum likelihood estimator is generally superior even for small sample sizes, except for $F(x)$ less than about 0.01, or greater than 0.25. Furthermore, the bias in the m.l.e. is slight over much of the domain where this estimator has smaller mean squared error.

As a practical solution to the estimation problem. it is suggested that the m.v.u.e. be calculated, and if this estimate is between 0.01 and 0.25, it should be replaced with the m.l.e. This combined estimator is shown to be nearly as efficient as the better of the m.v.u.e. and m.l.e. throughout the domain of $F(x)$.

KEY WORDS

Estimation
Mean Squared Error Efficiency
Normal Distribution Function
Composite Estimators

1. INTRODUCTION

A common point estimation problem occurs in industrial statistics when manufactured items with some normally distributed property, such as tensile strength, are required to meet a minimum specification. A preliminary estimate of the proportion of product which will not meet the specification is often required to determine

(a) cost estimates for the manufacturing process,

(b) the relative efficiency of two or more alternative methods of manufacture.

A first estimate of fraction defective will usually be based upon a small sample of product, and, in the pilot stage of the process, we may assume that the parameters (μ, σ) of the normal distribution are both unknown. We are not interested in "best" estimates of these parameters, but in an estimate of fraction defective, defined as

$$F(x_0 \, ; \mu, \sigma) = \int_{-\infty}^{(x_0-\mu)/\sigma} (2\pi)^{-\frac{1}{2}} \exp(-t^2/2)\, dt,$$

where x_0 is the minimum specification.

If a random sample, $(x_1 , x_2 , \cdots x_n)$ is available, the *maximum likelihood esti-*

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Received Jan. 1972; revised Aug. 1972

mator, $\hat{F}(x_0)$, for fraction defective may be constructed by evaluating the above integral with the maximum likelihood estimators,

$$\hat{\mu} = 1/n \sum_{i=1}^{n} x_i$$

and

$$\hat{\sigma} = \sqrt{1/n \sum_{i=1}^{n} (x_i - \hat{\mu})^2}$$

inserted in place of μ and σ.

An alternative estimator for $F(x_0 ; \mu, \sigma)$, the *minimum variance unbiased estimator,* $\tilde{F}(x_0)$, was developed by Kolmogorov [4]. The derivation is given in a paper by Lieberman and Resnikoff [5]. The estimator is obtained as the symmetrical incomplete beta function integral,

$$\tilde{F}(x_0) = \int_0^K \left[\frac{\Gamma(n-2)}{\Gamma\left(\frac{n-2}{2}\right)}\right]^2 [q(1-q)]^{(n-4)/2}\, dq,$$

where

$$K = \max\left[0, \tfrac{1}{2} + \tfrac{1}{2}\frac{x_0 - \hat{\mu}}{\hat{\sigma}}(n-1)^{-\frac{1}{2}}\right].$$

Guenther [1] has discussed methods of numerically evaluating this integral.

If unbiasedness is considered absolutely essential for an estimate of fraction defective, then the m.v.u.e. should be used. However, a more important consideration would appear to be the avoidance of large estimation errors. We propose to compare these two common estimators of fraction defective from this standpoint. It seems reasonable to assume that the cost associated with an error in estimating fraction defective is an increasing nonlinear function of the size of the error. For example, the effect of a small error in either direction might well be unimportant. A large overestimate of fraction defective might lead to a decision not to bid for the right to manufacture the product, while a significant underestimate might lead to extensive cost overruns when large scale manufacturing begins. The precise cost-error relationship would be most difficult to specify mathematically. We assume herein that cost is a quadratic function of estimation error. This quadratic cost function is tractable mathematically, and weights larger estimation errors more heavily than small errors. A choice between the m.l.e. and m.v.u.e. of fraction defective may then be made on the basis of expected cost of estimation error [8].

2. Comparison of Estimators

Let $F^*(x_0)$ be any estimator of $F(x_0)$ and let a loss function be defined as

$$L(F^*, F) = C \cdot [F^*(x_0) - F(x_0)]^2,$$

where C is a positive cost function, possibly dependent on $F(x_0)$, but not on $F^*(x_0)$. Then the expected loss from estimation is

$$E[L(F^*, F)] = C \cdot E[F^*(x_0) - F(x_0)]^2.$$

We define the *mean squared error efficiency* in our problem as the expected loss for the m.l.e., $\hat{F}(x_0)$, divided by the expected loss for the m.v.u.e., $\tilde{F}(x_0)$. When

this ratio exceeds unity, the m.v.u.e. is the preferred estimator. Clearly, the preference holds for all positive cost functions, C, defined as above.

The squared error of $\tilde{F}(x_0)$ or $\hat{F}(x_0)$ is a function of the two statistics, $\hat{\mu}$ and $\hat{\sigma}$, plus the four parameters, μ, σ, x_0 , and n. However, the statistics and parameters may be redefined to reduce the squared errors to a function of two statistics and two parameters. Let

$$z = (\hat{\mu} - \mu)\sqrt{n}/\sigma,$$

$$c = (x_0 - \mu)/\sigma,$$

$$W = n\hat{\sigma}^2/\sigma^2.$$

Then

$$\tilde{F}(x_0) = \int_0^{\max[0,\,\frac{1}{2}+\frac{1}{2}w^{-\frac{1}{2}}\cdot(n-1)^{-\frac{1}{2}}\cdot(n^{\frac{1}{2}}c-z)]} \frac{\Gamma(n-2)}{\Gamma\left(\frac{n-2}{2}\right)^2}[q(1-q)]^{(n-4)/2}\,dq;$$

$$\hat{F}(x_0) = \int_{-\infty}^{w^{-\frac{1}{2}}(n^{\frac{1}{2}}c-z)} (2\pi)^{-\frac{1}{2}} \exp(-t^2/2)\,dt;$$

and

$$F(x_0) = \int_{-\infty}^{c} (2\pi)^{-\frac{1}{2}} \exp(-t^2/2)\,dt.$$

Taking the expectation of the squared errors with respect to the random variables, z and w, reduces the problem to a two-dimensional exploration of $\{(c, n)\}$, the proposed policy space for choice of estimators.

The random variable, z, has a unit normal distribution, while w has a chi-square distribution with parameter $(n - 1)$. Furthermore, z and w are stochastically independent, so their joint density is the product of the unit normal and chi-square, or

$$h(z, w) = 2^{-(n-1)/2}\left[\Gamma\left(\frac{n-1}{2}\right)\right]^{-1}(2\pi)^{-\frac{1}{2}}w^{(n-3)/2}\exp(-z^2/2 - w/2];$$

$$-\infty < z < \infty, \quad 0 < w < \infty .$$

The mean squared error efficiency of the m.v.u.e. relative to the m.l.e. is

$$\frac{\int_{-\infty}^{\infty}\int_{0}^{\infty}[\hat{F}(x_0) - F(x_0)]^2 h(z, w)\,dz\,dw}{\int_{-\infty}^{\infty}\int_{0}^{\infty}[\tilde{F}(x_0) - F(x_0)]^2 h(z, w)\,dz\,dw}.$$

It was necessary to evaluate these integrals by numerical methods.

Common multiple numerical quadrature methods include the Monte Carlo, Newton Cotes, and Gaussian techniques. The Monte Carlo method excels at integration of unruly, but analytically given integrands of several variables [6]. Iterated Newton Cotes methods, notably Simpson's Rule, provide good results for the evenly spaced base points of tabulated functions [2]. Gaussian quadrature formulas, especially Gauss–Legendre, produce superior results for smooth integrands given analytically [9]. A comparison of these methods [3, 7] indicates the superiority of Gaussian quadrature methods for the problem at hand.

Integration of the mean squared error efficiency of the m.v.u.e. was performed with the Gauss–Legendre formula over a square lattice with vertices $(\pm 1, \pm 1)$.

Transformations of variables of integration to this domain involved replacement of infinite bounds by heuristic limits derived from the density functions of h, and z. All computations were performed with ten decimal digit accuracy, except for summation operations which carried fourteen. As checks of the accuracy and appropriateness of the method, the 16, 20, and 32-point Gauss–Legendre formulas were used for integration, and the expectation of the m.v.u.e. was evaluated for unbiasedness with agreement to five decimal digits.

5. Results

Mean squared errors were determined for $n = 5(5)30$, 50, and 100. At each value of n, c was assigned eighteen values from the unit normal table corresponding to fraction defective of .0005, .001, .002, .004, .005, .008, .010, .020, .05(.05).50. The mean squared error ratio is shown in Table 1. For fraction defectives above 0.50, the symmetry of the normal distribution permits use of $1 - F(x_0)$ in place of $F(x_0)$ to determine mean squared error.

It is clear from Table 1 that, unless the fraction defective is anticipated to be extremely small (below about .010), or extremely large (above about 0.25) the m.l.e. provides a smaller mean squared error over all the values of n. Furthermore, convergence with increasing n is slow, and even at sample sizes of 20 or more, the efficiency of the m.v.u.e. may be below 0.9. For very small fraction defectives, the advantage of the m.v.u.e. is substantial, and again, is maintained even with relatively large samples.

Table 2 shows the root mean square error of the m.l.e. as a ratio to $F(x_0)$, while Table 3 shows the percentage bias in the m.l.e. An entry in this table of zero would indicate unbiasedness, an entry of +17 would indicate that the m.l.e. is expected to overestimate fraction defective by 17 percent, etc.

The bias in the m.l.e. tends to be positive over the domain $F(x_0) < .05$ and negative in the domain $.05 - .50$. The greatest bias in the m.l.e. occurs for small fraction defective, where the m.v.u.e. is preferable. For large fraction defective,

Table 1

Mean Squared Error Ratio—M.L.E. to M.V.U.E. Enclosed Region Indicates Superiority of M.L.E.

Sample Size n	5	10	15	20	25	30	50	100
Fraction Defective								
.0005	1.81	2.04	1.98	1.89	1.80	1.72	1.51	1.30
.001	1.43	1.65	1.63	1.57	1.52	1.47	1.34	1.20
.002	1.15	1.34	1.35	1.33	1.30	1.28	1.20	1.12
.004	.94	1.12	1.15	1.15	1.14	1.13	1.10	1.06
.005	.89	1.06	1.09	1.10	1.09	1.09	1.07	1.04
.008	.79	.96	1.00	1.01	1.02	1.02	1.02	1.01
.01	.76	.91	.96	.98	.99	.99	1.00	1.00
.02	.68	.82	.88	.91	.92	.93	.96	.98
.05	.66	.79	.84	.88	.90	.91	.95	.97
.10	.73	.83	.88	.91	.93	.94	.96	.98
.15	.82	.90	.93	.95	.96	.97	.98	.99
.20	.91	.96	.97	.98	.99	.99	.99	1.00
.25	1.00	1.01	1.01	1.01	1.01	1.01	1.01	1.00
.30	1.07	1.06	1.04	1.03	1.03	1.03	1.02	1.01
.35	1.13	1.09	1.07	1.05	1.04	1.04	1.02	1.01
.40	1.18	1.12	1.09	1.06	1.05	1.04	1.03	1.01
.45	1.20	1.14	1.10	1.07	1.06	1.05	1.03	1.01
.50	1.21	1.14	1.10	1.07	1.06	1.05	1.03	1.01

TABLE 2

Root Mean Squared Error of M.L.E. as a Proportion of Fraction Defective Being Estimated

Sample Size n	5	10	15	20	25	30	50	100
Fraction Defective								
.0005	14.0	7.5	5.3	4.0	3.5	2.8	1.9	1.1
.001	9.3	5.4	3.9	3.0	2.4	2.2	1.4	1.0
.002	6.3	3.8	2.8	2.3	1.9	1.7	1.2	0.9
.004	4.3	2.8	2.1	1.8	1.5	1.3	1.0	0.7
.005	3.8	2.5	1.9	1.6	1.4	1.2	0.9	0.6
.008	3.0	2.0	1.6	1.3	1.2	1.1	0.8	0.6
.01	2.7	1.8	1.4	1.2	1.1	1.0	0.8	0.5
.02	1.9	1.3	1.1	1.0	0.9	0.8	0.6	0.4
.05	1.3	0.9	0.8	0.7	0.6	0.6	0.4	0.3
.10	1.0	0.7	0.6	0.5	0.5	0.4	0.3	0.2
.15	0.8	0.6	0.5	0.4	0.4	0.3	0.3	0.2
.20	0.7	0.5	0.4	0.4	0.3	0.3	0.2	0.2
.25	0.7	0.5	0.4	0.3	0.3	0.3	0.2	0.1
.30	0.6	0.4	0.3	0.3	0.3	0.2	0.2	0.1
.35	0.6	0.4	0.3	0.3	0.2	0.2	0.2	0.1
.40	0.5	0.3	0.3	0.2	0.2	0.2	0.1	0.1
.45	0.5	0.3	0.2	0.2	0.2	0.2	0.1	0.1
.50	0.4	0.3	0.2	0.2	0.2	0.2	0.1	0.1

the bias is virtually always below -10 percent. The m.l.e. is partciularly superior in the vicinity of 5 percent defective, where there is virtually no bias, and a noticeable mean squared error advantage.

Since each of the two estimators is superior over part of the domain of $F(x_0)$, we might consider, as a practical solution to the estimation problem, using the m.l.e. in others. For example, use the m.v.u.e. to obtain an unbiased estimate of $F(x_0)$. If this estimate is in the domain $.01 < \tilde{F}(x_0) < .25$, replace it with $\hat{F}(x_0)$. Table 4 shows the mean squared error efficiency of this combined estimator relative to the *better* of the m.v.u.e. and m.l.e. at each combination of $[n, F(x_0)]$. There

TABLE 3

Percentage Bias of M.L.E.

Sample Size n	5	10	15	20	25	30	50	100
Fraction Defective								
.0005	+339	+198	+138	+106	+86	+73	+44	+23
.001	+225	+136	+97	+76	+62	+52	+32	+17
.002	+146	+91	+66	+52	+43	+36	+23	+12
.004	+90	+58	+42	+33	+28	+24	+15	+8
.005	+76	+49	+36	+29	+24	+20	+13	+7
.008	+51	+33	+25	+20	+16	+14	+9	+4
.010	+41	+27	+20	+16	+13	+11	+7	+4
.020	+17	+11	+8	+7	+5	+5	+3	+1
.050	-3	-1	-1	-1	-1	-1	0	0
.10	-9	-6	-4	-3	-3	-2	-1	-1
.15	-10	-6	-5	-4	-3	-2	-1	-1
.20	-9	-6	-4	-3	-3	-2	-1	-1
.25	-8	-5	-3	-3	-2	-2	-1	-1
.30	-6	-4	-3	-2	-2	-1	-1	0
.35	-5	-3	-2	-1	-1	-1	-1	0
.40	-3	-2	-1	-1	-1	-1	0	0
.45	-1	-1	-1	0	0	0	0	0
.50	0	0	0	0	0	0	0	0

TABLE 4

Mean Squared Error Ratio—Smaller of (M.L.E., M.V.U.E.) to Combined Estimator. Enclosed Region Indicates Comparison to M.L.E., Remaining Region Comparison to M.V.U.E.

Sample Size n	5	10	15	20	25	30	50	100
Fraction Defective								
.0005	1.02	.72	.81	.88	.94	.97	.99	1.00
.001	.98	.79	.81	.87	.89	.93	.98	1.00
.004	1.16	.92	.87	.88	.89	.90	.92	.97
.008	.99	.95	.95	.93	.92	.93	.93	.95
.01	.98	.94	.94	.94	.94	.94	.94	.95
.02	.88	.93	.94	.95	.95	.95	.96	.97
.05	.81	.95	.98	.99	.99	.99	1.00	1.00
.10	.80	.94	.97	.99	1.00	1.00	1.00	1.00
.15	.84	.92	.96	.97	.98	.99	1.00	1.00
.20	.90	.93	.94	.95	.95	.96	.97	.99
.25	.95	.94	.94	.94	.94	.95	.96	.97
.30	.95	.93	.94	.95	.95	.96	.97	.99
.35	.95	.94	.96	.97	.98	.98	.99	1.00
.40	.95	.95	.97	.99	.99	.99	1.00	1.00
.45	.95	.97	.99	.99	1.00	1.00	1.00	1.00
.50	.96	.98	1.00	1.00	1.00	1.00	1.00	1.00

are 128 combinations of fraction defective and sample size tabulated, and the combined estimator is at least 93 percent efficient in 108 of the 128 cases. Lower efficiencies are restricted almost entirely to very small sample sizes and/or fraction defectives. We also examined the bias of this combined estimator (Table 5). Comparison with Table 3 reveals a substantial improvement over the m.l.e. in this respect.

4. Discussion

Since this paper was completed, Zacks and Milton [10] have published some results on the same topic; mean squared errors were calculated for 3 sample sizes

TABLE 5

Percentage Bias of Combined Estimator

Sample Size n	5	10	15	20	25	30	50	100
Fraction Defective								
.0005	+7	+17	+5	+2	+1	0	0	0
.001	+11	+13	+7	+3	+2	+1	0	0
.004	-3	+10	+10	+5	+4	+3	+2	0
.008	-4	+10	+7	+6	+5	+4	+3	+1
.01	-8	+6	+7	+5	+5	+4	+3	+2
.02	-9	+2	+2	+3	+2	+3	+2	+1
.05	-12	-4	-2	-1	-1	-1	0	0
.10	-10	-6	-4	-3	-3	-2	-1	-1
.15	-7	-5	-4	-3	-3	-2	-1	-1
.20	-5	-4	-3	-2	-2	-2	-1	-1
.25	-4	-2	-2	-1	-1	-1	-1	0
.30	-2	-1	-1	-1	0	0	0	0
.35	-1	-1	0	0	0	0	0	0
.40	-1	0	0	0	0	0	0	0
.45	-1	0	0	0	0	0	0	0
.50	0	0	0	0	0	0	0	0

(n = 6, 12, 18) and $F(x_0)$ in the domain .025 − .50. Their mean squared error ratios agree closely with those in this paper. However, these authors conclude erroneously that the m.l.e. is preferred for all values of $F(x_0)$ below 0.20. In fact, Tables 1 and 2 of the present paper show that the m.l.e. is especially poor for very small values of $F(x_0)$, being highly biased and having large mean squared error relative to the m.v.u.e.

5. Acknowledgment

We wish to thank the referees and the associate editor for several helpful suggestions to improve the manuscript.

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