III. Nonparametric Spectrum Estimation for Stationary Random Signals - Non-parametric Methods -

Goal: To estimate the Power Spectral Density (PSD) of a wss process.

- Two main types of approaches exist:
    - Non parametric methods:
        - do not rely on data model
        - based on FT and FFT

Section focus

    - Parametric methods:
        - based on different parametric models of the data:
            - Moving Average: MA,
            - AutoRegressive: AR,
            - AutoRegressive & Moving Average: ARMA

        - based on subspace methods

Classical methods (discussed here)

→ use FT of the data sequence or its correlation sequence

1) Periodogram

- Easy to compute
- Limited ability to produce accurate PS estimate

Recall PSD defined as:

$$\boxed{P_x\left(e^{j\omega}\right) = \sum_{k=-\infty}^{\infty} R_x(k) e^{-j\omega k}}$$

In practice: $R_x(k)$ is approximated with estimated correlation sequence $\hat{R}_x(k)$

Recall: Biased correlation estimate of *x(n), n=0,…N-1*, defined as

$$\hat{R}_x(k) = \frac{1}{N}\sum_{k=0}^{N-1-k} x(\ell+k)\,x^*(\ell) \qquad 0 \le k < N-1$$

$$\downarrow$$

$$\boxed{\hat{P}\left(e^{j\omega}\right) = \sum_{k=-L}^{L} \hat{R}_x(k)\,e^{-j\omega k}} \qquad L \le N$$

$L \le 10\%\ N$ $\Rightarrow$ PSD estimate sometimes known as *correlogram*

Assume L=N-1

$$\hat{P}\left(e^{j\omega}\right) = \sum_{k=-N+1}^{N-1} \hat{R}_x(k)\,e^{-j\omega k} \Rightarrow \textit{periodogram}$$

Recall

$$\hat{R}_x(k) = \frac{1}{N}\sum_{k=0}^{N-1-k} x(\ell+k)\,x^*(\ell), \quad 0 \le k < N-1$$

$$= \frac{1}{N} x(k) * x^*(-k), \text{ for } x(k) \neq 0 \quad 0 \le k < N-1$$

Define:

$$x_N(n) = x(n)\cdot w(n) \xrightarrow{\text{DFT}} X_N(k)$$

$$\Rightarrow \hat{P}_x(e^{j\omega}) = FT\left[\frac{1}{N} x_N(k) * x_N^{\;*}(-k)\right]$$

$$= \frac{1}{N} X_N(e^{j\omega})\, X_N(e^{j\omega})^*$$

$$\boxed{\hat{P}_x(e^{j\omega}) = \frac{1}{N}\left|X_N(e^{j\omega})\right|^2}$$

Periodogram directly defined in terms of data

❖ How to Compute the Periodogram

Rectangular window of length N

$$x_N(n) = x(n) \cdot w(n) \xrightarrow{\text{DFT}} X_N(k)$$

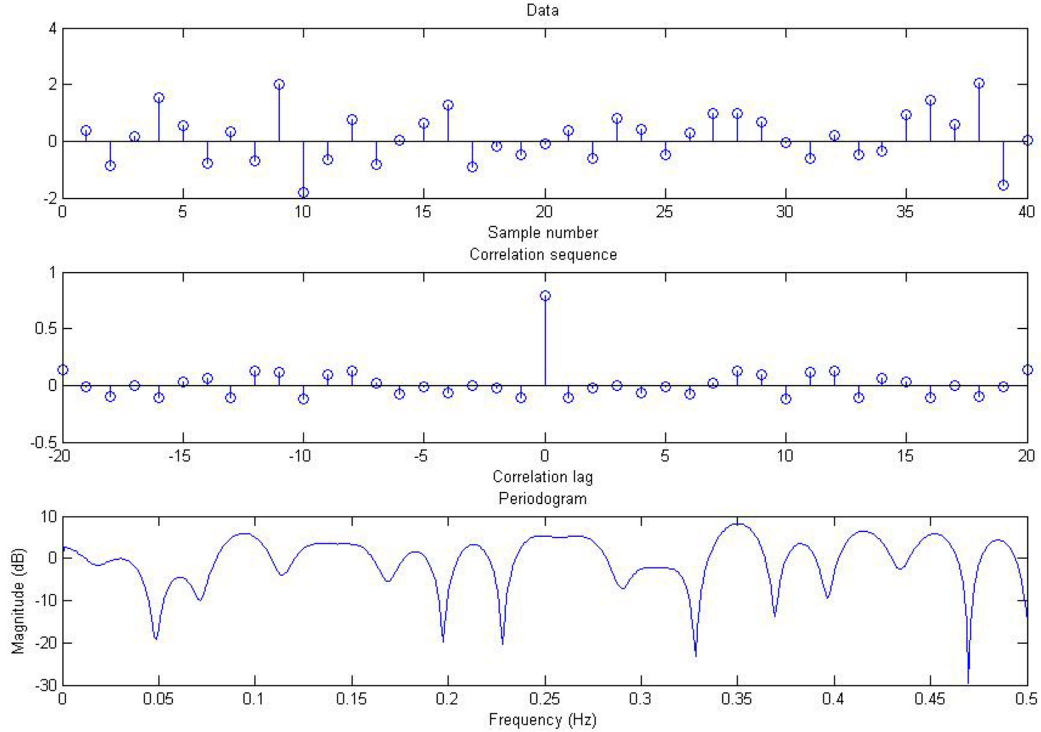
$$\hat{P}_x\left(e^{j(2\pi k/N)}\right) = \frac{1}{N}\left|X_N(k)\right|^2, k = 0, \ldots N-1$$

Zero padding $x_N(n)$ in the time-domain increases the number of frequency PSD points available

❖ Example: *x(n)* white noise

$$R_x(k) =$$

$$\Rightarrow P_x\left(e^{j\omega}\right) =$$



❖ Average behavior of the periodogram

- $$E\left\{\hat{P}_x\left(e^{j\omega}\right)\right\} = E\left\{\sum_{-N+1}^{N-1} \hat{R}_x(k)e^{-jk\omega}\right\}$$
$$= \sum_{k=-N+1}^{N-1} E\left\{\hat{R}_x(k)\right\}e^{-j\omega k}$$

- $$E\left\{\widehat{R}_x(k)\right\} = \frac{1}{N}\sum_{\ell=0}^{N-1-k} E\left[x(\ell+k)x^*(\ell)\right]$$
$$= \frac{N-k}{N}R_x(k)$$
$$= R_x(k)\cdot w_B(k) \qquad w_B(k) = \begin{cases} \dfrac{N-|k|}{N} & |k| \le N \\ 0 & ow \end{cases}$$

- $$E\left\{\hat{P}_x\left(e^{j\omega}\right)\right\} = \sum_{k=-N+1}^{N-1} E\left\{\hat{R}_x\left(k\right)\right\} e^{-j\omega k}$$

- $$E\left\{\hat{R}_x\left(k\right)\right\} = R_x\left(k\right)\cdot w_B\left(k\right), w_B\left(k\right) = \begin{cases} \dfrac{N-|k|}{N} & |k| \le N \\ 0 & ow \end{cases}$$

$$\Rightarrow \quad E\left\{\hat{P}_x\left(e^{j\omega}\right)\right\} = \sum_{k=-N+1}^{N-1} \left(R_x\left(k\right) w_B\left(k\right)\right) e^{-j\omega k}$$

$$\boxed{E\left\{\hat{P}_x\left(e^{j\omega}\right)\right\} = \frac{1}{2\pi} P_x\left(e^{j\omega}\right) * W_B\left(e^{j\omega}\right)}$$

$$\frac{1}{N}\left[\frac{\sin(N\omega/2)}{\sin(\omega/2)}\right]^2$$

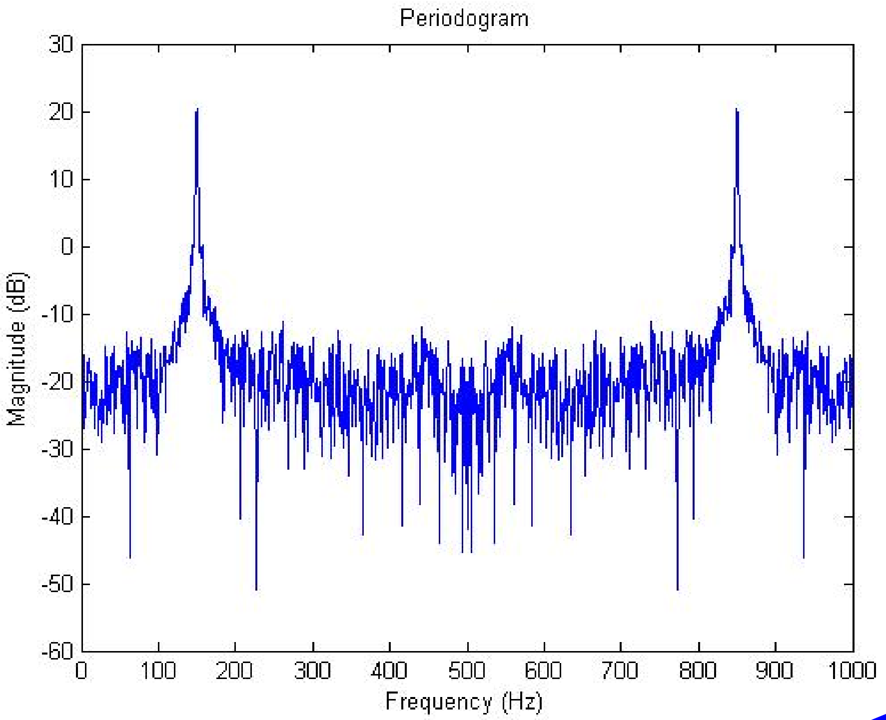
Note: the MATLAB function *periodogram.m* computes a "normalized" periodogram defined as

$$x_N(n) = x(n)\cdot w(n) \xrightarrow{\text{DFT}} X_N(k)$$

$$\hat{P}_x\left(e^{j(2\pi k/N)}\right) = \frac{1}{F_s N}\left|X_N(k)\right|^2, k = 0,\ldots N-1$$

Normalization parameter (sampling frequency)

❖ **Example**: *x(n)* is 1 tone in white noise



$$x(n) = A\sin(\omega_0 n + \phi) + v(n)$$

$$E\left\{\hat{P}_x(e^{j\omega})\right\} = \frac{1}{2\pi} P_x(e^{j\omega}) * W_B(e^{j\omega})$$

$$= \sigma_v^2 + \frac{A^2}{4}\left[W_B(e^{j(\omega+\omega_0)}) + W_B(e^{j(\omega-\omega_0)})\right]$$

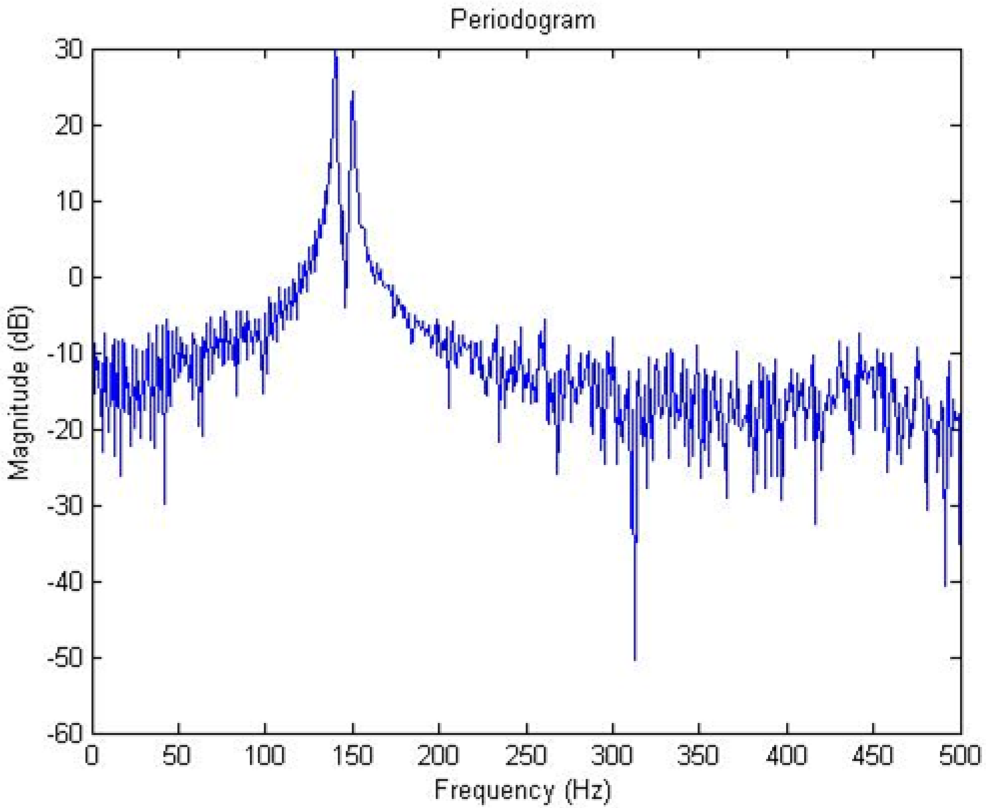
Computes the two-sided periodogram
(note: *better to stick to one-sided when dealing with real-data*)

```
[Px,f]=periodogram(x,[],'twosided',1024,fs);
plot(f,10*log10(fs*Px))
title('Periodogram'),ylabel('Magnitude (dB)')
xlabel('Frequency (Hz)')
```

To compensate for the MATLAB sampling frequency normalization (*optional*)

❖ **Example**: *x(n)* is 2 tones in white noise

$x(n) = \sin(2\pi*150*t) + 2*\sin(2*\pi*140*t) + 0.1*randn(size(t)); f_s=1KHz$




```
[Px,f]=periodogram(xn,[],'onesided',1024,fs);
 plot(f,10*log10(fs*Px))
title('Periodogram'),ylabel('Magnitude (dB)')
xlabel('Frequency (Hz)')
```

❖ Properties of the Periodogram

1) Periodogram is biased when dealing with finite windows
2) Periodogram is asymptotically unbiased

i.e., $$\lim_{N\to+\infty} E\left[\hat{P}_x(e^{j\omega})\right] = P_x(e^{j\omega})$$

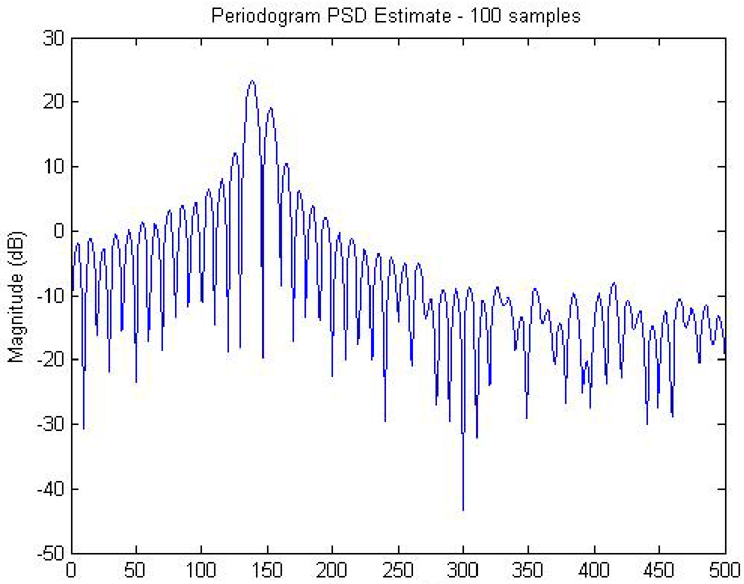
3) Periodogram has limited ability to resolve closely spaced narrowband components (limiting factor due to the rectangular window, the shorter the data length, the worse off) (*not good*)

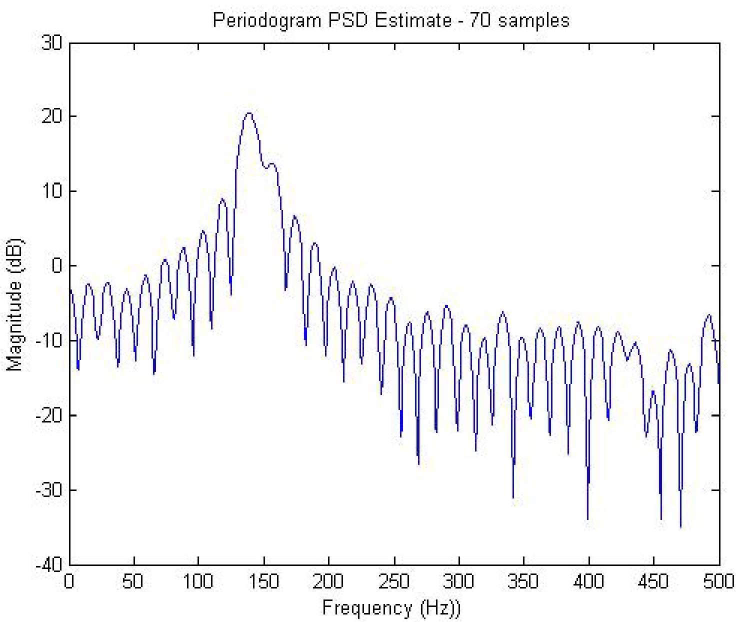
4) Values of the periodogram spaced in frequency by integer multiples of $2\pi/N$ are approximately uncorrelated . As data length increases, rate of fluctuation in the periodogram increases (*not good*).

5) Variance of the periodogram does not decrease as data length increases (*not good*).

➔ In statistical terms, it means *the periodogram is not a consistent estimator of the PSD*. Nevertheless, the ***periodogram can be a useful tool for spectral estimation in situations where SNR is high, and especially if the data is long***.

```
xn = sin(2*pi*150*t) + 2*sin(2*pi*140*t) + 0.1*randn(size(t)); fs=1000; nfft=1024;
```





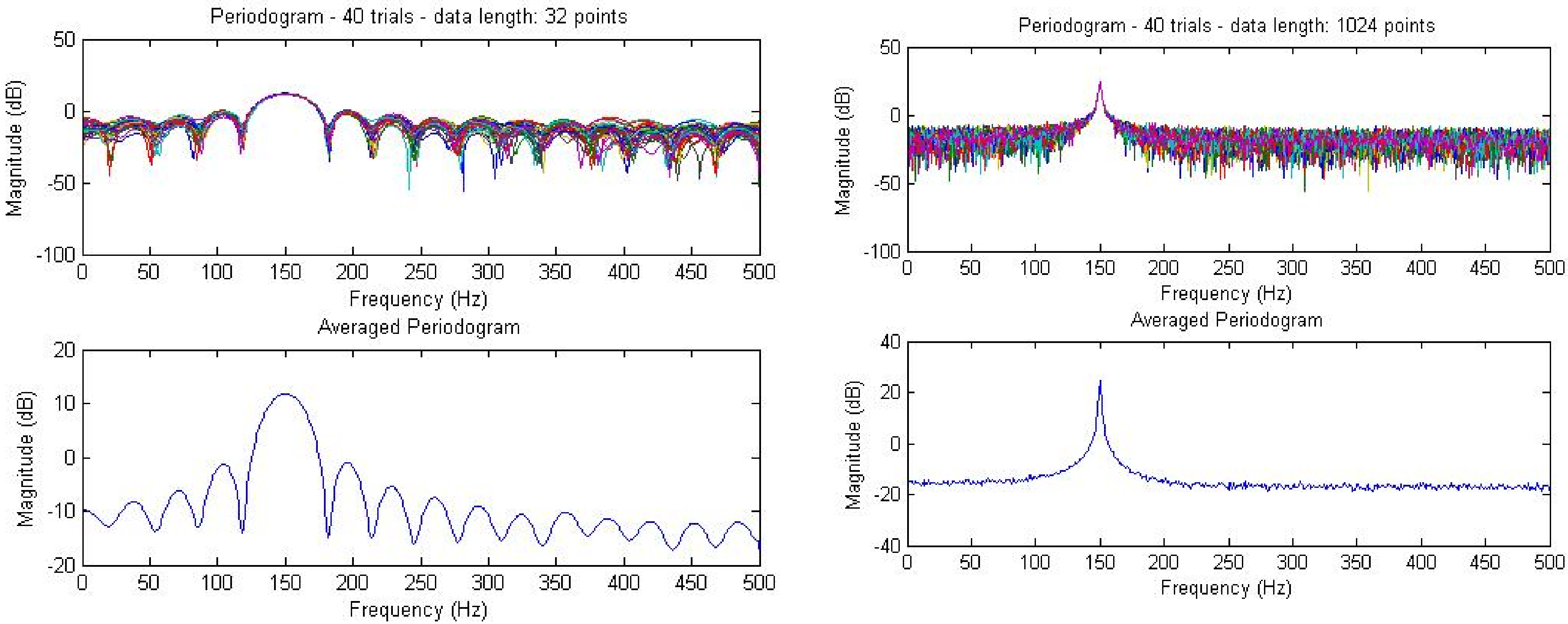
Comments (1)

Periodogram has ***limited ability to resolve closely spaced narrowband components*** (limiting factor due to the rectangular window, the shorter the data length, the worse off) (*not good*)

[6]

Comments (2)

Variance of the periodogram *does not* decrease as the data length increases (*not good*).



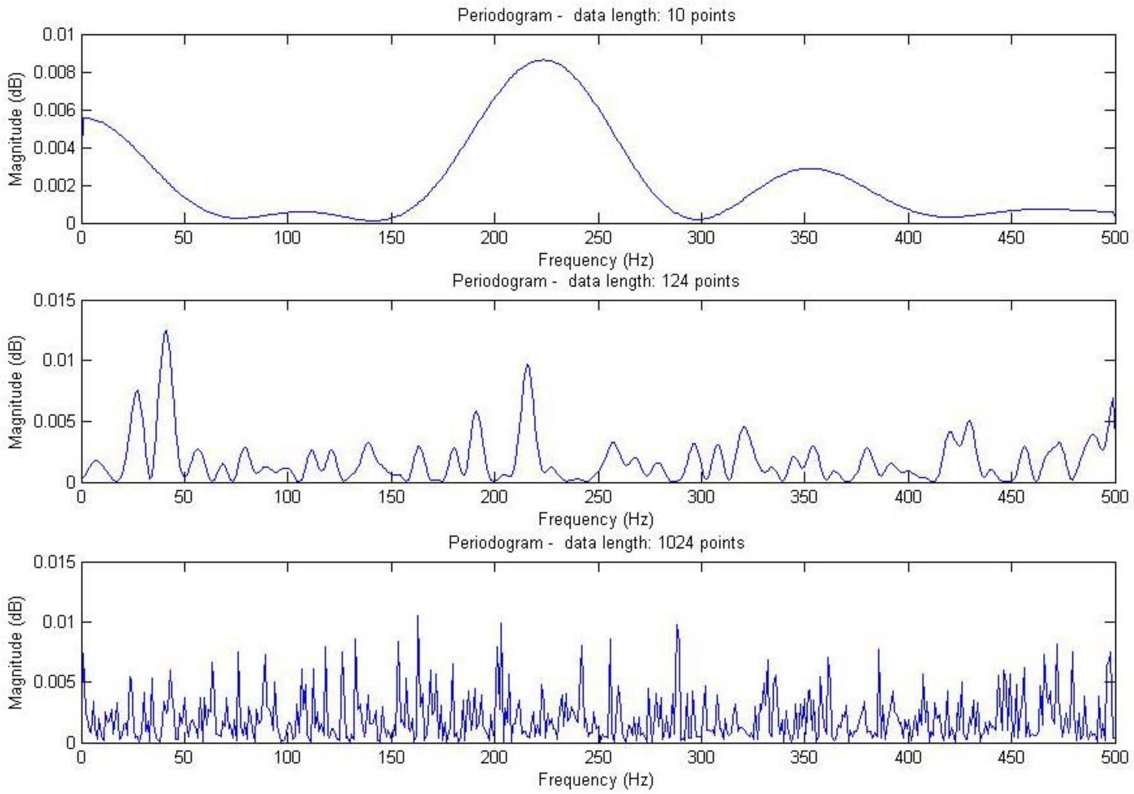
For white noise *x(n)* one can show [Therrien, p.590]

$$\lim_{N \to +\infty} \text{var}\left[\widehat{P}_x\left(e^{j\omega}\right)\right] \neq 0$$

$$\text{var}\left[\widehat{P}_x(e^{j\omega})\right] = \sigma_x^4$$

Comments (3)

As data length increases, rate of fluctuation in the periodogram increases (*not good*).



2. Modified Periodogram

– Apply a general window $w_R(n)$ to $x(n)$ prior to computing the periodogram (periodogram extension)

$$\hat{P}_x\left(e^{j\omega}\right)=\frac{1}{N}\sum_{n=-\infty}^{+\infty}\left|x(n)\,w_R(n)\,e^{-jn\omega}\right|^2$$

$$\boxed{E\left[\hat{P}_x\left(e^{j\omega}\right)\right]=\frac{1}{2\pi N}P_x\left(e^{j\omega}\right)*\left|W_R\left(e^{j\omega}\right)\right|^2}$$

Note: amount of smoothing is determined by the window type

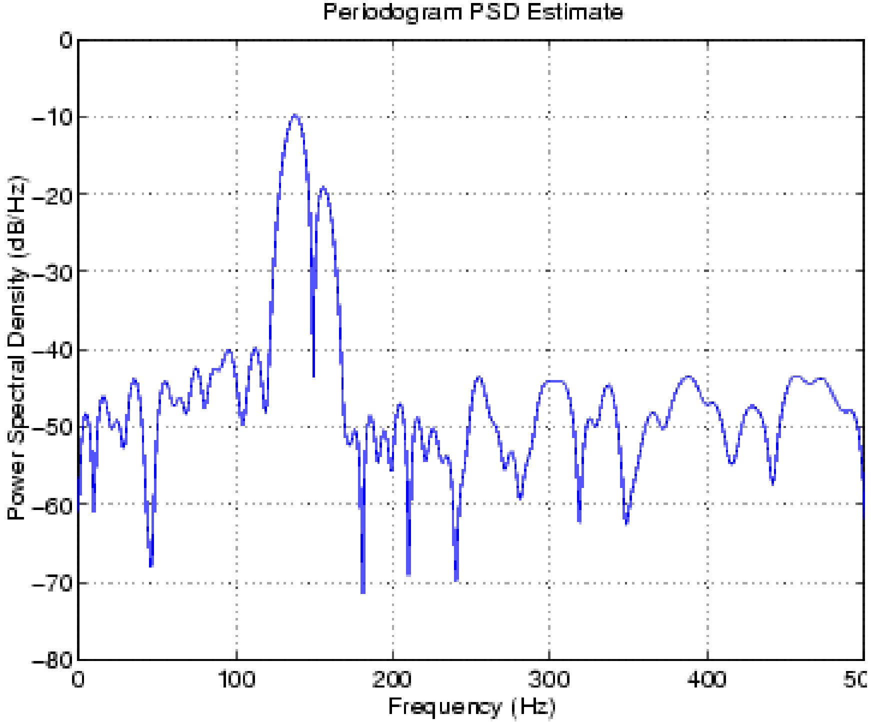
Matlab implementation:

Hs = spectrum.periodogram('Hamming'); ← *create modified periodogram object*

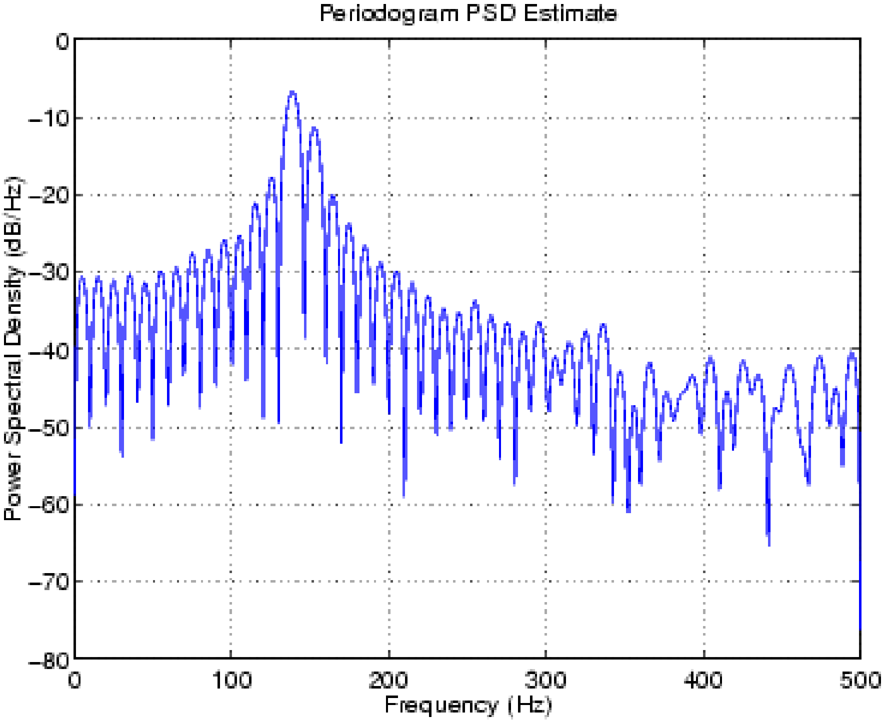
psd(Hs,xn,'Fs',fs,'NFFT',1024,'SpectrumType',‘onesided') ← *plot*

❖ Example:

xn = sin(2*pi*150*t) + 2*sin(2*pi*140*t) + 0.1*randn(size(t)); fs=1000; nfft=1024; data length=100.



Hamming window



Rectangular window

Note: *lower Hamming window sidelobes*, but *wider mainlobes* with the Hamming window than with the rectangular window

[6]

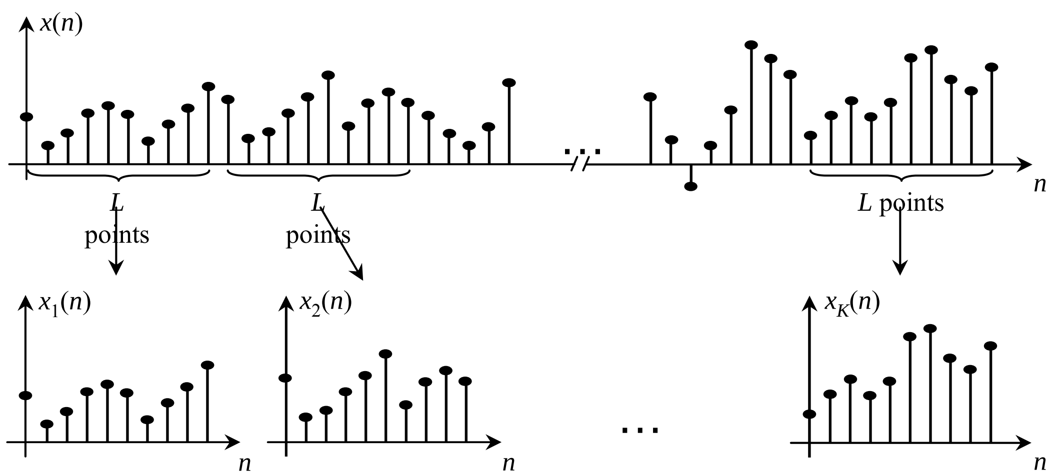
3. Bartlett's Method

- Averages several periodograms (rectangular window applied to each)
- Leads to consistent estimate of the PSD

$$\text{Define: } \hat{P}_B\left(e^{j\omega}\right) = \frac{1}{K}\sum_{k=1}^{K} \hat{P}_x^{(k)}\left(e^{j\omega}\right)$$

where K: number of segments in the data.

$$\boxed{\widehat{P_B}(e^{j\omega}) = \frac{1}{KL}\sum_{k=1}^{K}\left|\sum_{n=0}^{L-1} x(n+(k-1)L)e^{-jn\omega}\right|^2}$$



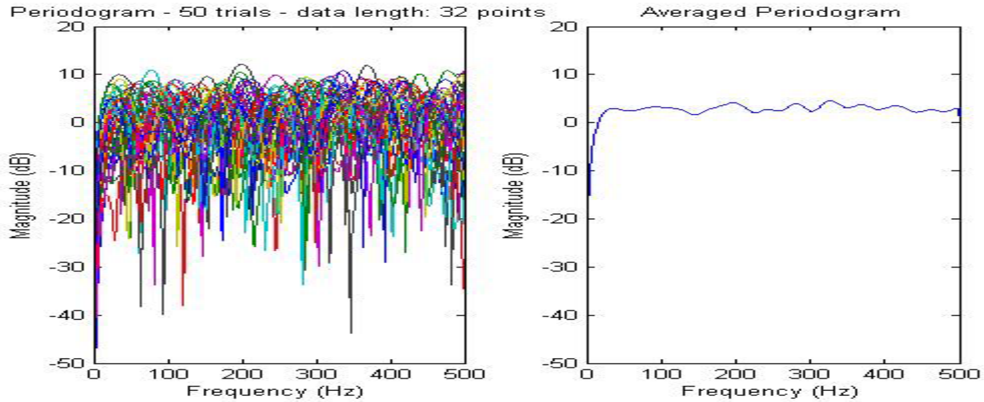
Note:

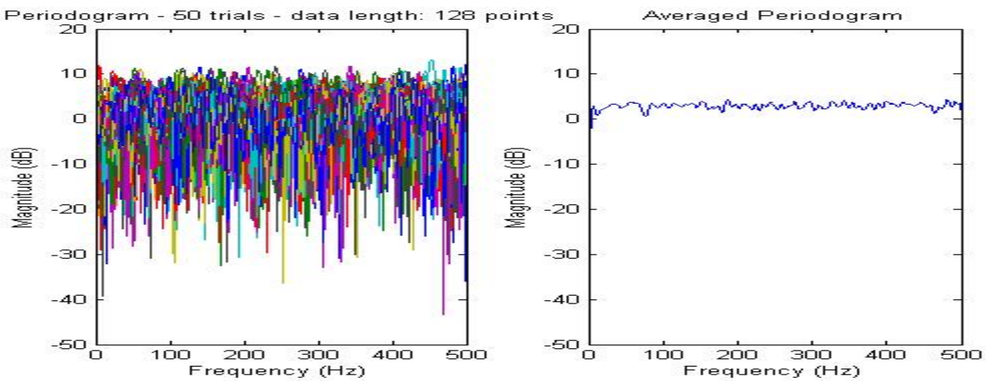
$$\operatorname{var}\left[\hat{P}_B\left(e^{j\omega}\right)\right] \simeq \frac{1}{K}\operatorname{var}\left[\hat{P}_x^{(k)}\left(e^{j\omega}\right)\right]$$

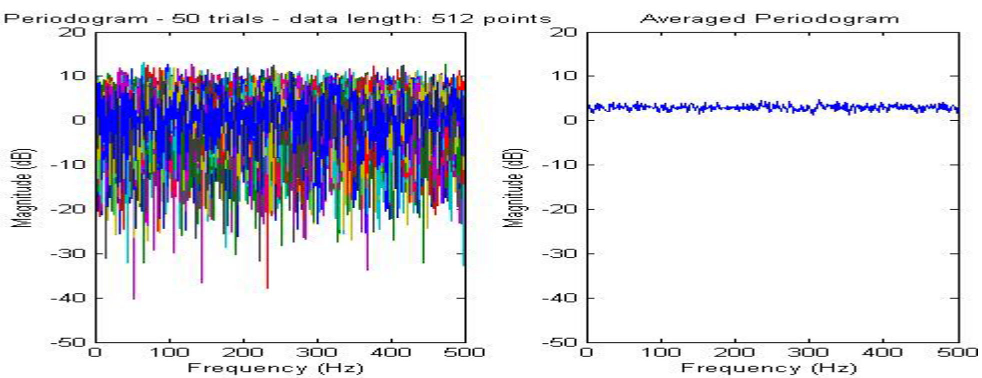
Examples: – white noise

– two sinusoids in noise

❖ **Compare:** basic periodograms / Bartlett estimates





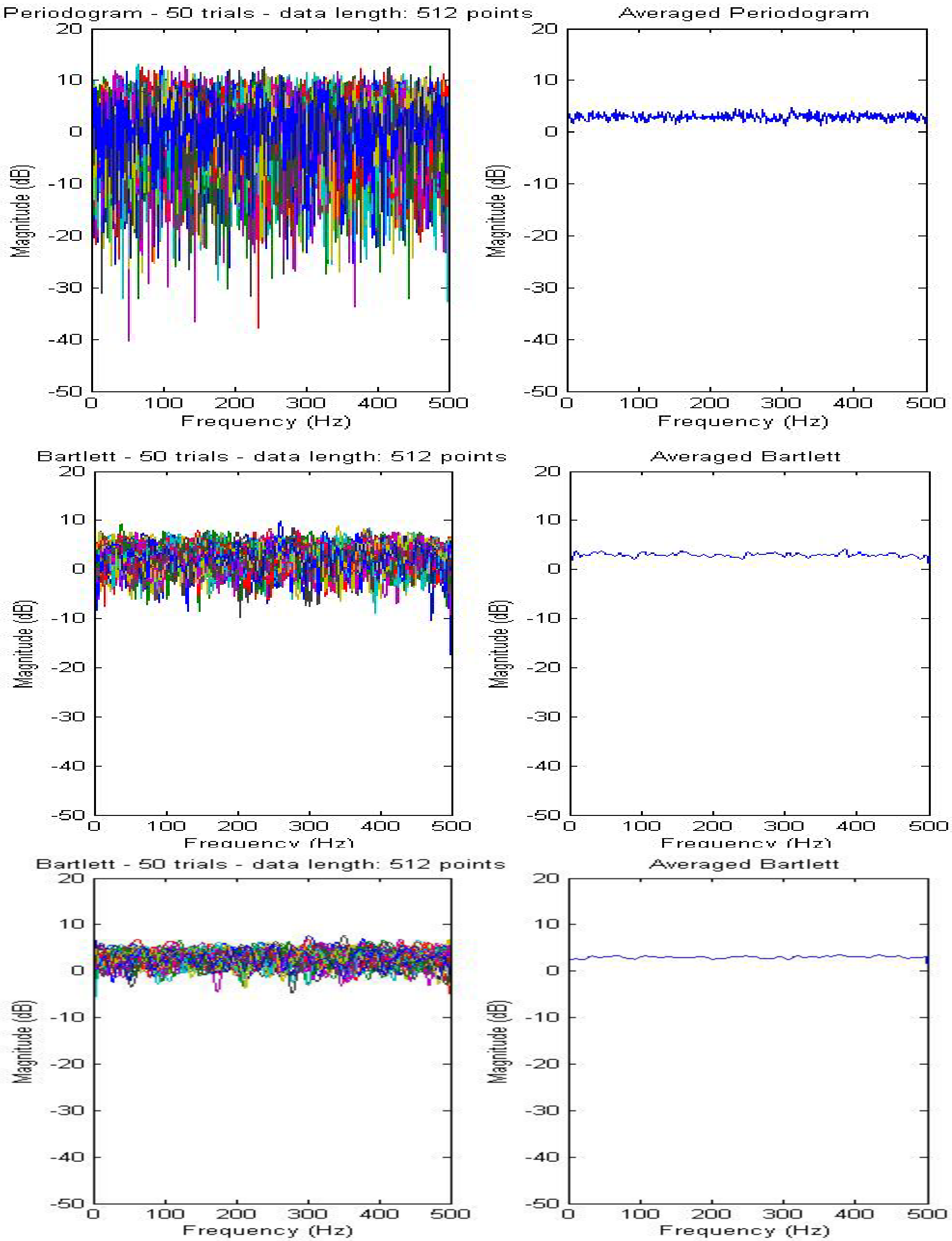


Periodogram of white noise with variance 1.

- Overlay 50 plots, N=32 & average.
- Overlay 50 plots, N=128 & average.
- Overlay 50 plots, N=256 & average.

Following [2, Fig. 8.9]

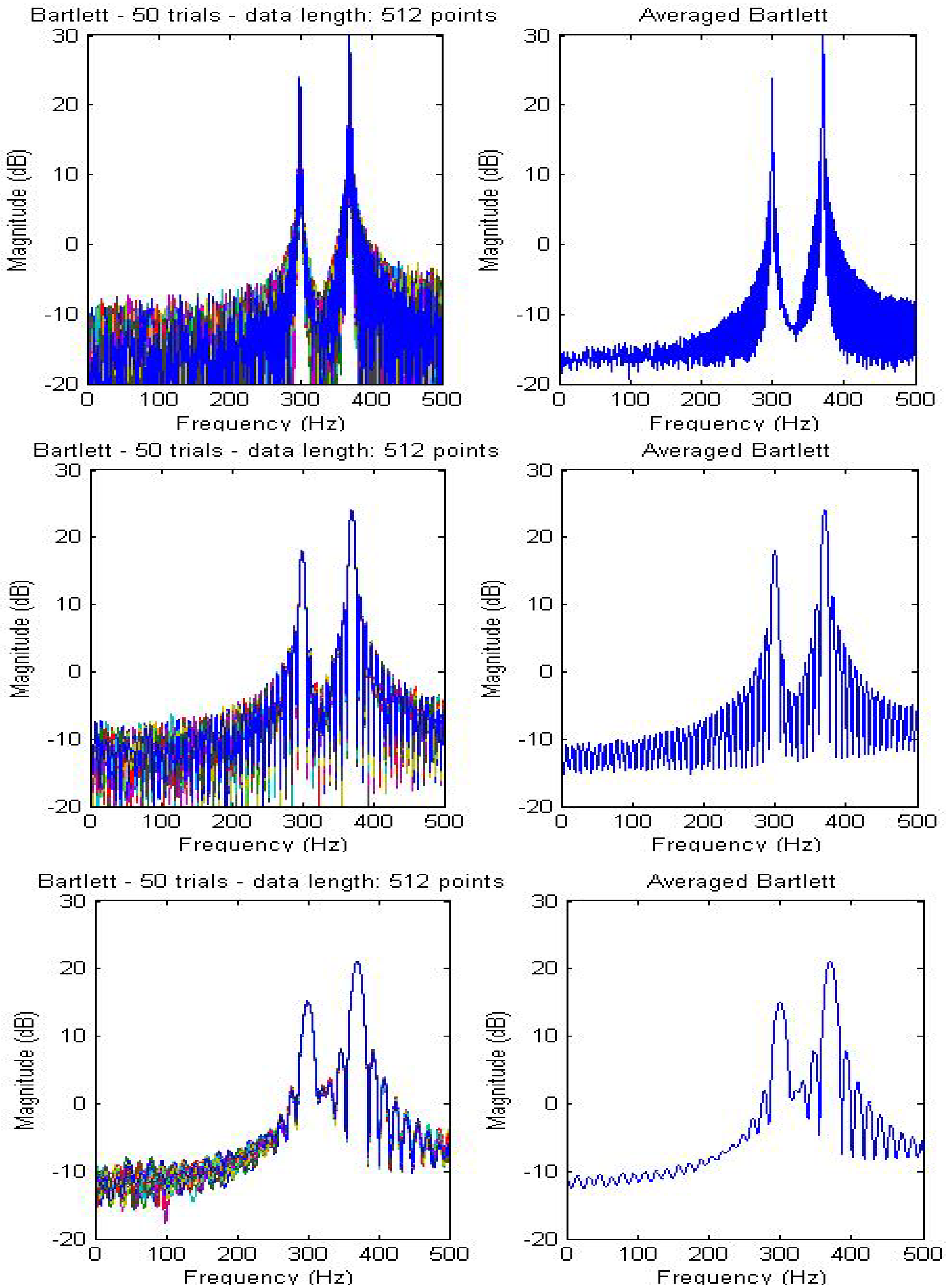
Bartlett estimates
of white noise N(0,1)



- 50-plot overlay, N=512 & average.
- Overlay of 50 Bartlett estimates with K=4 & L=128, N=512, & average.
- Overlay of 50 Bartlett estimates with K=8 & L=64, N=512, & average.

K: number of segments
L: segment length

Following [2, Fig. 8.14]



$$x(n) = \sin\left(2\pi(300/1000)n\right) + 2\sin\left(2\pi(370/1000)n\right)$$
$$+v(n), \quad v(n) \sim N(0, 0.1)$$

Bartlett estimates of 2 tones in white noise

A) 50-plot overlay, N=512, average.

B) Overlay of 50 Bartlett estimates with K=4 & L=128, N=512, & average.

C) Overlay of 50 Bartlett estimates with K=8 & L=64, N=512, & average.

*Note: For a given data length N=K*L, tradeoff between good spectral resolution (large L) and reduction in variance (Large K)*

Note: Spectral peaks broadening, why?

Following [2, Fig. 8.15]

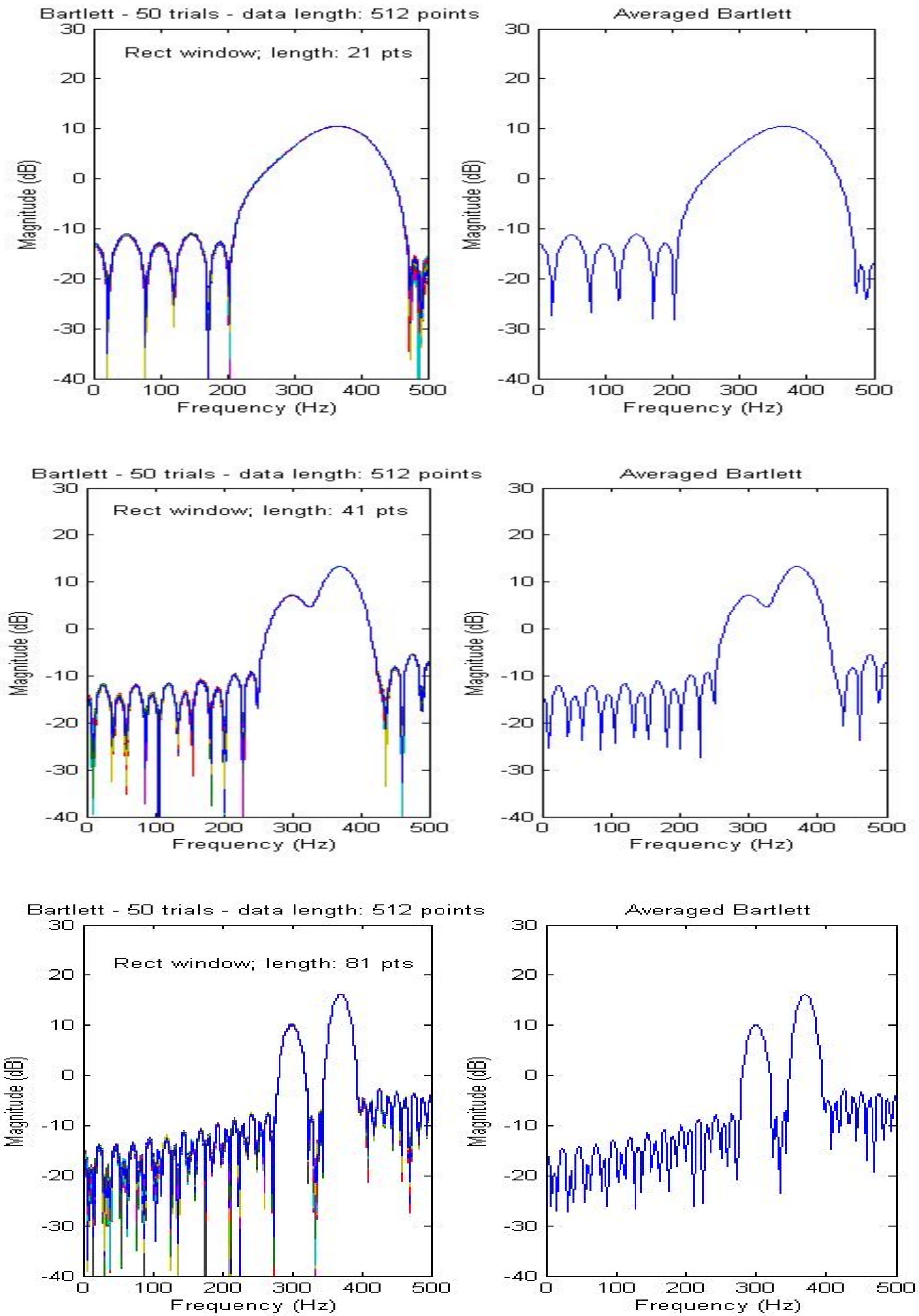
4. Blackman-Tukey (BT) Procedure

– apply windowing to correlation samples prior to transformation to reduce variance of the estimator

$$\hat{P}_{BT}\left(e^{j\omega}\right) = \sum_{k=-M}^{M} w(k)\,\hat{R}_x(k)\,e^{-j\omega k}$$

where the window $w(k)$ length is much smaller than data length, $w(k) = 0$ for $|k| > M$ where $M << N$

Maximum recommended window length $M<N/5$

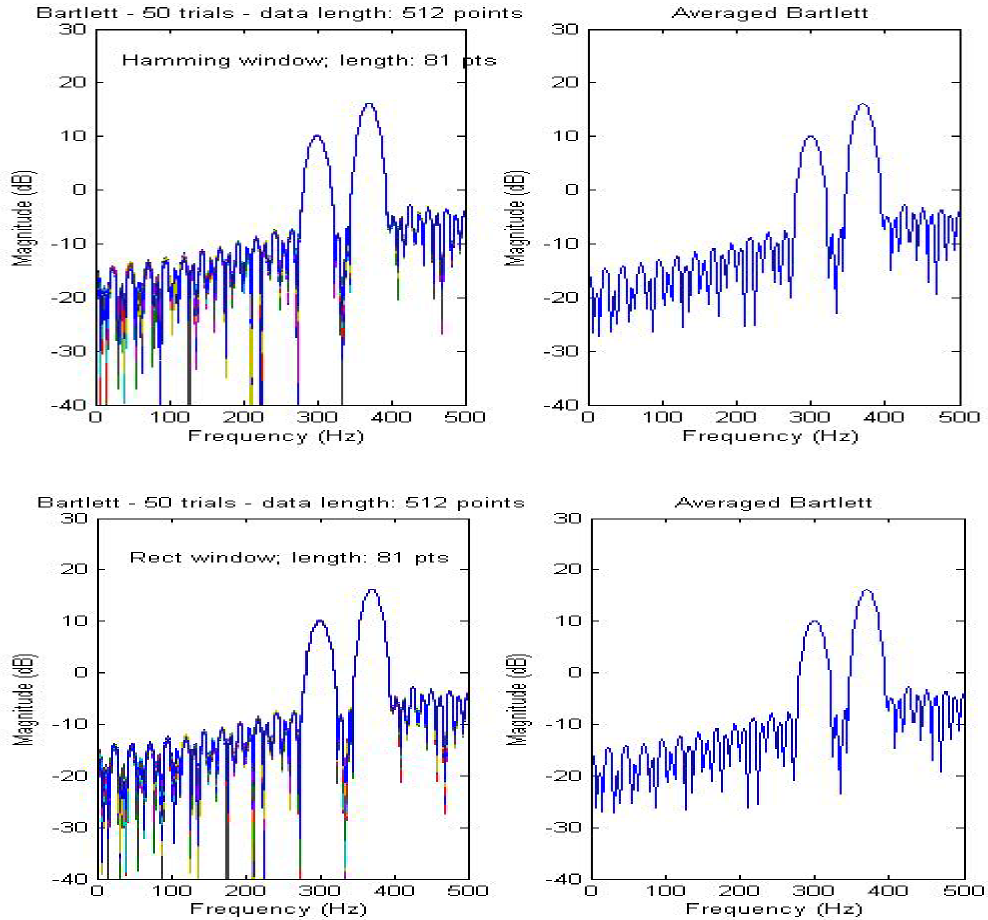


$$x(n) = \sin\left(2\pi(300/1000)n\right) + 2\sin\left(2\pi(370/1000)n\right) + v(n), \quad v(n) \sim N(0, 0.1)$$

BT estimates of 2 tones in white noise, 50-plot overlay, & average, data size=512, NFFT=512

- Rectangular wind. length=21
- Rectangular wind. length=41
- Rectangular wind. length=81

$$x(n) = \sin\left(2\pi(300/1000)n\right) + 2\sin\left(2\pi(370/1000)n\right) + v(n), \quad v(n) \sim N(0, 0.1)$$



BT estimates of 2 tones in white noise, 50-plot overlay, & average, data size=512, NFFT=512

- Hamming wind. length=81
- Rectangular wind. length=81

5. Welch's Procedure

- Modification to the Bartlett procedure: Averaging modified periodograms
- Dataset split into K **possibly overlapping** segments of length L
- Window applied to each segment
- All K periodograms are averaged

$$\hat{P}_w\left(e^{j\omega}\right) = \frac{1}{K}\sum_{k=1}^{K}\hat{P}_x^{(k)}\left(e^{j\omega}\right)$$

where

$$\hat{P}_x^{(k)} = \frac{1}{N}\sum_{n=0}^{L-1}\left|w(n)\underbrace{x^{(k)}(n)}e^{-j\omega n}\right|^2$$

window

data $x(n)$ located in K^{th} segment

MATLAB

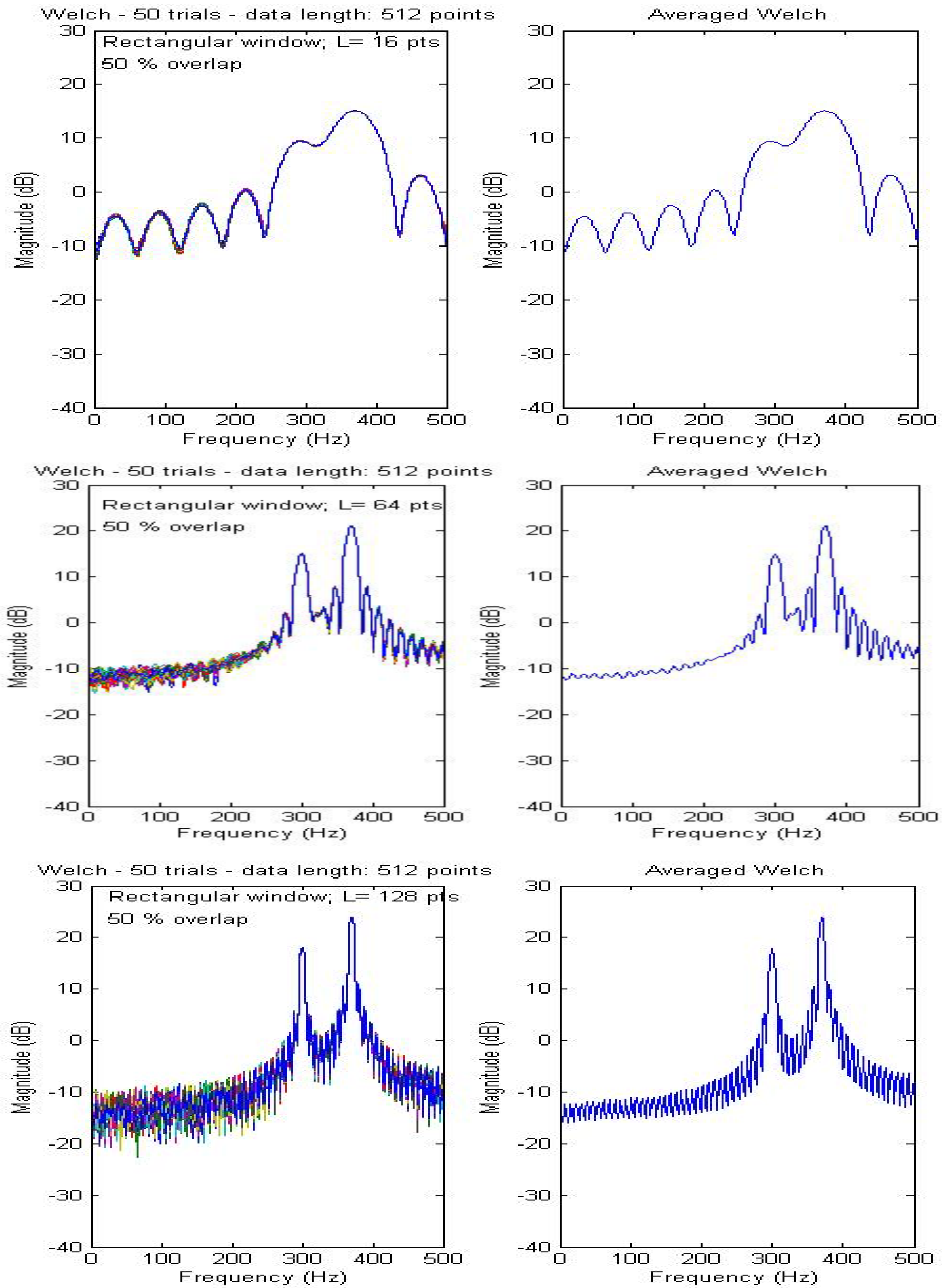
Hs = spectrum.welch('rectangular', segmentLength, overlap %); ← define Welch spectrum object

psd(Hs,xn,'Fs',fs,'NFFT',512); ← plot

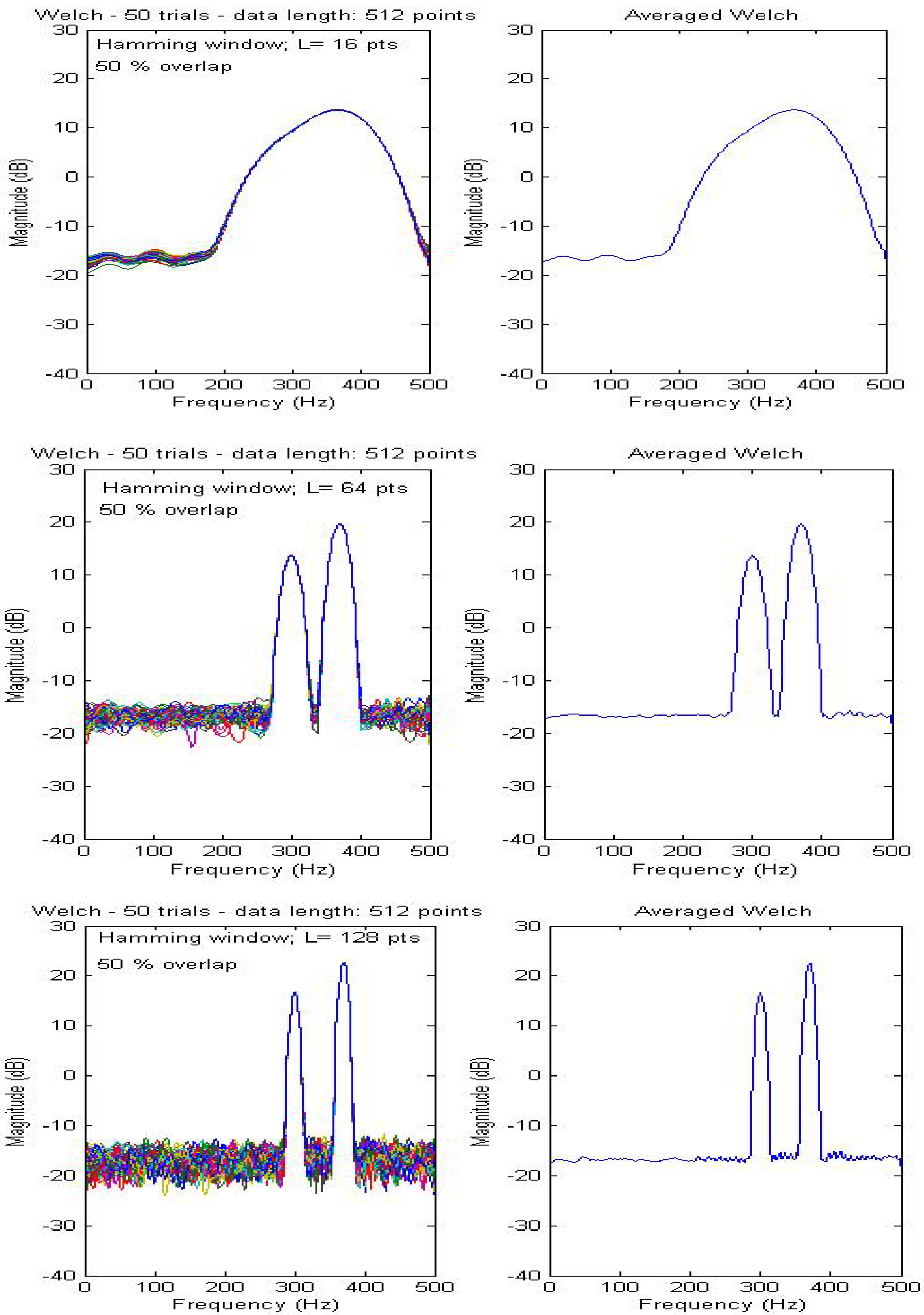
$$x(n) = \sin\left(2\pi(300/1000)n\right) + 2\sin\left(2\pi(370/1000)n\right)$$
$$+ v(n), \quad v(n) \sim N(0, 0.1)$$

Welch estimates of 2 tones in white noise, 50-plot overlay, & average, data size=512, NFFT=512, overlap: 50%

- Rectangular wind. length=16
- Rectangular wind. length=64
- Rectangular wind. length=128



Following [2, Fig. 8.17]

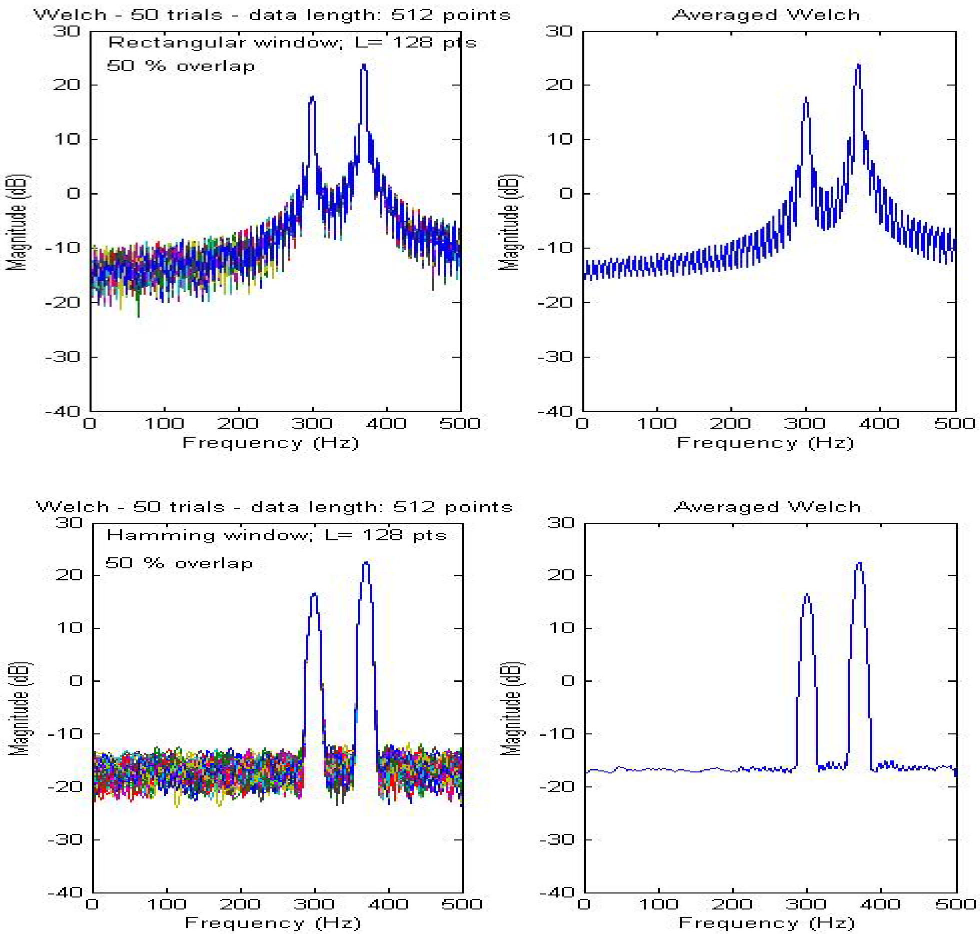


$$x(n) = \sin\left(2\pi(300/1000)n\right) + 2\sin\left(2\pi(370/1000)n\right)$$
$$+v(n), \quad v(n) \sim N(0, 0.1)$$

Welch estimates of 2 tones in white noise, 50-plot overlay, & average, data size=512, NFFT=512, overlap: 50%

- Hamming wind. length=16
- Hamming wind. length=64
- Hamming wind. length=128

Following [2, Fig. 8.17]



Note: what do we gain with the Welch's method?

→ Reduction in spectral leakage that takes place through the sidelobes of the data window

Performance comparison

[2, Table 8.7]

	variability	resolution	Figure of merit
Periodogram	1	0.89 (2π/N)	0.89 (2π/N)
Bartlett	1/K	0.89 K(2π/N)	0.89 (2π/N)
Welch (50% overlap, triangular window)	9/8K	1.28 (2π/L)	0.72 (2π/N)
BT	2M/3N	0.64 (2π/M)	0.43 (2π/N)

K: number of data segments
N: overall data length
L: data section length
M: BT window length

All schemes limited by data length

Definitions:

$$\text{variability} \triangleq \frac{\text{var}\left\{\hat{P}_x(e^{j\omega})\right\}}{E^2\left\{\hat{P}_x(e^{j\omega})\right\}}$$ (normalized variance)

$$\text{figure of merit} \triangleq \text{variability} \times \text{resolution}$$

$\Rightarrow$ should be as small as possible.

resolution $\triangleq$ mainlobe width at its 1/2 power (6dB) down level

6. Minimum Variance (MV) Spectrum Estimate – Capon PSD

• PSD estimate adapts itself to the data characteristics and exhibits smaller leakage amounts than previous schemes discussed.

• Provides better resolution than DFT based schemes.

• Basic idea behind MV PSD scheme:

→ *Estimate the power content estimated at a certain frequency without being influenced by the power at other frequencies.*

• Design PSD estimate as a filterbank where each filter of length P<=N (data length) adapts itself to data characteristics, i.e.,

- has center frequency f_i (spread between 0 and f_s)
- has bandwidth $\Delta \sim f_s/P$ (filters spread over $[0,f_s]$)
- depends on data correlation matrix R_x
- designed so components at f_i are passed without distortion and rejects maximum amount of out-of-band power [1, p.474]

MV Spectrum Estimate, cont'

- Resulting PSD estimate is given as:

$$\hat{P}_{MV}\left(e^{j\omega}\right)=\frac{P}{\underline{e}^{H}R_x^{-1}\underline{e}},$$

$$\underline{e}=[1,e^{j\omega},\ldots,e^{j(P-1)\omega}]^{T}$$

- MV PSD advantages/drawbacks:
  - o Achieves higher resolution than that obtained with DFT schemes

  - o Very sensitive to frequency partitioning selected → need very fine frequency spacing to accurately measure PSD (computational costs may be an issue)

[1, Section 9.5, p.474]

MATLAB implementation

$$\hat{P}_{MV}\left(e^{j\omega}\right)=\frac{P}{\underline{e}^H R_x^{-1}\underline{e}}, \quad \underline{e}=[1,e^{j\omega},\ldots,e^{j(P-1)\omega}]^T$$

Note:

$$\underline{e}^H R_x^{-1}\underline{e}=\underline{e}^H(U\Lambda U^H)^{-1}\underline{e}=\underline{e}^H U\Lambda^{-1}U^H\underline{e}$$

$$=(\underline{e}^H U)\Lambda^{-1}(\underline{e}^H U)^H$$

$$\underline{e}^H U=[1,e^{-j\omega},\ldots,e^{-j(P-1)\omega}]\begin{bmatrix}\vdots & & \vdots\\ \underline{u}_1 & \cdots & \underline{u}_P\\ \vdots & & \vdots\end{bmatrix}$$

$$=\left[\mathrm{FT}(\underline{u}_1), \quad \cdots \quad, \mathrm{FT}(\underline{u}_P)\right]$$

$$\underline{e}^H R_x^{-1} \underline{e} = (\underline{e}^H U)\Lambda^{-1}(\underline{e}^H U)^H$$

$$= \left[\mathrm{FT}(\underline{u}_1), \quad \cdots \quad, \mathrm{FT}(\underline{u}_P)\right]\Lambda^{-1}$$

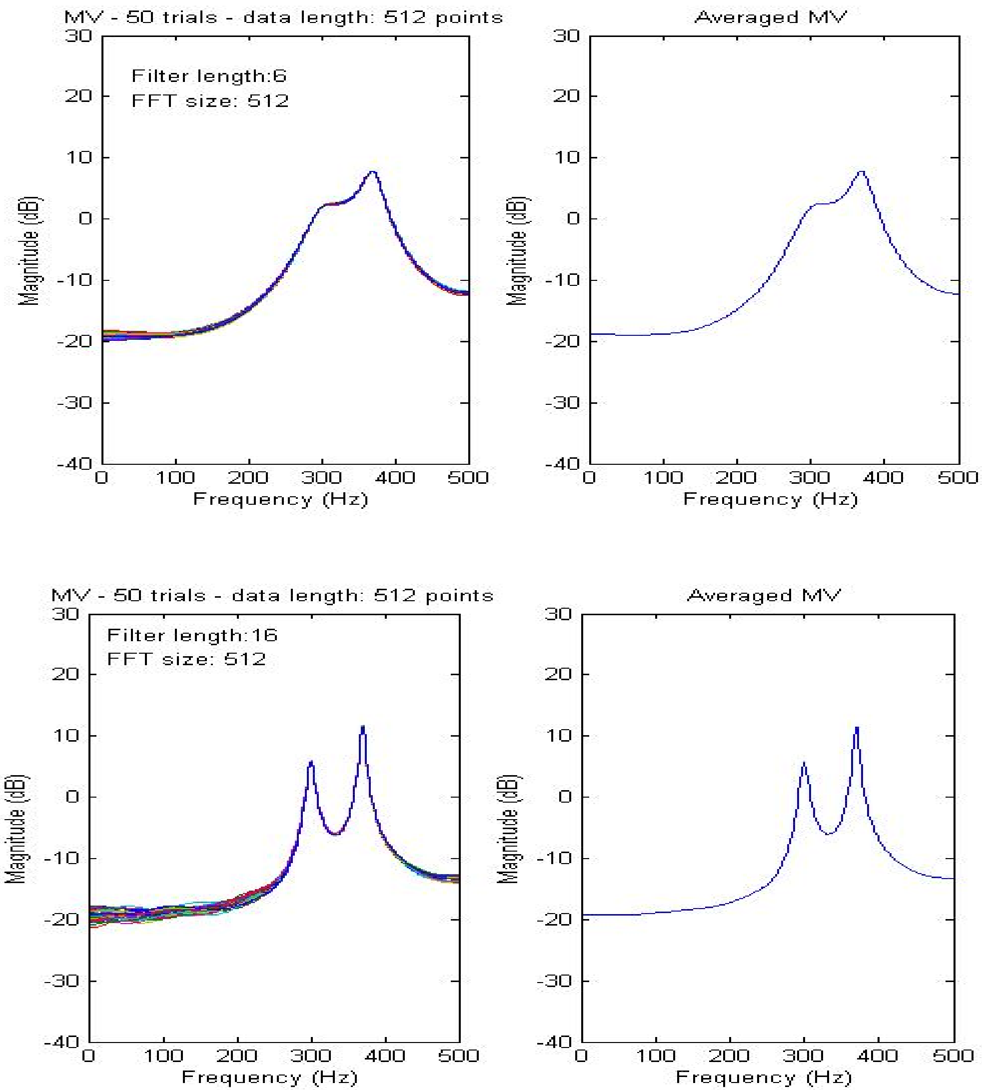
$$\times\left[\mathrm{FT}(\underline{u}_1), \quad \cdots \quad, \mathrm{FT}(\underline{u}_P)\right]^H$$

$$= \left[(1/\lambda_1)|\mathrm{FT}(\underline{u}_1)|^2 + \quad \cdots \quad + (1/\lambda_\mathrm{P})|\mathrm{FT}(\underline{u}_P)|^2\right]$$

$$\Rightarrow \quad \underline{e}^H R_x^{-1} \underline{e} = \sum_{\mathrm{k}=1}^{\mathrm{P}} (1/\lambda_\mathrm{k})|\mathrm{FT}(\underline{u}_k)|^2$$

$$\boxed{\hat{P}_{MV}\left(e^{j\omega}\right) = \frac{P}{\sum_{\mathrm{k}=1}^{\mathrm{P}} (1/\lambda_\mathrm{k})|\mathrm{FT}(\underline{u}_k)|^2}}$$

$$x(n)=\sin\left(2\pi(300/1000)n\right)+2\sin\left(2\pi(370/1000)n\right)$$
$$+v(n),\quad v(n)\sim N(0,0.1)$$



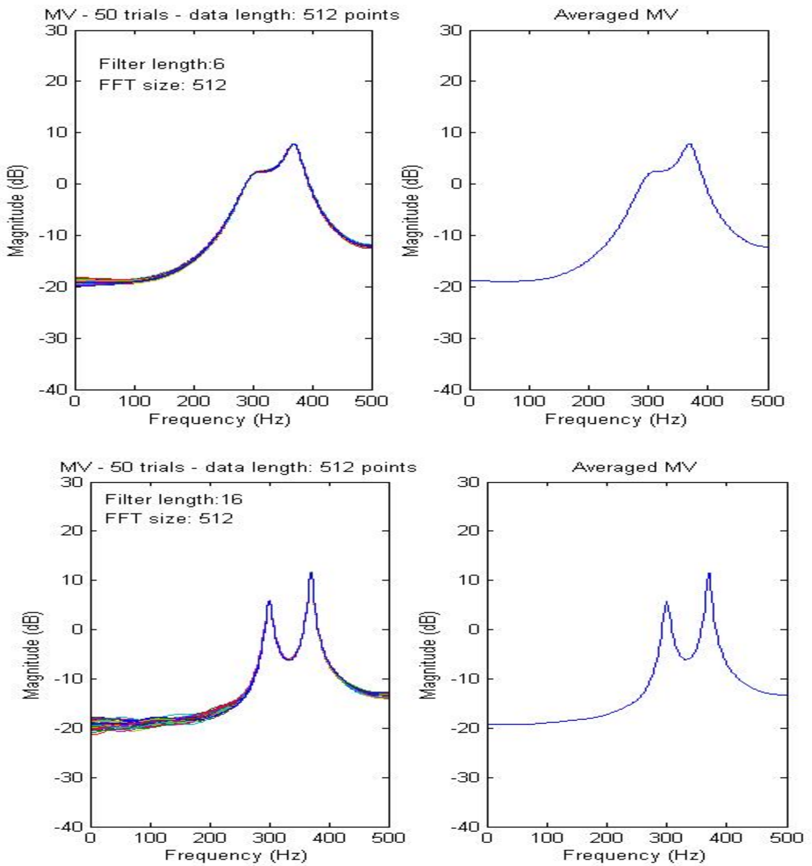
MV estimates of 2 tones in white noise, 50-plot overlay, & average, data size=512, NFFT=512,

- Filter length: 6
- Filter length: 16

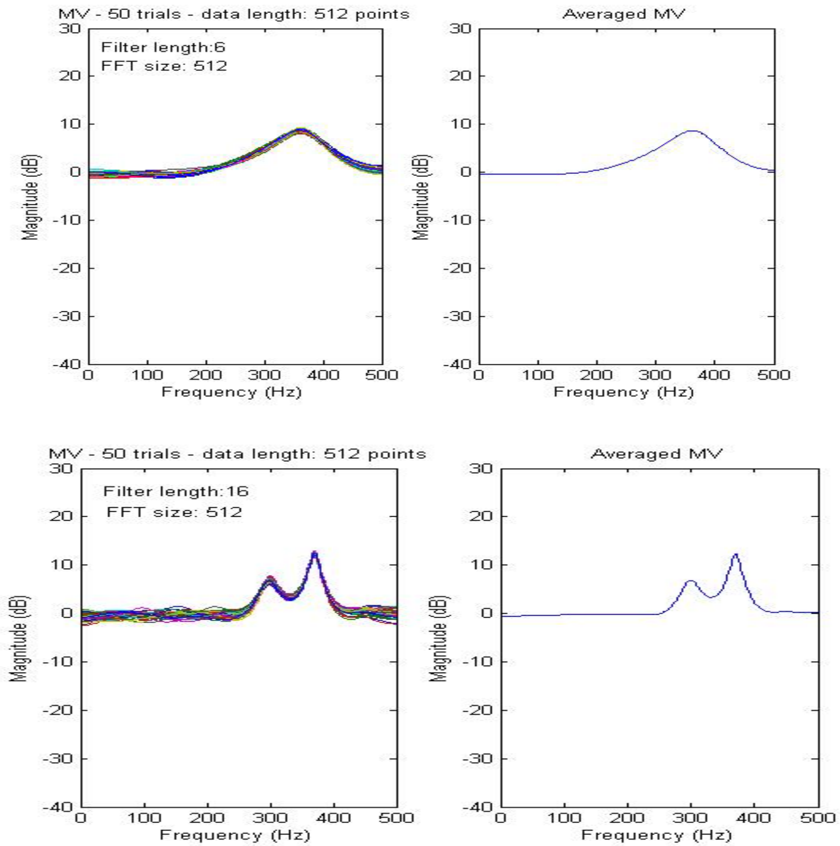
MV estimates of 2 tones in white noise, 50-plot overlay, & average, data size=512, NFFT=512,variable noise power

$$x(n) = \sin\left(2\pi(300/1000)n\right) + 2\sin\left(2\pi(370/1000)n\right) + v(n), \quad v(n) \sim N(0,K)$$

Noise ~N(0,0.1)



Noise ~N(0,1)



❖ MATLAB Implementation

```
x=x-mean(x);    % remove mean to prepare for covariance estimation
xc=xcorr(x,P,'biased'); % compute covariance sequence
R=toeplitz(xc(P+1:end)); % compute covariance matrix
[v,d]=eig(R);
UI=diag(inv(d)+eps); % extract eigenvalues

VI=abs(fft(v,nfft)).^2;
Px=10*log10(P)-10*log10(VI*UI);
```

In practice

- Pick P <N
- Zero pad FFT to insure more accurate measurements

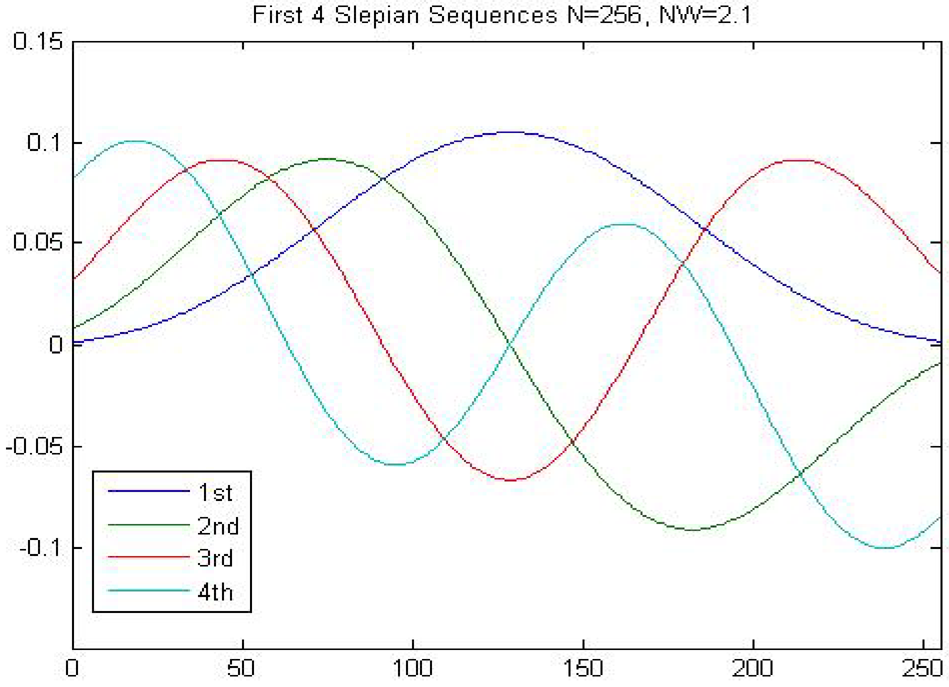
7. MultiTaper Spectrum Estimate Method (MTM) (Thompson)

• Periodogram can be viewed as obtained by using a filterbank structure, (using rectangular windows).

• Multitaper method (MTM) can be viewed the same way, with different filters, i.e., different orthogonal windows applied to the full data length. Thompson (1982) proposed to use *discrete prolate spheroidal sequences (DPSS)*. Other sequences can also be used.

• Several independent PSD estimates are computed by pre-multiplying the data by orthogonal tapered windows designed to minimize spectral leakage.

• Final PSD estimate obtained by combining set of PSD estimates



[6, statistical signal processing, spectral analysis, non parametric methods]   [4, 5, 7]

7. MultiTaper Method (MTM) cont'

• PSD estimate variance is reduced by using the full data over different windows. Since the data length is not shortened, the overall PSD bias is smaller than that obtained using data segments instead.

• PSD estimate uses K_m DSPS sequences of length N, bandwidth $[-W, W]$ and order 0 to K_m-1, where $K_m \leq 2NW-1$, typical choices for NW are 2, 2.5, 3, 4, 3.5.

•The time-bandwidth parameter NW balances the PSD estimate variance and resolution. MATLAB implementation uses $2*NW-1$ sequences to form the estimate.

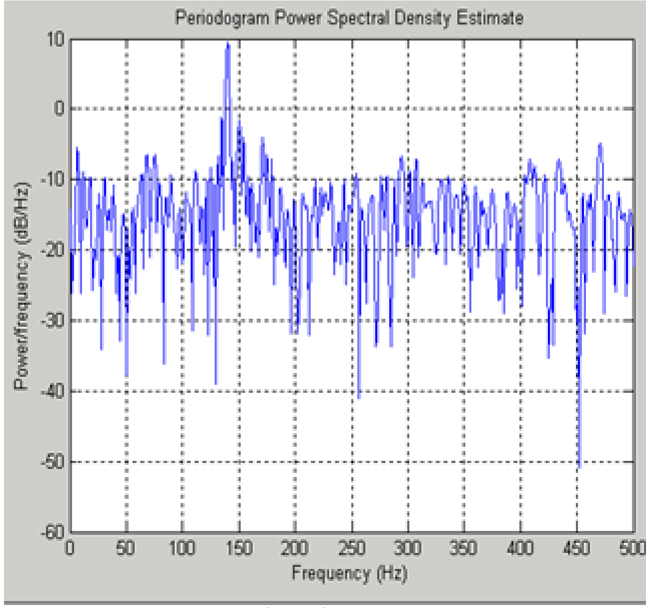
• As NW increases,
  - More estimates of the power spectrum are combined to form the MT PSD estimate, and the variance of the PSD decreases,
  - each estimate exhibits more spectral leakage (i.e., wider peaks) and the overall spectral estimate is more biased.

• For each data set, there is usually a value for NW that allows an optimal trade-off between bias and variance.

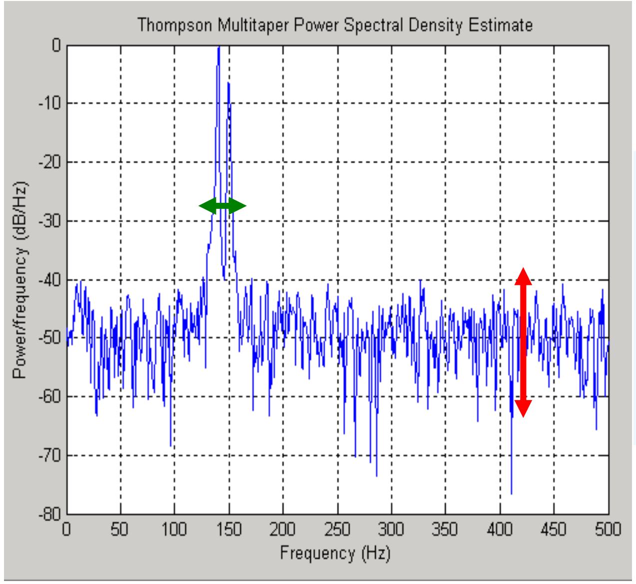
[6, statistical signal processing, spectral analysis, non parametric methods]

MT estimates of 2 tones in white noise, data size=512, NFFT=512,

fs = 1000Hz; N=1000; 2 sine components,
Amplitude = [1 2];
Sinusoid frequencies f = [150;140];
Hs1 = spectrum.mtm(4,'adapt');
psd(Hs1,xn,'Fs',fs,'NFFT',1024)



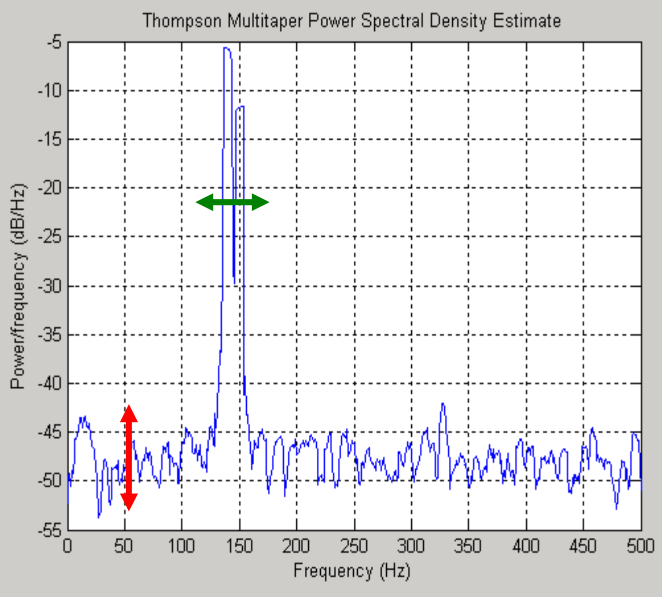
Periodogram



MTM, NW=3/2

Note, as NW increases

- ***Variance decreases***
- ***Peaks get larger***



MTM, NW=4

Appendices

❖Appendix A: Connections between Periodogram and filterbank

- The periodogram can be viewed as obtained by using a filterbank structure.

Recall periodogram defined for *x(n)*, *n=0,…N-1*, as:

$$x_N(n) = x(n)\cdot w(n) \xrightarrow{\text{DFT}} X_N(k) = \sum_{p=-\infty}^{+\infty} x_N(p) e^{-j2\pi kp/N}$$

$$X_N(k) = \sum_{p=0}^{N-1} x_N(p) e^{-j2\pi kp/N}$$

$$\Rightarrow \quad \hat{P}_x\left(e^{j(2\pi k/N)}\right) = \frac{1}{N}\left|X_N(k)\right|^2, k = 0, \ldots N-1$$

For $x(n)$ defined over a finite time window $\{x(n-N+1),\ldots x(n)\}$

$$x_N(n) = x(n)\cdot w(n) \xrightarrow{\text{DFT}} X_N(k) = \sum_{p=0}^{N-1} x_N(n-p)\underbrace{e^{-j2\pi kp/N}}_{h(p)}$$

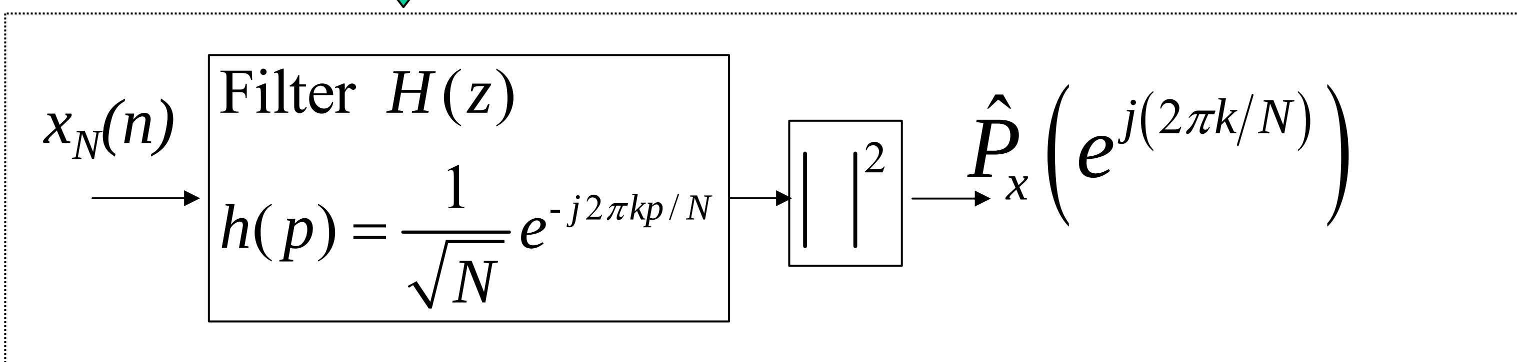
$$x_N(n) = x(n)\cdot w(n) \xrightarrow{\text{DFT}} X_N(k) = \sum_{p=0}^{N-1} x_N(n-p)h(p)$$

$k=0, \ldots N-1$

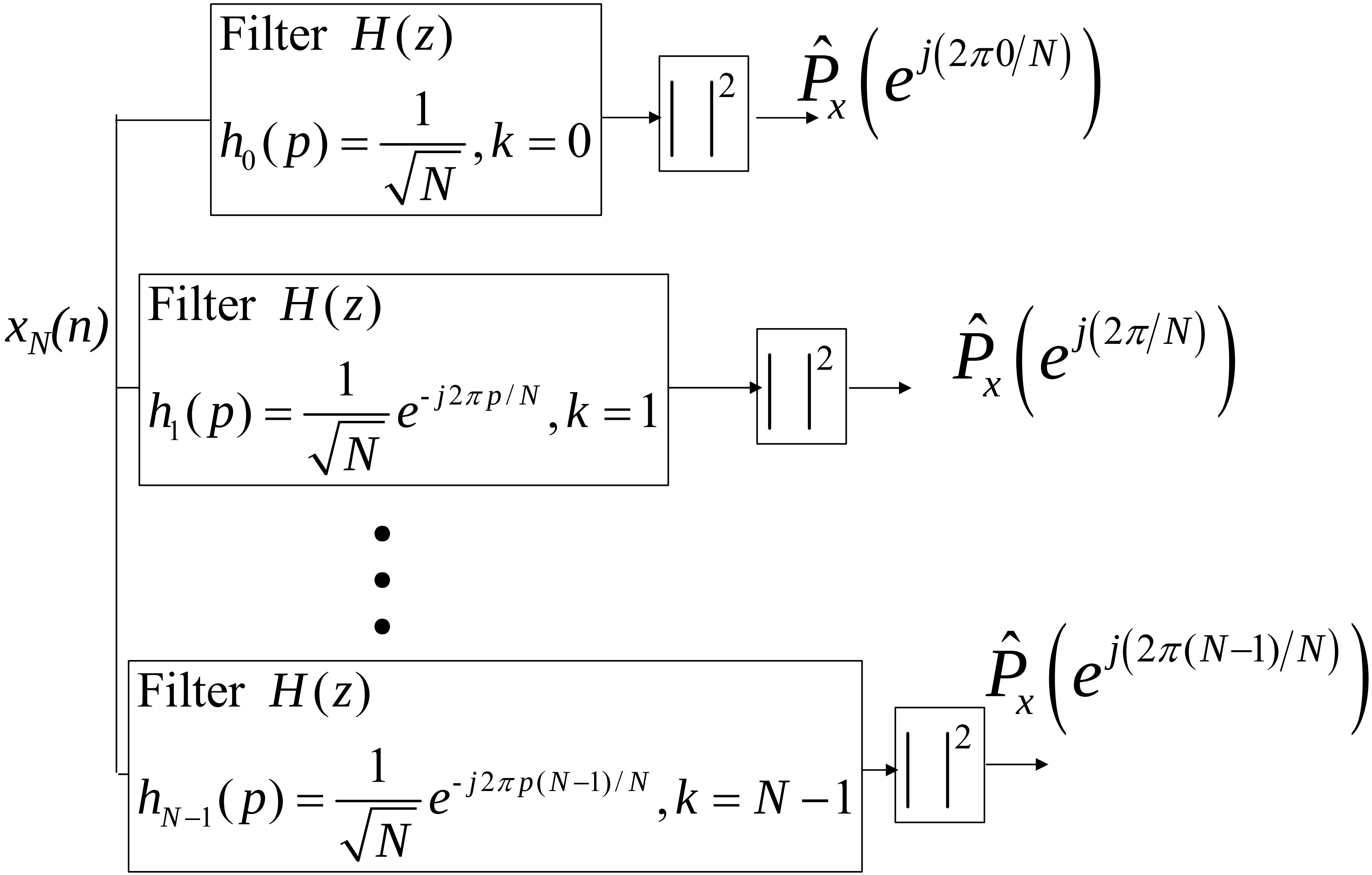
For $x(n)$ defined over a finite time window $\{x(n-N+1),\ldots x(n)\}$

$$x_N(n) = x(n)\cdot w(n) \xrightarrow{\text{DFT}} X_N(k) = \sum_{p=0}^{N-1} x_N(n-p)\underset{\uparrow \atop e^{-j2\pi kp/N}}{h(p)}$$

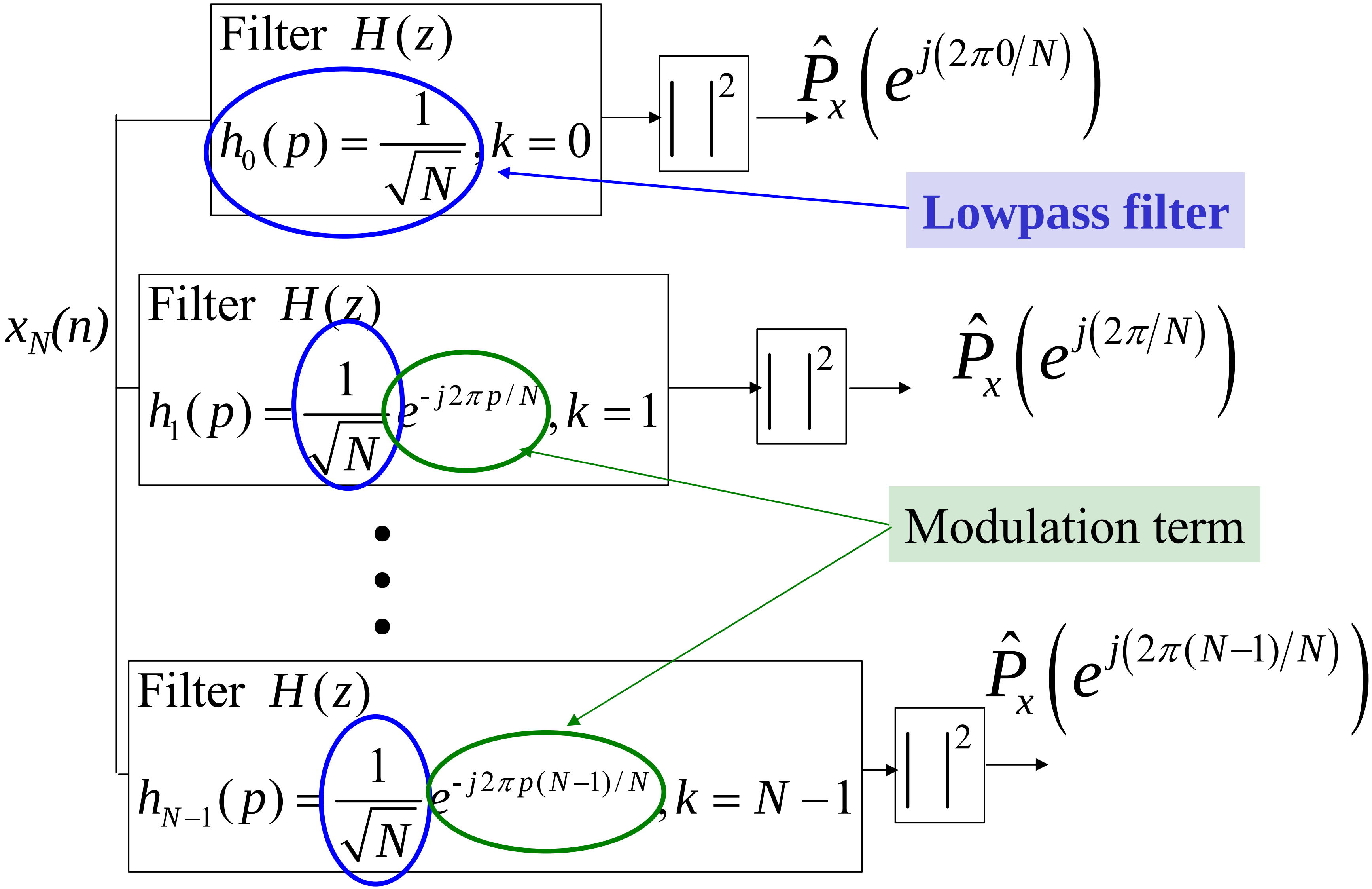
$$\hat{P}_x\left(e^{j(2\pi k/N)}\right) = \frac{1}{N}\left|X_N(k)\right|^2 \qquad k=0,\ldots N\text{-}1$$



Periodogram $\Longleftrightarrow$ DFT filterbank



$$\hat{P}_x\left(e^{j(2\pi k/N)}\right) = \frac{1}{N}\left|X_N(k)\right|^2$$



$$\hat{P}_x\left(e^{j(2\pi k/N)}\right) = \frac{1}{N}\left|X_N(k)\right|^2$$

Periodogram DFT matrix formulation

$$
\begin{array}{r}
k=0 \rightarrow \\ \\ \\ k=N-1 \rightarrow
\end{array}
\begin{bmatrix} X_N(0) \\ X_N(1) \\ \vdots \\ X_N(N-1) \end{bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & e^{-j2\pi/N} & & e^{-j2\pi(N-1)/N} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & e^{-j2\pi(N-1)/N} & & e^{-j2\pi(N-1)^2/N} \end{bmatrix}
$$

$$
\times \left(\begin{bmatrix} 1/\sqrt{\mathrm{N}} \\ 1/\sqrt{\mathrm{N}} \\ \vdots \\ 1/\sqrt{\mathrm{N}} \end{bmatrix} \begin{bmatrix} x(n) \\ x(n-1) \\ \vdots \\ x(n-N+1) \end{bmatrix} \right)
$$

May be viewed as a rectangular window applied to data before FT operation

$$
\hat{P}_x\left(e^{j(2\pi k/N)}\right) = \frac{1}{N}\left|X_N(k)\right|^2
$$

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