A COST ANALYSIS OF SAMPLING INSPECTION UNDER MILITARY STANDARD 105D

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ABSTRACT

Military Standard 105D has been almost universally adopted by government and private consumers for the lot-by-lot sampling inspection of product which may be inspected on a dichotomous basis.

The plan specifies, for each lot size, a random sample size and set of acceptance numbers (maximum allowable number of defectives in each sample). The acceptance numbers are based upon the binomial distribution and depend upon the quality required by the purchaser. Where several consecutive lots are submitted, a shift to less severe ("reduced") inspection or more severe ("tightened") inspection is specified when the ongoing quality is very high or low. Further experience permits a return to normal sampling from either of these states.

This paper examines the long range costs of such a sampling scheme. The three inspection types are considered as three distinct Markov chains, with periodic transitions from chain to chain. The expected sample size and the expected proportion of rejected product are determined as a function of the two parameters under control of the manufacturer, lot size and product quality. Some numerical examples are given which illustrate how to compute the overall cost of sampling inspection. Suggestions are made concerning the choice of parameters to minimize this cost.

1. INTRODUCTION

One of the major fields of statistical quality control is acceptance sampling. An increasing number of private and government purchasers of manufactured products require the sampling inspection of each shipment, or lot, and will accept the lot only if a quality standard is passed by the sample. Rejected lots may be returned to the manufacturer, purchased with a price concession, subjected to 100 percent screening, or scrapped. Clearly, there are costs involved for both the sampling inspection and the disposal of rejected lots. In this paper, we examine the cost aspects of sampling inspection according to the most widely used plan in the United States—the attributes sampling scheme designated Military Standard–105D [6].

In attributes sampling plans, a random sample of n individual items is selected from a lot, and each item is classified simply as acceptable or unacceptable. The lot is accepted if c or fewer items out of n are unacceptable. The conditional probability of acceptance, P_a, given that a proportion, p, of the lot is unacceptable, is given approximately by the cumulative binomial probability.

(1) $$P_a = \sum_{y=0}^{c} \binom{n}{y} p^y (1-p)^{n-y}.$$

This assumes that the sample size is small relative to the lot size.

The long-run cost contribution of such a sampling plan may be computed as a function of overall product quality, p, as follows:

Let

C = cost per manufactured item as a result of sampling inspection;

C_1 = cost of inspecting a single item;

C_2 = cost per rejected item for the disposal of rejected lots, e.g., for 100 percent inspection of all items in the lot;

f = fraction of product inspected as sample;

q = fraction of product rejected;

N = size of each lot submitted;

then

$$C = C_1 f + C_2 q;$$

$$E[C] = C_1 n/N + C_2 \left(\frac{N-n}{N}\right)(1 - P_a).$$

For example, lots of size 500 from a process producing 5-percent defectives are submitted to sampling inspection according to the plan $n = 20$, $c = 2$. The cost of sampling inspection is \$2.00 per item and rejected lots must be 100 percent screened also at a cost of \$2.00 per item. By using Equation (1), binomial probability tables [9] give $P_a = 0.925$ and

$$E[C] = 2.00(20/500) + 2.00\left(\frac{480}{500}\right)(0.075)$$

$$= 0.08 + 0.144$$

$$= \$0.224.$$

This is the expected cost per manufactured item as a consequence of sampling inspection.

In practice, most attribute sampling plans have somewhat more complicated rules for their application. Dodge and Romig [1] have devised a set of attribute plans based upon minimum cost, assuming a desired incoming quality, with specific requirements for consumer protection against very bad lots. Hald [3] has greatly expanded this idea, and developed sampling plans which minimize cost for any prior distribution. However, today by far the most popular set of plans is that published by the Department of Defense in 1963, known as MIL–STD–105D. It is estimated that more than 13,000 industrial plants in the western world are currently involved with this standard, either as vendors or purchasers.

The sampling plans in MIL–STD–105D are not based upon the minimum-cost concepts of Hald. The plans are indexed by lot size and by a number designated "Acceptable Quality Level." The AQL is always specified by the purchaser, and is defined as the value of p which will lead to a high conditional probability of acceptance, P_a. The precise value of P_a corresponding to the AQL varies with lot size and AQL. The domain for P_a is about 0.89–0.99.

TABLE 1. *Some representative sample sizes and acceptance numbers (shown in body of table) from MIL–STD–105D, Table II–A, for various lot sizes and* AQL *percent defective. The arrow signifies to use the plan on the row at the end of the arrow. Plans shown are for general inspection level II in MIL–STD–105D.*

Lot size	Sample size	*AQL*							
		0.40	0.65	1.0	1.5	2.5	4.0	6.5	10.0
2–8	2						↓	0	
9–15	3					↓	0	↑	↓
16–25	5				↓	0	↑	↓	1
26–50	8			↓	0	↑	↓	1	2
51–90	13		↓	0	↑	↓	1	2	3
91–150	20	↓	0	↑	↓	1	2	3	5
151–280	32	0	↑	↓	1	2	3	5	7
281–500	50	↑	↓	1	2	3	5	7	10
501–1,200	80	↓	1	2	3	5	7	10	14
1,201–3,200	125	1	2	3	5	7	10	14	21
3,201–10,000	200	2	3	5	7	10	14	21	↑
10,001–35,000	315	3	5	7	10	14	21	↑	
35,001–150,000	500	5	7	10	14	21	↑		
150,001–500,000	800	7	10	14	21	↑			
500,001 up	1,250	10	14	21	↑				

Table 1 shows some typical plans for normal sampling inspection from MIL–STD–105D. Once a sampling plan has been selected, the user is required to keep a historical record of lot-by-lot experience. Criteria are presented in MIL–STD–105D for an alteration of the values of n and c when the experience over several lots shows either unusually good or unusually bad quality. The rules are as follows [6]:

1. A switch from the normal values of n and c to "reduced" inspection is permissible when:

a. Ten consecutive lots have been accepted.

b. The total number of defectives in the 10 lots does not exceed a critical value supplied in Table VIII of MIL–STD–105D. Some examples of these critical values are shown in Table 2.

c. Production is continuous.

d. Reduced inspection is considered desirable by the responsible authority.

Under reduced inspection, n is substantially decreased. Two numbers, c and $r(>c)$ are supplied. Lots are accepted if the number of defectives is less than r. However, if a lot has more than c defectives, normal inspection must be resumed on the next lot.

2. A switch to "tightened" from normal inspection is required when two of the most recent five lots have been rejected. Under tightened inspection, n is the same as for normal inspection, but c is reduced. A return to normal is permitted when five consecutive lots have been accepted. If tightened inspection is still in use for 10 consecutive lots, however, sampling inspection must be discontinued entirely.

Hence, in attributes sampling by this scheme, the long run costs will be dependent on the proportion of lots inspected under normal, reduced, and tightened sampling plans. Koyama [4] has examined the severity of the switching rules in MIL–STD–105D, using signal flow graph theory. Mendizabal [5] has performed some limited Monte Carlo studies in the same area. The Koyama results indicate that, for lots with consistent *AQL* quality, the switching rules are severe, with the probability of entering tightened inspection relatively high.

TABLE 2. *Some limit numbers from MIL–STD–105D, Table VIII for transfer from normal to reduced inspection. Numbers in body of table are maximum number of defectives in most recent 10 lots.*

Number of items in last 10 lots	*AQL*							
	0.40	0.65	1.0	1.5	2.5	4.0	6.5	10.0
20–29								0
30–49							0	0
50–79						0	0	2
80–129					0	0	2	4
130–199				0	0	2	4	7
200–319			0	0	2	4	8	14
320–499		0	0	1	4	8	14	24
500–799	0	0	2	3	7	14	25	40
800–1,249	0	2	4	7	14	24	42	68
1,250–1,999	2	4	7	13	24	40	69	110
2,000–3,149	4	8	14	22	40	68	115	181
3,150–4,999	8	14	24	38	67	111	186	
5,000–7,999	14	25	40	63	110	181		
8,000–12,499	24	42	68	105	181			
12,500–19,999	40	69	110	169				
20,000–31,499	68	115	181					
31,500–49,999	111	186						
50,000 up	181	301						

In the present paper, we develop, for MIL–STD–105D an expression for expected inspection cost as a function of process quality, p, and lot size, N, the two parameters which may be controlled, to some extent, by the manufacturer. We consider on-going inspection as a dynamic process, where recycling among normal, reduced, and tightened inspection will occur as a large number of lots is submitted. The solution produces the expected proportion of lots on normal, reduced, and tightened inspection, plus the expected proportion of rejected lots, for each (AQL, p, N) combination. With appropriate assumptions on the disposal of rejected lots and the cost of inspection, the expected cost of a MIL–STD–105D plan may then be obtained.

2. THEORY

We make two assumptions throughout this section. First, it is assumed that rejected lots incur a known cost per item for every item in the lot as a consequence of 100 percent screening. The second assumption concerns the re-institution of sampling inspection after it has been discontinued. MIL–STD–105D, section 8.4, states "In the event that 10 consecutive lots or batches remain on tightened inspection (or such other number as may be specified by the responsible authority), inspection under the provisions of this document should be discontinued pending action to improve the quality of submitted material." We assume that this cessation is in force for the next 10 lots, which will be screened 100 percent; normal sampling inspection will be resumed after this screening period.

The following additional notation will be needed in this section:

N = lot size;

n = sample size under normal inspection;

n_T = sample size under tightened inspection;

n_R = sample size under reduced inspection;

$P_{A,N}$ = probability of accepting a lot in normal inspection;

r^* = maximum number of defectives in 10 accepted lots under normal inspection to qualify for reduced inspection (Table VIII, MIL–STD–105D);

$P'_{A,N}$ = probability of meeting, at the first opportunity, the maximum defectives criterion of Table VIII for a transfer to reduced inspection, given that the most recent 10 lots are accepted;

$P_{A,T}$ = probability of accepting a lot in tightened inspection;

$P'_{A,R}$ = probability of accepting a lot in reduced inspection, but with a return to normal required;

$P_{A,R}$ = probability of accepting a lot in reduced inspection, with a continuation of reduced inspection;

X_N = number of adoptions of normal sampling during the inspection of L lots;

X_R = number of adoptions of reduced sampling during inspection of L lots;

X_T = number of adoptions of tightened sampling during inspection of L lots.

After L lots have been inspected under the MIL–STD–105D scheme, the total number of items in these lots will be $N \cdot L$. Suppose that l_n of these lots have been under normal inspection, l_r under reduced, and l_t under tightened. Then the fraction of product inspected by sampling will be

$$f = [l_R n_R + l_N n + l_T n_T]/NL. \tag{2}$$

Note that $l_N + l_R + l_T \leq L$, in general, since sampling inspection may be discontinued prior to the Lth lot.

The fraction of product subjected to 100-percent screening will be

$$E[q \mid l_N, l_T, l_R, X_R] = \left\{L - l_N P_{A,N} - l_T P_{A,T} - l_R + X_R\left[\frac{1 - P_{A,R} - P'_{A,R}}{1 - P_{A,R}}\right]\right\}/L. \tag{3}$$

Finally, the expected cost of sampling will be

$$E[C] = C_1 E[f] + C_2 E[q \mid l_N, l_T, l_R, X_R]. \tag{4}$$

Examining f and q, we note that these variables are functions of the parameters N, $P_{A,N}$, $P_{A,R}$, $P'_{A,R}$, $P_{A,T}$ and are linear functions of the variables l_N, l_R, l_T, X_R. It remains to find the numerical values for the parameters and the expectations of the four random variables.

Once a lot size, N, AQL, and inspection level have been specified, MIL–STD–105D provides both the sample size and acceptance number for normal, tightened, and reduced inspection. When product of quality p is supplied, the values of $P_{A,N}$, $P_{A,T}$, $P_{A,R}$, $P'_{A,R}$ may be computed directly from Equation (1) or other appropriate binomial sums. In practice, the Poisson approximation,

$$P_A = \sum_{y=0}^{c} (np)^y \exp(-np)/y!, \tag{5}$$

is usually employed, since p is small and n reasonably large. Molina [7] gives extensive tables of these Poisson probabilities. In the numerical examples which follow, Poisson probabilities were used in place of the binomial.

The computation of $P'_{A,N}$, the probability that a sequence of 10 accepted lots on normal inspection will meet the criterion of Table VIII in MIL–STD–105D for a transfer to reduced inspection, can best be shown through a numerical example. Suppose that an AQL of 1.5 and lot size of 400 is specified, and that material of AQL quality is submitted. The sampling plan is $n=50$, $c=2$; r^*, the limit for transfer to reduced inspection is 3, or fewer in 10 lots. Now, each of the 10 accepted lots contains two, one, or zero defectives. Hence, a truncated Poisson distribution is appropriate for the distribution of defectives in these lots:

c	Poisson probability	Conditional probability
0	0.4724	0.4923
1	0.3543	0.3692
2	0.1328	0.1385

The probabilities in column 3 were obtained by dividing the values in column 2 by their sum.

Next, $P'_{A,N}$ is the sum of the following multinomial probabilities:

Ten-sample Σc_i	Probabilities
3	$90(0.4923)^8(0.3692)(0.1385)+120(0.4923)^7(0.3692)^3$
2	$10(0.4923)^9(0.1385)+45(0.4923)^8(0.3692)^2$
1	$10(0.4923)^9(0.3692)$
0	$(0.4923)^{10}$

which gives, for this example, $P'_{A,N}=0.0890$.

For the general case, let

(6) $$f(c_i),\ c_i=0, 1, \ldots, c$$

be the truncated Poisson probability distribution for lot i, accepted under normal inspection, and

$$g_k(s),\ s=0, 1, \ldots, kc$$

be the distribution for the total number of defectives in k accepted lots. Then the probability distribution for s is the convolution

$$g_{10}(s)=f(c_1)*f(c_2)* \ldots *f(c_{10});$$

$$\sum_{i=1}^{10} c_i = s;\ s=0, 1, \ldots, 10c;$$

while

(7) $$P'_{A,N}=\sum_{s=0}^{r^*} g_{10}(s).$$

The calculation of $P'_{A,N}$ becomes lengthy, particularly for large lots and/or *AQL* values, where the number of partitions of Σc_i into 10 sample points is large. However, these cases may be handled with a normal approximation, as a consequence of the central limit theorem. If μ and σ are the parameters of a truncated Poisson distribution, then the sum of 10 random observations is approximately normally distributed with parameters 10μ and $\sqrt{10}\,\sigma$. We have tested this approximation and found it to be excellent for large values of the limit numbers in Table VIII. Even for the numerical example above, where c and r^* are 2 and 3, respectively, the normal approximation is reasonably good. The mean and standard deviation for the conditional Poisson distribution are 0.6462 and 0.7111. Hence, for the sum of 10 such observations, μ is 6.462 and σ is 2.248. Then

$$z = \frac{3.5 - 6.462}{2.248} = -1.32;$$

$$P'_{A,N} = P\left(\sum_{i=1}^{10} c_i \leqslant 3\right) \simeq .093,$$

while the exact multinomial probability is 0.089.

Consider, now, the state of the system when 10 consecutive lots are accepted under normal inspection, but the sum of the number of defectives in the 10 lots exceeds r^* in Table VIII. In the calculations of the next section, we will require two types of probabilities for these lots. First, the probability distribution for the total number of defectives in the 10 lots will be

$$g_{10}(s') = g_{10}(s \mid s > r^*) = g_{10}(s) / \sum_{r^*+1}^{10c} g_{10}(s),\ s' = r^* + 1, \ldots, 10c. \tag{8}$$

Secondly, the probability distribution for the number of defectives in a random lot chosen from the 10 will depend on the numerical value of s':

$$f(c_i') = f(c_i \mid s') = f(c_i) \cdot g_9(s' - c_i) / \sum_{c_i=0}^{c} f(c_i) \cdot g_9(s' - c_i),\ c_i' = 0, 1, \ldots, c. \tag{9}$$

Continuing the previous numerical example, where $n = 50$, $c = 2$, $r^* = 3$, and the process quality is 1.5-percent defective, assume that 10 consecutive lots have been accepted, but that these lots contain a total of exactly 5 defectives. By using the reasoning which lead to Equations (6) and (9), we have

c_i	$f(c_i)$	$g_9(5 = c_i)$	Joint	$f(c_i')$
0	0.4923	0.1697	0.0835	0.5743
1	0.3692	0.1403	0.0518	0.3562
2	0.1385	0.0729	0.0101	0.0695

3. MARKOV CHAIN SOLUTION OF THE INSPECTION SCHEME

The three inspection plans may be separately formulated as finite state, discrete parameter Markov chains [8]. Table 3 illustrates the state space and transition probability matrix for normal inspection. The entries, ρ_a, represent $P_{A,N}$, and

$$\rho_r = 1 - \rho_a.$$

This matrix may be partitioned as indicated into the canonical form

$$\left(\begin{array}{c|c} I & 0 \\ \hline R & Q \end{array}\right)$$

The experiencing of a lot rejection during normal inspection forces entry into one of the transition states provided for all permutations of one rejection in the most recent j lot inspections, $j=1, 2, 3, 4$. Four acceptances subsequent to a rejection cause transition back to the "4*A*" state. A second failure in five, or less, consecutive lot inspections forces absorption into tightened inspection.

Transition states, labeled "*KA*," represent K consecutive lot acceptances, $K=1, 2, \ldots 9$. From state 9*A*, we have indicated a probability, ρ_a^* of absorption into reduced inspection. This is the joint probability that a tenth lot is accepted and that the limit number of Table VIII is not exceeded:

$$\rho_a^* = P_{A,N} P'_{A,N}. \tag{10}$$

From state 9*A*, we enter either reduced inspection with probability ρ_a^*, state *AAAR* with probability ρ_r, or state 10*A* with probability $1-\rho_a^*-\rho_r$. State 10*A* implies 10 consecutive lot acceptances with the total defectives in the 10 lots greater than the limit number of Table VIII.

State 10*A* is a transient, not an absorbing state, since it is possible with additional inspections to reject a lot (moving to state *AAAR*) or to reach the limit number, r^*, for the 10 most recent lots, moving to reduced inspection. These are the only possible exits from state 10*A*. The probability of emerging to reduced inspection has been designated in Table 3 as ρ_a^{**}, and to *AAAR*, $1-\rho_a^{**}$.

Calculation of Probabilities For State 10A

Entrance into state 10*A* for the first time implies that the total number of defectives in the latest 10 lots is $r^*+1, r^*+2, \ldots, 10c$. The probability distribution for these states is given by equation (8). Now, as a new lot is sampled, 1 of the initial 10 lots containing $0, 1, \ldots, c$ defectives is replaced with a lot containing $0, 1, \ldots, n$ defectives. If this new lot has more than c defectives, there is an exit from 10*A* to *AAAR*. This probability is simply ρ_r. Otherwise, the new lot is accepted, and we must examine the possible transitions from r^*+1, r^*+2, etc., to each other and to reduced inspection. A limited number of such transitions are possible, since the total number of defectives cannot change by more than $\pm c$ as a single lot is replaced.

The process of replacing the original 10 lots, 1 by 1, with new lots may be represented as a step-by-step alteration of the probabilities in a transition matrix whose columns and rows represent the number of defectives in the "old" and "new" lots. We may illustrate this semi-Markov process with the numerical example presented in section 2 of this paper ($n=50$, $c=2$, $r^*=3$, process quality 1.5-percent defective). Assume that state 10*A* has just been entered for the first time. Then the possible states are 4, 5, . . ., 20 defectives in the 10 lots. Using exact multinomial probabilities, Equations (6), (7), and (8) yield, for those states with probability at least 0.001,

Number of defectives in 10 old lots										
4	5	6	7	8	9	10	11	12	13	14
0.117	0.166	0.191	0.182	0.145	0.098	0.056	0.028	0.011	0.004	0.001

Deletion of a lot randomly chosen from these 10 reduces the total number of defectives by 0, 1, or 2, and produces a new probability vector for states 2, 3, . . ., 18. To calculate these probabilities, we must examine each starting state, 4, 5, . . ., 20 individually, since $f(c_i')$, as shown in Equation (9), is a function of the exact number of defectives in the 10 lots. For example, using the numerical result calculated in section 2 for 5 defectives in 10 lots, we have

c_i'	$f(c_i')$
0	0.5743
1	0.3562
2	0.0695

These are precisely the conditional probabilities of moving from state 5 to 5, 4, and 3, respectively, after one lot has been deleted; when multiplied by 0.166, the prior probability for state 5, we have the contribution of prior state 5 to the probability vector for the number of defectives in 9 lots. The process must be repeated for each prior state.

The deleted lot will be replaced with a new unconditional lot with probability distribution

c	Poisson probability
0	0.4724
1	0.3543
2	0.1328
(Reject)	0.0405

Since the probability distributions for defectives in this new lot and in the nine old lots are independent, we may multiply them to form a transition matrix. A portion of the resulting matrix is shown below:

Defectives in new lot	Defectives in 9 old lots									
	2	3	4	5	6	7	8	9	10	11
0	0.003	0.018	0.067	0.090	0.096	0.083	0.058	0.032	0.015	0.005
1	0.002	0.016	0.050	0.067	0.072	0.062	0.043	0.024	0.011	0.004
2	0.001	0.006	0.019	0.025	0.027	0.023	0.016	0.009	0.004	0.001
(Reject)	(sum of this row = 0.0405)									

TABLE 3. *Normal inspection state space and transition probabilities*—Continued

States	Transition probabilities																						
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
Reduced 1	1																						
Tightened 2		1																					
10A 3	ρ_a^{**}															$1-\rho_a^{**}$							
9A 4	ρ_a^*		$1-\rho_a^*-\rho_r$													ρ_r							
8A 5				ρ_a												ρ_r							
7A 6					ρ_a											ρ_r							
6A 7						ρ_a										ρ_r							
5A 8							ρ_a									ρ_r							
4A 9								ρ_a								ρ_r							
3A 10									ρ_a							ρ_r							
2A 11										ρ_a									ρ_r				
1A 12											ρ_a										ρ_r		
RAAA 13		ρ_r							ρ_a														
ARAA 14		ρ_r											ρ_a										
AARA 15		ρ_r												ρ_a									
AAAR 16		ρ_r													ρ_a								
RAA 17		ρ_r											ρ_a										
ARA 18		ρ_r												ρ_a									
AAR 19		ρ_r													ρ_a								

TABLE 3. *Normal inspection state space and transition probabilities* — Continued

RA 20		ρ_r															ρ_a					
AR 21		ρ_r																ρ_a				
R 22		ρ_r																		ρ_a		
Start 23												ρ_a									ρ_r	

The events enclosed by the broken line will lead to a transfer to reduced inspection, while the entire last row will lead to a transfer to normal inspection state *AAAR*.

We may now use the remainder of the transition matrix as a new prior state, delete a second old lot and add a second new lot. For elements in each column, recalculation of $f(c_i')$ is necessary using $g_8(s' - c_i)$, etc. The row probabilities are altered by applying unconditional Poisson probabilities for a single lot. For example, the prior state (1, 4) can transfer to (1, 4), (1, 3), (1, 2), (2, 4), (2, 3), (2, 2), (3, 4), (3, 3) (3, 2), the six states representing 0, 1, or 2 deletions, and 0, 1, or 2 additions.

The next matrix will have the appearance:

Defectives new lots	Defectives in 8 old lots							
	0	1	2	3	4	5	. . .	16
0								
1	(Reduced)							
2								
3								
4 (Reject)								

In general, the dimensionality of this matrix will begin at 1 by $10c+1$, change to $c+1$ by $9c+1$, $2c+1$ by $8c+1$, etc., until the tenth deletion yields a $10c+1$ by 1 array. After 10 deletions, this final column vector will represent the probability of states r^*+1, r^*+2, . . ., $10c$ for the beginning of a second cycle of deletions and additions. Calculations will be identical to those for the first cycle, except for the initial probability vector. Recycling may be continued until the initial probability vector for states r^*+1, . . ., $10c$ contains negligible elements. At that point, the total probability of transfer to reduced inspection and to normal state *AAAR* will be available for insertion into the normal transition matrix (Table 3).

The expected number of lots processed per entry into state 10*A* may be readily computed from the probabilities of exiting to reduced inspection and normal state *AAAR* at each stage of deletion and addition. Thus, state 10*A* in Table 3 may be treated exactly the same as the other transient states in the normal transition matrix, except that, in any analysis of the expected number of lots processed under each transient state, we must multiply the number of entries into state 10*A* by the expected number of lots under "10*A*" inspection per entry.

Analysis of the Canonical Matrix

The expected transitions, $v_{i,j}$, for any starting state i and transient state j may be found by forming the fundamental matrix [8]

$$\mathbf{V} = \mathbf{I} + \mathbf{Q} + \mathbf{Q}^2 + \ldots.$$

It may be shown that

$$\mathbf{V} = (\mathbf{I} - \mathbf{Q})^{-1}$$

and that the expected number of lots inspection during a single adoption of normal inspection is

$$E[m_n] = \sum_j v_{\text{start}, j}.$$

When departure from normal inspection occurs, the probabilities, $b_{i,j}$, of absorption into state j, given starting state i, are found by

$$\mathbf{B} = \mathbf{V}\ \mathbf{R}.$$

For normal inspection, the probability of eventual absorption into reduced inspection is

$$a_{N,R} = b_{\text{start, reduced}} = v_{\text{start}, 9A}\ \rho_a^* + v_{\text{start}, 10A}\ \rho_a^{**},$$

and the probability of absorption into tightened inspection is

$$a_{N,T} = 1 - a_{N,R}.$$

TABLE 4. *Tightened inspection*

States		Transition probabilities 1	2	3	4	5	6
Normal	1	1					
4A	2	ρ_a					ρ_r
3A	3		ρ_a				ρ_r
2A	4			ρ_a			ρ_r
1A	5				ρ_a		ρ_r
Start	6					ρ_a	ρ_r

Table 4 shows the state space and transition probability matrix for tightened inspection. ρ_a represents $P_{A,T}$,

$$\rho_r = 1 - \rho_a.$$

States are provided for K consecutive lot acceptances, $K = 0, 1, \ldots, 5$, with $K = 5$ an absorption back to normal inspection, and $K = 0$ the starting state.

The expected number of lots inspected in an adoption of tightened inspection, $E[m_t]$, may be found for the maximum of 10 lots inspected on this plan by borrowing notation from the normal inspection model and determining

$$\mathbf{V} = \mathbf{I} + \mathbf{Q} + \mathbf{Q}^2 + \ldots + \mathbf{Q}^9;$$

$$E[m_T] = \sum_j v_{\text{start}, j}.$$

The probability of return to normal inspection is

$$a_{T,N} = b_{\text{start, normal}} = v_{\text{start, }4A} \cdot P_{A,T}.$$

The probability of discontinuation of inspection is

$$1 - a_{T,N}.$$

TABLE 5. *Reduced inspection*

States		Transition probabilities 1	2	3
Normal/Accept	1	1		
Normal/Reject	2		1	
Start	3	p'_a	p_r	p_a

Reduced inspection transition probabilities and state space are shown in Table 5. p_a represents $P_{A,R}$, p'_a is $P'_{A,R}$, and

$$p_r = 1 - p_a - p'_a.$$

Using theory equivalent to that for normal inspection, the expected number of lots inspected for each adoption of reduced inspection may be simply determined as

$$E[m_R] = (1 - P_{A,R})^{-1}.$$

The probability of return to normal inspection with a rejected lot is

$$a_{R,N} = (1 - P_{A,R} - P'_{A,R})E[m_R],$$

$$a_{R,N} = (1 - P_{A,R} - P'_{A,R})/(1 - P_{A,R}).$$

For the duration of a MIL–STD–105D inspection scheme, the expected number of adoptions of normal, tightened, and reduced inspection may be obtained from the "meta" transition matrix shown in Table 6. Solution of this canonical matrix yields (assuming a start on normal inspection):

TABLE 6. *Meta-transition matrix*

States		1	2	3	4
Discontinue	1	1			
Tightened	2	$1 - a_{T,N}$		$a_{T,N}$	
Normal	3		$a_{N,T}$		$a_{N,R}$
Reduced	4			1	

Expected number of adoptions of	Equation
Reduced	$E[X_R] = a_{N,R}/(1 - a_{N,R} - a_{N,T}a_{T,N})$
Tightened	$E[X_T] = a_{N,T}/(1 - a_{N,R} - a_{N,T}a_{T,N})$
Normal	$E[X_N] = 1/(1 - a_{N,R} - a_{N,T}a_{T,N})$.

Then

$$E[l_R] = E[m_R]E[X_R],$$

$$E[l_T] = E[m_T]E[X_T],$$

$$E[l_n] = E[m_N]E[X_N].$$

4. SOME NUMERICAL RESULTS

Table 7 shows, for several AQL values and lot sizes, the results of submitting lots of consistent AQL quality to the MIL–STD–105D inspection scheme. We note from the Markov chain analysis that the probability of eventual discontinuation of sampling inspection is 1.0 if $p > 0.0$ since this is the final absorbing state in the Markov chains. The expected number of lots to absorption, the propor-

TABLE 7. *Some representative results from Markov solution of MIL–STD–105D inspection schemes when lots of* AQL *quality are submitted. Percentages shown are for the lots inspected up to the discontinuation point.*

AQL	Lot size	Expected number of lots before discontinuation	Expected percentage of lots on—			Expected percentage of lots rejected
			Normal	Tight	Red	
.25	90–1,200	107	74.2	25.8	0.0	21.5
	1,201–10,000	204	76.3	12.6	11.1	14.1
	10,001–35,000	763	63.5	3.2	33.3	4.9
1.00	16–150	104	73.4	26.6	0.0	22.0
	151–500	198	76.3	12.6	11.1	14.1
	501–1,200	671	65.2	3.6	31.2	5.3
	1,201–3,200	1,324	86.0	2.9	11.1	4.4
	3,201–10,000	5,465	85.1	0.6	14.3	1.7
2.50	9–50	107	74.2	25.8	0.0	21.5
	51–150	198	76.3	12.6	11.1	14.1
	151–280	666	65.2	3.6	31.2	5.3
	281–500	1,324	86.0	2.9	11.1	4.4
	501–1,200	5,465	85.1	0.6	14.3	1.7
	1,201–3,200	12,921	84.3	0.5	15.2	1.4
4.0	6–15	110	75.0	25.0	0.0	20.9
	16–50	179	76.1	13.9	10.0	15.2
	51–150	671	65.2	3.6	31.2	5.3
	151–280	1,125	82.1	3.2	14.7	4.7
	281–500	5,465	85.1	0.6	14.3	1.7
	501–1,200	8,623	86.6	0.6	12.8	1.7

tion of lots on normal, reduced, and tightened inspection, and the proportion of rejected lots has been indicated for each lot size. The results confirm the observation by Koyama [4] that switching rules in MIL–STD–105D are severe. Tightened inspection is frequent for small lot sizes, and the bulk of *AQL*-quality lots will undergo normal, rather than reduced, inspection.

A curious feature of MIL–STD–105D is that for some *AQL*'s the proportion on reduced inspection tends to be higher for small lots than for large lots. This is primarily a consequence of more stringent requirements to remain on reduced inspection. For example, Table IIC of MIL–STD–105D shows that, at $AQL = 1.0$, lots of size 1,200 require one or less defectives in 32 observations, while lots of size 3,200 require one or less defectives in 50 observations to remain on reduced inspection.

Table 8 shows the impact of incoming quality on the proportion of lots under normal, tightened, and reduced inspection for a particular (AQL, N) combination. The table reveals that more than one-half of the lots will be rejected, and sampling inspection will be discontinued in short order if p is twice the *AQL*. If p is half the *AQL* value, nearly all lots will be under reduced inspection, with virtually no rejections.

5. EXAMPLE OF COST CALCULATIONS

Consider the following problem. A manufacturer produces metal castings. The quality characteristic to be examined is the presence, or absence, of hot cracks to be fluoroscopically detected. Production is continuous and the manufacturer has some control, through negotiations with the purchaser, over lot size, which may vary from 50 to 550 castings.

The purchaser has specified MIL–STD–105D sampling with an Acceptable Quality Level of 4.0 percent defectives. Sampling inspection costs are estimated at $1.50 per casting inspected. Rejected lots must be subjected to 100 percent screening, also at a cost of $1.50 per casting.

The purchaser specifies that if sampling inspection is discontinued, the manufacturer must subject 10 subsequent lots to 100 percent screening in order to resume normal inspection again.

Suppose, for example, that the manufacturer produces material at the *AQL* level of quality (4.0 percent defectives), and submits lots of size 150. From MIL–STD–105D, the appropriate sampling plans are:

Type of sampling	n	c	r	r^*
Normal	20	2	–	4
Reduced	8	1	3	–
Tightened	20	1	–	–

TABLE 8. *Some results from Markov solution of MIL–STD–105D inspection with lot size at 150 and AQL at 4.0. Percentages shown are for the lots inspected up to the discontinuation point.*

Incoming quality, percent defective	Expected number of lots before discontinuation	Expected percentage of lots on—			Expected percentage of lots rejected
		Normal	Tight	Red	
2.0	664,666	12.8	0.0	87.2	0.2
3.0	8,800	36.8	0.5	62.7	1.1
4.0	671	65.2	3.6	31.2	5.3
5.0	154	78.0	11.6	10.4	15.3
6.0	69	74.4	23.0	2.6	28.8
7.0	45	65.0	34.5	0.5	41.7
8.0	35	55.5	44.4	0.1	52.2

The Markov chain analysis revealed that 671 lots are expected to be examined before discontinuation of sampling inspection, and that

431 lots are expected to be on normal inspection,
24 lots are expected to be on tightened inspection,
206 lots are expected to be on reduced inspection.

The expected number of rejected lots is 37. Hence, 47 lots out of 671 will be expected to be given 100 percent screening.

When these expected values are used, Equations (2), (3), and (4) yield:

$$E(f)=0.107;$$

$$E(q)=0.053;$$

$$E(c)=1.50(0.107)+1.50(0.053)$$

$$=\$0.24 \text{ per casting.}$$

Figure 1 shows the average inspection cost per unit, *E(C)*, as a function of lot size, *N*. The discontinuities in cost occur at the lot sizes where a new acceptance number and sample size take effect.

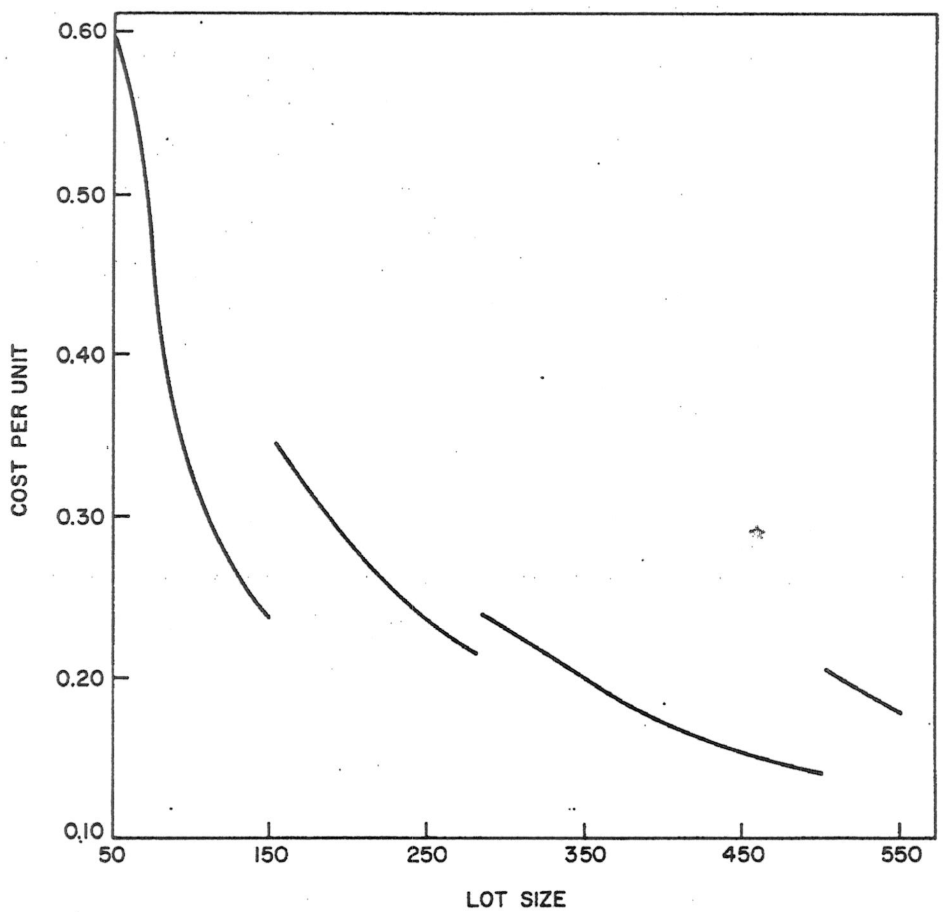

FIGURE 1. Cost per manufactured item as a function of lot size, $AQL=4.0$, $p=AQL$.

These critical lot sizes are shown in Table 1 and are the same for normal, tightened, and reduced inspection in MIL–STD–105D.

Figure 1 shows that the set of interesting alternatives occurs in the vicinity of lot sizes equal to the upper boundaries shown in Table 1, namely 50, 90, 150, 280, 500. The manufacturer should produce in lots of 500, and the cost per casting will be $0.14. Small lot sizes, say 60, could result in a cost per casting more than triple this figure.

Table 9 shows average cost per unit as a function of incoming quality, at a lot size of 150. Costs are extremely sensitive to p. Submission of 5 percent defective material, rather than 4 percent, to a 4.0 percent AQL plan causes a 67 percent increase in inspection costs.

TABLE 9. *Cost per Unit for Sampling Inspection with Lot Size 150* AQL = 4.0. *Calculations assume that rejected lots are screened 100 percent and that discontinuation of sampling requires screening of 10 lots.*

Incoming percent defective	Expected cost per unit
2.0	$0.10
3.0	0.14
4.0	0.24
5.0	0.40
6.0	0.60
7.0	0.78
8.0	0.93

6. CONCLUSION

Analysis of these inspection schemes over realistic domains for AQL, p, and N reveals that the solution set will always occur in the vicinity of critical lot sizes as in the example. Thus, having estimated the costs of inspection for a product, a manufacturer need only perform this analysis for feasible lot sizes from the set 50, 90, 150, . . ., and in cases for which the maximum feasible lot size is not a critical lot size, this maximum should be examined as well.

The cost of inspection is very sensitive to lot quality, principally because to achieve and remain on reduced inspection very high quality lots must be submitted. Even when consistent AQL quality material is submitted to a MIL–STD–105D inspection scheme, a substantial number of lots may be subjected to tightened inspection when lot sizes are small. This implies that the switching rules in the document may be too severe, as suggested by other authors.

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