

Birth-death-suppression Markov process and wildfiresGeorge Hulsey^{1,*}, David L. Alderson², and Jean Carlson^{1,†}¹*Department of Physics, UC Santa Barbara, Santa Barbara, California 93106, USA*²*Operations Research Department, Naval Postgraduate School, Monterey, California 93943, USA*

(Received 24 July 2023; accepted 12 December 2023; published 11 January 2024)

Birth and death Markov processes can model stochastic physical systems from percolation to disease spread and, in particular, wildfires. We introduce and analyze a birth-death-suppression Markov process as a model of controlled culling of an abstract, dynamic population. Using analytic techniques, we characterize the probabilities and timescales of outcomes like absorption at zero (extinguishment) and the probability of the cumulative population (burned area) reaching a given size. The latter requires control over the embedded Markov chain: this discrete process is solved using the Pollazcek orthogonal polynomials, a deformation of the Gegenbauer/ultraspherical polynomials. This allows analysis of processes with bounded cumulative population, corresponding to finite burnable substrate in the wildfire interpretation, with probabilities represented as spectral integrals. This technology is developed to lay the foundations for a dynamic decision support framework. We devise real-time risk metrics and suggest future directions for determining optimal suppression strategies, including multievent resource allocation problems and potential applications for reinforcement learning.

DOI: [10.1103/PhysRevE.109.014110](https://doi.org/10.1103/PhysRevE.109.014110)**I. INTRODUCTION**

Markov processes provide simple but robust models of many real-world stochastic phenomena. The birth and death process is a continuous-time Markov process describing the dynamics of an abstract population including its possible extinction. Birth-death processes have been applied to queueing theory, epidemiology, mathematical biology, and even thermodynamic diffusion; see Refs. [1–4] for general review and applications. In a recent work [5], the authors introduce a modified birth-death process as a model of the dynamics of a wildfire under suppression, i.e., firefighting. Here, we extend this work and establish the analytical solution of the specific Markov process.

The technical focus of this work is on the solution of the birth-death-suppression process, but a guiding motivation is wildfire modeling. Dealing with wildfires at a regional level requires allocation of a finite amount of suppression resources—firefighters, trucks, aircraft—across potentially many active wildfire events of varying intensity and risk. There is an ongoing need for decision support tools that incorporate new conditions and allow for optimization-based assessment of trade-offs in suppression strategies. Markov processes have previously been applied to multiple facets of wildfire management, including fire spread, risk to infrastructure, and fire mitigation, among others [6–14].

We stress that the introduction of the present model is not meant to replace or even compare to the high-fidelity, physics-based fire modeling tools employed by agencies today. Instead, the model is meant to capture important

trade-offs in time-dependent resource management in a simple environment, where simulation is cheap and analytical results are available. As it contains no spatial information about the spread of a wildfire, the birth-death-suppression model is effectively a ‘mean-field’ approach to describing wildfire dynamics.

More generally, the birth-death-suppression process can model any population with a natural and an artificial source of eradication: the rate of the former being proportional to the size of the population and the latter independent of it. For example, one could interpret the process as modeling the size of an infected population in the presence of treatment efforts. Indeed, the dynamical equations of the susceptible-infectious-recovered (SIR) model of an epidemic are known to reduce to those of a birth-death process in certain regimes [15]. The model we consider describes populations in a linear growth stage, where no dynamical limit to their growth exists, and where there is no chance of resurrection after complete eradication.

The main subject of our model is a discrete stochastic variable $j(t)$ we call the *population*, representing in the wildfire interpretation the number of actively burning (spatial) parcels of fire, or *firelets*; one may take a single firelet to represent, e.g., a burning acre of land. In Fig. 1, the population $j(t)$ is shown in red (light gray), both as a stochastic variable and schematically as the actively burning area of a wildfire.

As time progresses, the variable j undergoes stochastic nearest-neighbor transitions $j \rightarrow j \pm 1$ with probabilities dependent only on the previous state. The transitions $j \rightarrow j \pm 1$ represent births and deaths in the population. There is an aggregate birth rate $\beta \cdot j$ of parcels per unit time which may change to reflect local conditions: fuel and weather in the fire interpretation or virulence in an epidemic interpretation. There is also a suppression parameter γ which reduces the

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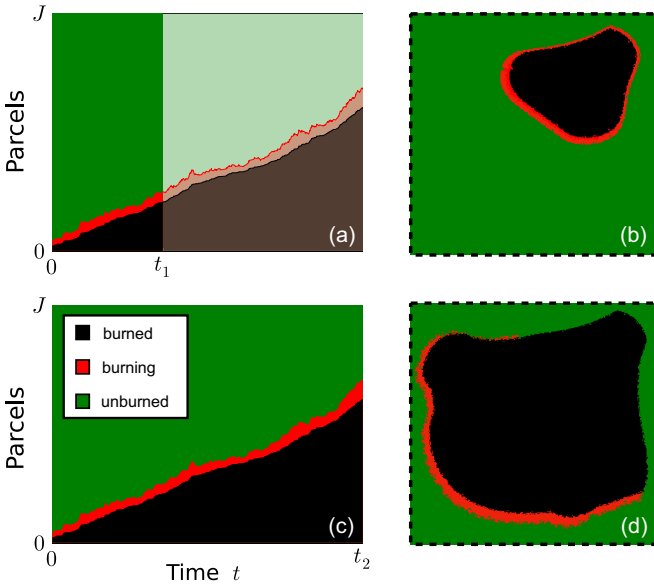


FIG. 1. A simulated birth-death process (left) alongside the schematic representation as a wildfire (right). (A) The state of the simulated process at the time t_1 , with all J parcels in one of the three states: unburned, burning, burned. (B) The corresponding spatial representation: a small fire at an early time t_1 , the active population shown in red (light gray). (C) Most parcels of the simulated process have entered the burned state. (D) Depiction of the same fire at the later time t_2 ; the footprint (black) has grown to almost fill the total burnable region (dark gray).

size of the overall population, representing the application of eradication resources; firefighting or treatment. A reference table of these parameters, their definitions and interpretations is included in Table I.

If at some time the process reaches the state $j = 0$, then the population has died out and the process is terminated. Estimating the probability of this event, and the time at which it happens, is a primary concern. However, fires propagate, and their associated risk is related to how much total area they burn, not just their active size. A fire could maintain a small active size but still propagate and burn a very large area before going out. Similarly, the severity of an epidemic is generally defined as the cumulative number of cases, not the number of currently infected individuals. The cumulative population

is a new stochastic variable $F(t)$, which we refer to as the *footprint*. This variable shares all the positive transitions of the population $j(t)$: if $j \rightarrow j + 1$, then $F \rightarrow F + 1$, but if $j \rightarrow j - 1$, F is unchanged. The footprint, as the area burned by a wildfire, is shown in black in Fig. 1, where one can see its monotonic increase in size as the process (left) or fire (right) progresses in time.

To be sensitive to the cost associated with a large burned area, one can ask whether the footprint F likely remains below some threshold value J . Asking whether the footprint satisfies $F \leq J$ is the type of question relevant for fires in the wildland-urban interface. When the footprint reaches $F \geq J$, one interprets the fire as having reached the built infrastructure. While the model is purely temporal, the threshold J incorporates the spatial notion of a burned area: the correspondence between this implicit spatial representation and the continuous-time Markov process is the subject of Fig. 1, with snapshots of the “fire” juxtaposed with a simulated birth-death process. The purpose of suppression is to keep $F \leq J$, thereby containing the cumulative size of the fire or epidemic. To this end we aim to compute the *escape* probability that $F \geq J$ at the end of the process, for any finite J .

Some immediate credence for the introduction of the birth-death-suppression model in the context of fire dynamics is supplied by the known statistics of fire footprints: if $P(F \geq J)$ is the probability of a given fire having footprint F greater than or equal to J , a power law $P(F \geq J) \sim J^{-\alpha}$ is observed. In particular, measurements of the area burned by real fires suggests that the exponent α of the empirical footprint distribution is close to $\alpha \approx 1/2$ [16]. This power-law distribution is reproduced in our birth-death model for fire close to the critical threshold where the birth and death rates are equal.

Power-law distributions have heavy tails, which means that large or rare events dominate the total cost of such events. The presence of a power-law distribution is common to many types of natural phenomena and various mechanisms exist which explain the origin of such scaling [17–22]. These notably include the idea of “highly optimized tolerance,” a model of systems which have evolved or been organized in some fashion [23,24]. Intuitively, highly optimized systems are tolerant to common events but unprepared for rare, catastrophic events. This results in the “cost” or “loss” associated to such rare events comprising the majority of the total loss

TABLE I. Reference of the parameters ($\beta, \delta, \gamma, N, J$) and variables (j, F, T) of the birth-death-suppression model. Included are their interpretations as describing wildfire, as in Ref. [5]; and epidemic dynamics, as in Ref. [15]. Setting $\delta = 1$ fixes the units of time. The lifetime T is defined as the exact time of absorption (reaching the state $j(T) = 0$). The process pictured in Fig. 1 has parameters $N = 10, J = 500, \beta/\delta = 1.1$, and $\gamma = 0$.

Quantity	Symbol	Definitions	Wildfires	Epidemics
Birth rate	β	$\beta > 0$	Spread rate	Infection rate
Death rate	δ	set $\delta = 1$	(Natural) Extinguish rate	Recovery rate
Suppression rate	γ	$\gamma \geq 0$	Firefighting rate	Treatment rate
Initial population	N	$j(0) = F(0) = N$	Seed ignition size	Initial outbreak size
Threshold	J	$F \leq J$	Burnable area	Susceptible population
Population	$j(t)$	$j(t) \geq 0$	Actively burning area	Infected population
Footprint	$F(t)$	$F(t) \geq j(t)$	Burned area	Total epidemic size
Lifetime	T	$j(t) = 0, t \geq T$	Extinguishment time	Eradication time

experienced. The suppression-free birth-death model recreates the desired footprint distribution, and the suppression parameter, once introduced, allows one to explicitly compute the effect of policy choices—allocation of suppression resources—on the distribution of possible outcomes of a fire event.

Birth-death processes were initially studied by Kendall [25], who introduced the cumulative population; and Karlin and McGregor [26,27], who developed the method of orthogonal polynomials as a means of solution. Linear birth-death processes, which include the birth-death-suppression process, were further studied by Karlin and McGregor [28,29], where the notion of the cumulative population (footprint) was addressed. However, they did not determine the dynamics of the cumulative population away from the critical point. The analytical tools for doing so were applied by Askey and Ismail [30] and involve analyzing the embedded discrete birth-death process, a random walk. However, the solution of the birth-death-suppression process—including the cumulative population—has not been fully synthesized in the literature. For other work on birth-death processes and their properties, see also Refs. [31–34].

The birth-death-suppression process is notably similar to the birth-death-immigration (BDI) process [1,30,35]. In the latter one has an immigration rate parameter which represents the growth of a population due to migration rather than births. This plays an antipodal role to the suppression rate in our model. The analytical methods of this work closely follow the approach of Askey and Ismail to solving the BDI process, and many potential applications of the birth-death-suppression have equally strong analogs that have been studied for the BDI process.

In fact, the birth-death-suppression process is the dual, in a technical sense, of the BDI process [1]. The duality mapping takes ergodic/recurrent processes (which do not terminate), like the BDI process, to transient processes (which may terminate in finite time) such as the birth-death-suppression process. The BDI process contains as special cases many other familiar continuous-time Markov processes: Poisson, pure birth, pure death, and the $M/M/\infty$ queueing model [33]. While all the aforementioned have been well-studied in the literature, the birth-death-suppression model—their image under the duality mapping—has garnered less direct attention. This is surely related to the fact that the presence of suppression complicates the standard approaches to solving the process.

A. Outline of paper

The general goal of this work is to gain a more analytic understanding of the birth-death-suppression process as a simple fire model, an interpretation suggested in Ref. [5] to which the reader should refer for more background. Throughout the paper, we will use the language of wildfire to describe the process, though we emphasize its generality in describing branching processes in phases of linear growth.

The paper is intended to be readable with different focuses. Section II reviews the general theory of birth-death processes and defines precisely the model under consideration. In Sec. III we review the solution of the continuous-time

TABLE II. The main results and corresponding equation numbers.

Quantity	Eq.
Absorption probability	(41)
Transition probability matrix	(B27)
“Firewalk” polynomials	(C20)
“Firewalk” orthogonality relation	(C55)
Transition count at absorption	(68)
Escape probability	(72)

process: this includes the characterization of the absorption times and the explicit formulas for transition probabilities. In Sec. IV we solve the embedded random walk; it is here that the novel results of the work are developed, in particular the escape probability. We develop a theory of orthogonal polynomials, which we refer to as the “firewalk” polynomials, to forecast outcomes of the random walk. After solving the model in question, in Sec. V we show some basic applications and sketch future directions for the technology developed in this work. The mathematically inclined reader may focus on Secs. III and IV which contain the bulk of the material on the specific Markov process itself. In the Appendices we include a thorough review on background material, a majority of the technical details, and review of techniques applied in the paper such that this work is relatively self-contained. As a concise summary, in Table II we include a reference of the main results and their equation numbers in the text.

II. BIRTH AND DEATH PROCESSES

Defining the stochastic birth-death process begins with the state space, which is composed of discrete parcels. These parcels represent individual elements of the population, e.g., acres of land in the wildfire interpretation. Each parcel can be in one of three states: unburned, burning, or burnt. A firelet is a burning parcel; $j(t)$ therefore counts the number of firelets at time t , and $F(t)$ represents the total burned area: the sum of burning and burnt parcels. We take the convention that $N \equiv F(0) = j(0)$, so that $F(t) \geq j(t)$ in general.

As time progresses, unburned parcels can become firelets ($j \rightarrow j + 1, F \rightarrow F + 1$) and burning parcels can become burnt ($j \rightarrow j - 1, F \rightarrow F$). Both the variables j, F are non-negative integers, with $F \geq N \geq 1$. Figure 2 shows an example simulation of the process, with $j(t), F(t)$ displayed in red (gray), black, respectively.

Recall the idea of a threshold J on the total number of available parcels. Any process has $F \leq J$ initially; reaching the point $F = J$ means the fire has consumed a burnable area of size J . Beginning such a process with $j(0) = F(0) = N$ active firelets, one of two possible outcomes will occur first: the process extinguishes, reaching $j = 0$; or it remains active and grows its footprint until it reaches $F = J$. In the Markov process literature, the state $j = 0$ is said to be *absorbing*: once the process reaches this state it remains there for all time. Therefore, we refer to the event of reaching the state $j = 0$ as *absorption*. Conversely, if the footprint F exceeds the bound J , then we say the process has *escaped*. “Solving” the process means characterizing the probabilities of each of

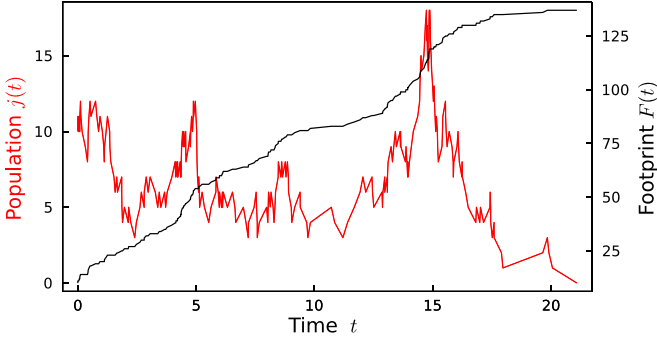


FIG. 2. An example birth-death fire process. Here, the population is initially of size 10, and hence $j(0) = F(0) = 10$. The process pictured has equal birth and death rates $\beta = \delta$ and reaches $j = 0$, ending in finite time.

these outcomes—absorption or escape—in terms of the birth and death rates of the population.

The transition probabilities define the dynamics of the system. Consider a single, actively burning unit out of the whole population $j(t)$. Let β be the rate (per unit time) at which this single parcel ignites a new firelet. This parameter might reflect the weather, fuel, and topographic conditions of a wildfire. In some short time interval Δt , the probability that the entire population $j(t)$ sees a single birth (new ignition) is

$$\Pr\{j \rightarrow j + 1\} = \beta j(t) \Delta t + \mathcal{O}(\Delta t^2), \quad (1)$$

which defines the aggregate birth rate $\lambda_j \equiv \beta j$. Define also the aggregate death rate $\mu_j \equiv \delta j + \gamma$. The aggregate death rate is the rate per unit time of deaths in the population, so that the probability of a single death (firelet burning out) in the interval Δt is

$$\Pr\{j \rightarrow j - 1\} = (\delta j(t) + \gamma) \Delta t + \mathcal{O}(\Delta t^2). \quad (2)$$

Here, the parameter δ is the death rate for an individual firelet per unit time; it represents the natural process of eradication, i.e., extinguishment by fuel exhaustion. The second parameter γ is the rate of suppression. It represents the action of eradication resources (firefighting) on the fire, again per unit time. The effect of suppression on the aggregate death rate is independent of the size of the fire; the suppression affects the aggregate population, while all the individuals feel the effect of fuel exhaustion. One can already see that the suppression rate γ affects the aggregate death rate $\delta j + \gamma$ more when the population $j(t)$ is small.

In some short time interval Δt , the transition probabilities are

$$\Pr\{j \rightarrow j + 1\} = \lambda_j \Delta t, \quad (3a)$$

$$\Pr\{j \rightarrow j - 1\} = \mu_j \Delta t, \quad (3b)$$

$$\Pr\{j \rightarrow j\} = 1 - (\lambda_j + \mu_j) \Delta t, \quad (3c)$$

where $\lambda_j = \beta j$, $\mu_j = \delta j + \gamma$. This defines the continuous-time dynamics of the population $j(t)$. Processes begin with an initial population $j(0) = N > 0$, after which stochastic transitions begin to occur.

As a choice of units, we set the death rate $\delta = 1$. The parameters β, δ, γ are defined in terms of some unit time; setting $\delta = 1$ is effectively a choice of time units, i.e., “minutes until

death.” Equivalently one can consider this a redefinition $\beta \rightarrow \beta/\delta$, $\gamma \rightarrow \gamma/\delta$. One must also specify some boundary conditions for the process. When $j = 0$, the process has absorbed. In this case $\lambda_0 = 0$, i.e., there is zero probability of another birth. For consistency one must set $\mu_0 = 0$ for any γ so no further deaths can occur. There is also an issue of boundary conditions as we move towards infinity in state space. While we consider the probability of the footprint reaching some finite size J , the analysis does not bound the population $j(t)$ in any normal way, and the parameter J does not enter into the dynamics.

A. General theory of the birth-death process

The most general birth-death process is defined by the transition probabilities above in Eq. (3), and is studied in its generality in, e.g., Refs. [1,2,26]. The birth-death-suppression model is just a specific instantiation with λ_j, μ_j being affine functions of the population $j(t)$. Consider first the population $j(t)$. A natural statistic to introduce is the probability of the population taking a certain value at a specific time:

$$p_n(t) = \Pr\{j(t) = n\}. \quad (4)$$

These population probabilities $p_n(t)$ should satisfy $0 \leq p_n(t) \leq 1$ and $\sum_n p_n(t) \leq 1$.¹ For an initial population of definite size $j(0) = N$, one would write $p_n(0) = \delta_{n,N}$ (here, the Kronecker delta). As time progresses, these probabilities will change in accordance with the transition probabilities as

$$\frac{d}{dt} p_n(t) = \lambda_{n-1} p_{n-1}(t) + \mu_{n+1} p_{n+1}(t) - (\lambda_n + \mu_n) p_n(t), \quad (5)$$

defining a differential-difference equation. If one can solve this equation to determine $p_n(t)$ for an arbitrary initial distribution $p_n(0)$, then one can deduce any expectations about the population dynamics. To this end, construct a related object: the transition probability matrix $P(t)$. This matrix has elements which are the (conditional) probability of being in state m at time t , conditioned on starting in state n at time $t = 0$:

$$P_{nm}(t) = \Pr\{j(t+s) = m | j(s) = n\}, \quad (6)$$

where $n, m \geq 1$. The Markov property ensures that this definition is independent of s . Note the restriction $n, m \geq 1$; this is done because zero is a special (absorbing) state in the process. Despite this exclusion, the probability $p_A(t)$ of absorbing at $j = 0$ can be calculated from $p_1(t)$ via Eq. (5), giving

$$p_A(t) = \mu_1 \int_0^t p_1(\tau) d\tau. \quad (7)$$

The absorption probability $p_A(t)$ will be of central interest.

For an initial population of definite size $j(0) = N$, the probabilities $p_n(t)$ form a row of the matrix $P(t)$:

$$p_n(t) = P_{Nn}(t). \quad (8)$$

¹In a finite state space, this would be a strict equality. The issue is accumulation of probability measure at infinity.

One can view $p_n(t)$ for $n \geq 1$ as a row vector $\langle p(t) |$. The initial conditions of the system $p_n(0)$ then form a vector. The transition matrix can be used to write

$$\langle p(t) | = \langle p(0) | P(t), \quad \frac{d}{dt} \langle p(t) | = \langle p(t) | \mathcal{A}, \quad (9)$$

also writing the differential-difference Eq. (5) in vector notation. The (possibly infinite) matrix \mathcal{A} is

$$\mathcal{A} = \begin{pmatrix} -(\lambda_1 + \mu_1) & \lambda_1 & 0 & 0 \\ \mu_2 & -(\lambda_2 + \mu_2) & \lambda_2 & 0 \\ 0 & \ddots & \ddots & \ddots \end{pmatrix}. \quad (10)$$

One should note that the probabilities $p_n(t)$, forming an infinite row vector, depend on a specific initial configuration $p_n(0)$, whereas the transition matrix $P(t)$ (forming an infinite matrix) does not. Given the dynamical equation for the $p_n(t)$, linearity implies that the transition matrix $P(t)$ satisfies, for any initial configuration $p_n(0)$, the first-order equation

$$\frac{d}{dt} P(t) = P(t) \mathcal{A}, \quad (11)$$

with the initial condition $P(0) = \text{Id}$, where Id is the identity matrix. In the Markov process literature, this is called the Kolmogorov backward equation. For a physicist, it is a first-order (Hamiltonian) dynamical equation, what statistical mechanics would refer to as a master equation.² Solving this equation determines the dynamics of the system for any initial configuration. The analogy between Markovian evolution and traditional statistical mechanics has been made precise in recent work, e.g., Ref. [36], or in Ref. [37], where the generality of the master equation is discussed.

If the state space is finite, then P and \mathcal{A} are finite-size matrices and the master equation admits the formal solution [38,39]

$$P(t) = \exp(\mathcal{A}t). \quad (12)$$

The matrix exponential is very cumbersome to compute for state spaces of any reasonable size, numerically or otherwise. In this paper, we take the state space to be infinite, so that P and \mathcal{A} are infinite matrices. In this case the matrix exponential solution is not well-defined; one must take a different approach. Thankfully, in an infinite state space, certain new analytical tools are available.

We assume throughout that β and γ are time-independent. Given this, the process is homogeneous and Markov. Physically, one might be interested in a situation in which the birth rate changes for some time: this would represent, for example, the effect of a high wind event increasing the spread rate of a fire. This is compatible with our assumptions so long as the birth rates are piecewise-constant. Say a process has birth rate β for $t < \tau$ and a different birth rate β' for $t > \tau$. The transition matrices may be stitched together, taking advantage

of the Markov property to do so:

$$\langle p(\tau + t) | = \langle p(0) | P(\tau; \beta) P(t; \beta'). \quad (13)$$

With this in mind, we take the birth and suppression rates to be time-constant, mostly without loss of generality. Note however that the footprint $F(t)$, being a cumulative variable, does not have the same Markovian properties as the population $j(t)$.

Given an expression for the transition matrix $P(t)$, and hence the probabilities $p_n(t)$, one can compute various probabilistic expectations for the system: the average population, the variance in population, or the absorption probability $p_A(t)$. Note that the absorption probability $p_A(t)$ is the probability that the population satisfies $j(T) = 0$ for any $T \leq t$; it is monotonically increasing. The actual time T of absorption (cf. extinguishment, eradication), which we refer to as the *lifetime*, is itself a variable of interest [25]. Since the absorption probability $p_A(t)$ is the integrated probability that $T \leq t$, the lifetime T is distributed as

$$d\sigma(T) \propto \frac{d}{dt} [p_A(T)] dT = \mu_1 p_1(T) dT \quad (14)$$

writing $d\sigma(T)$ for the normalized probability measure on lifetimes T . Determining this distribution $d\sigma(T)$ allows computation of statistics like the average $\langle T \rangle$ or median T_m lifetimes. We proceed to calculate these types of quantities in an effort to characterize the physical predictions of the Markov model.

B. The suppression-free process

Without suppression, the model reduces to the pure linear birth-death process with aggregate birth-death rates

$$\lambda_j = \beta j, \quad \mu_j = \delta j. \quad (15)$$

The dynamics of this process are well-known; they are solved by Kendall [25] using a generating function approach. The solution by this method and its details are reviewed in Appendix A. In this section, we discuss the phenomenology of this suppression-free birth-death process, using the language of the wildfire interpretation to add physical relevance to the formulas.

First, consider the evolution of the population. Let the process have some constant birth rate β and begin with an initial population of $j(0) = N$ firelets. With a death rate $\delta = 1$, the mean population is characterized by exponential growth or decay depending on the value of the birth rate:

$$\langle j(t) \rangle = N e^{(\beta-1)t}. \quad (16)$$

With $\beta < 1$, births happen more slowly than deaths and the population $j(t)$ reaches zero with probability 1: the fire is extinguished after some finite (but possibly long) time. This defines a phase of the process that we refer to as *subcritical*. When $\beta > 1$, the average population diverges in size exponentially and the phase is *supercritical*. Finally, in the *critical* $\beta = 1$ phase, the average population is constant for all time. The dynamics of the critical population consist of fluctuations in size around a constant average; indeed, when $\beta = 1$ the population variance $\Delta^2 j(t) \propto t$ increases linearly in time. This type of behavior is familiar from random walks

²Since P is an infinite matrix, one can regard it as a (positive semidefinite) operator on a Hilbert space, and interpret \mathcal{A} as a Hamiltonian: like a Hamiltonian, \mathcal{A} generates time translations of the “density” matrix of our system, $P(t)$.

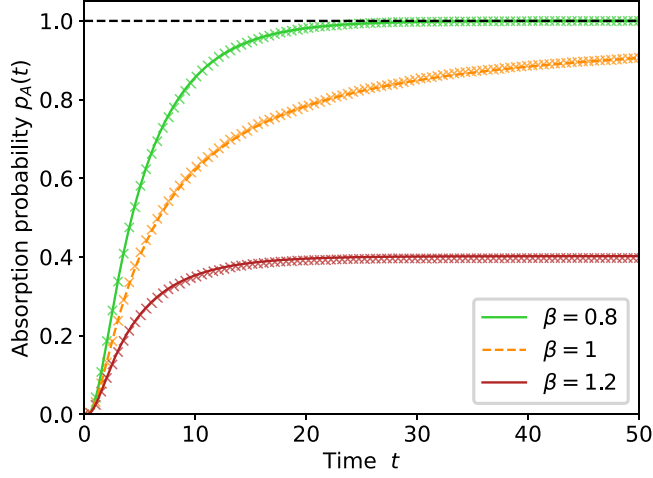


FIG. 3. Absorption probability—the probability of being extinguished—versus time. Here, $j(0) = N = 5$ with curves pictured for various birth rates, alongside scatter plots of simulated data (marked with an “x”) in an ensemble of 10^4 processes. The super- and subcritical curves (upper, lower) quickly approach their asymptotic limits, whereas the critical curve (middle) does so much more slowly. Both subcritical and critical processes approach an absorption probability of one as $t \rightarrow \infty$.

and is essentially Brownian motion in one dimension. Such a fire tends to develop a large footprint while maintaining a roughly even active size, potentially still posing a large risk.

To keep a small (or even finite) footprint, the fire must be extinguished—the population must reach the state $j(t) = 0$. The likelihood of this is characterized by the absorption probability introduced earlier. Here, it is given by

$$p_A(t) = \left(\frac{1 - e^{(\beta-1)t}}{1 - \beta e^{(\beta-1)t}} \right)^N, \quad (17)$$

$$\lim_{\beta \rightarrow 1} p_A(t) = \left(\frac{t}{1+t} \right)^N, \quad (18)$$

and is plotted in Fig. 3.

In the subcritical phase, absorption almost always occurs after long times. Specifically, taking the asymptotic limit $t \rightarrow \infty$ one finds

$$p_A(\infty) = \lim_{t \rightarrow \infty} G_0(0, t) = \begin{cases} \beta^{-N}, & \beta > 1, \\ 1, & \beta \leq 1. \end{cases} \quad (19)$$

This result states that asymptotically, fires in the critical or subcritical phases eventually extinguish. The probability of a supercritical fire extinguishing decreases exponentially in the initial size of the fire and is in general less than unity.

To better understand the timescales of absorption, consider the distribution of lifetimes, where the lifetime T is defined as the exact time of absorption. With a subcritical fire, it is most likely that absorption occurs early; the lifetime distributions have medians near $T_m \sim \log N$. The critical process, however, tends to take a much longer time to absorb, with the median critical lifetime going like $T_m \sim N$. Such long-lived fires would generically result in a large footprint. Note that because the distributions in question are conditioned on absorption occurring, the median lifetime in the supercritical

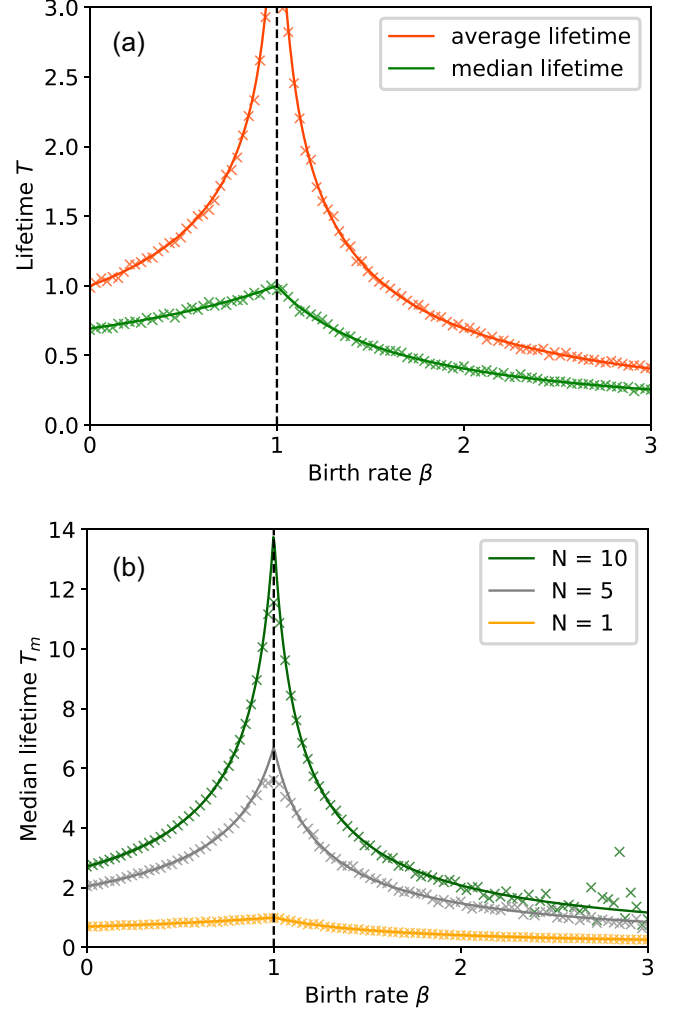


FIG. 4. Lifetime statistics versus birth rate for processes which end in absorption; exact results alongside simulated data (marked with an “x”) from an ensemble of 10^4 processes. In the case $N = 10$, 10^5 simulated processes were needed to obtain sufficiently conditioned data. (A) Average and median lifetimes versus birth rate β and with $j(0) = N = 1$. At criticality, the average lifetime diverges. (B) Median process lifetime for various initial population sizes, on a logarithmic scale. The dependence on the initial population N is strongest at criticality $\beta = 1$.

phase also goes like $T_m \sim \log N$, with $T_m \rightarrow 0$ as $\beta \rightarrow \infty$. These properties are demonstrated by the lifetime statistics in Fig. 4.

The distributions of absorption times for the suppression-free process share universal behavior with other absorbing Markov processes, and are in fact Gumbel-distributed [40]. This universality class is defined by processes where the birth/death rates decrease linearly as one approaches the absorbing state. This is true for the zero-suppression process in all phases, as was noted in Ref. [41].

It may seem counterintuitive that the supercritical median and average lifetimes are so comparable to the subcritical ones. This is because most supercritical fires are *not* absorbed at zero; those that do become absorbed likely reached $j = 0$ quickly, as a rare downward fluctuation. Hence the lifetime

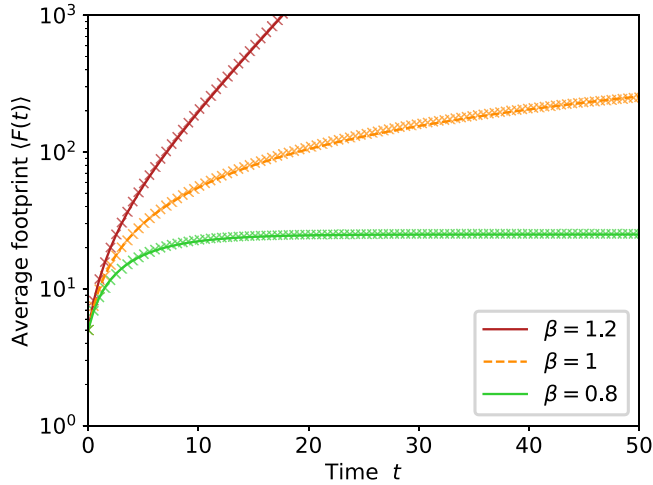


FIG. 5. Average footprint $\langle F(t) \rangle$ versus time on a logarithmic scale for various birth rates β , compared with simulated data (marked with an “x”) from an ensemble of 10^3 processes. Each curve has $N = 5$. The subcritical processes terminate with a finite average footprint while in the critical and supercritical phases, the average footprint continues to grow for all time, exceeding any finite bound J .

distributions, being predicated on absorption occurring, are quite similar away from criticality. In fact, it can be shown that supercritical processes with birth rate β , when conditioned on eventual absorption, behave like subcritical processes with birth rate $1/\beta$ [42,43]. When absorption does not happen, the processes are formally of infinite lifetime.

Basic statistics of these lifetime distributions are shown in Fig. 4, where one can see that proximity to criticality leads to the longest-lived eventually absorbing processes. For high supercritical birth rates, the vast majority of processes do not absorb. The distributions of lifetimes are therefore conditioned on extremely rare events, which means they have little physical importance. This is evidenced by the degeneration of the numerical data on the right-hand side of Fig. 4(b), which was generated from an ensemble of 10^5 simulated processes.

Along with the timescale of absorption, one ideally would understand the timescale and probability of escape: when (and if) the footprint reaches some given size J . Exact results on the joint distribution of $j(t)$ and $F(t)$ can be obtained using the method of the generating function, though most calculations using this method are out of practical reach [25]. The average footprint $\langle F(t) \rangle$, a tractable quantity, evolves as

$$\langle F(t) \rangle = \frac{N}{\beta - 1} (\beta e^{(\beta-1)t} - 1), \quad (20)$$

$$\lim_{\beta \rightarrow 1} \langle F(t) \rangle = N(t + 1). \quad (21)$$

In the subcritical phase there is a finite asymptotic limit whereas in the critical and supercritical phases the average footprint grows without bound, surpassing any finite size J . This behavior is demonstrated in Fig. 5. However, this quantity only minimally describes the distribution of footprints.

Finally, consider the asymptotic distribution of footprints. Empirically it is observed that the cumulative distribution of wildfire footprints is a power law; the probability $P(F \geq J)$

that a given fire has footprint larger than or equal to J goes like $\sim J^{-1/2}$ [16]. The same is true for the near critical birth-death process. In the large footprint F , $J \gg 1$ regime, one approximately has³

$$P(F_\infty \geq J) \approx \frac{J^{-1/2}}{\sqrt{\pi}} + \frac{(1 - \beta)}{2} \left(\frac{J^{-1/2}}{\sqrt{\pi}} - 1 \right), \quad (22)$$

expanding to first order below the critical point $\beta = 1$. By F_∞ we mean the asymptotic limit of $F(t)$, i.e., $F_\infty = \lim_{t \rightarrow \infty} F(t)$. The power-law scaling is correctly reproduced, and is stable to leading order in $(1 - \beta)$.

These results characterize the probability of absorption, its timescales, and the asymptotic distribution of footprints for the linear birth-death process without suppression. Since absorption occurs with probability 1 in the subcritical and critical phases, we have described these dynamics of these phases completely. In the following sections, we repeat this analysis for the birth-death process with suppression.

III. THE POPULATION DYNAMICS

We now turn to the full birth-death-suppression process, where we solve the master equation

$$\frac{d}{dt} P(t) = P(t) \mathcal{A}, \quad (23)$$

in which the matrix \mathcal{A} , defined in Eq. (10), has elements given by the birth and death rates $\lambda_j = \beta j$, $\mu_j = \delta j + \gamma$ (with $\delta = 1$) for each value of the population $j(t)$. For reasons of convention, we introduce a new variable k to index the state space defined simply as $k = j - 1$. In terms of this new index the birth-death rates are

$$\lambda_k = \beta(k + 1), \quad \mu_k = k + \gamma + 1, \quad k \geq 0. \quad (24)$$

The reason for this relabeling is to be consistent with the wider literature on birth-death processes and the conventions for orthogonal polynomials. Note that now the state $j = 0$ or $k = -1$ is absorbing, and that a process with one initial unit of the population $N = 1$ has $j(0) = 1$, $k(0) = 0$.

With zero suppression the method of generating function can be used to find the occupation probabilities for the population $j(t)$. In the presence of a nonzero-suppression factor $\gamma \neq 0$, an inhomogeneous term is introduced to the generating function equation preventing a direct solution by this method, as demonstrated in Appendix A. Given this obstacle, to compute the transition matrix $P(t)$ one must turn to a different approach: the method of Karlin and McGregor, making use of orthogonal polynomials, which is nothing more than solving Eq. (23) by eigenvector decomposition.

A. The method of Karlin and McGregor

The (infinite) matrix \mathcal{A} , and the matrix $P(t)$, are almost symmetric. To be precise, there exists a set of constants π_n such that the matrix $P(t)$ satisfies the symmetry condition

$$\pi_n P_{nm} = \pi_m P_{mn}, \quad (25)$$

³To get this result one approximates a sum by an integral and enters the asymptotic regime $J, F \gg 1$.

where the π_n are defined as

$$\pi_0 = 1, \quad \pi_n = \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n}, \quad n, m \geq 0. \quad (26)$$

This symmetry relation (also referred to as the condition of detailed balance) describes the intuitive fact that the process is time-reversed if one swaps $\lambda \leftrightarrow \mu$. Together, the backward Eq. (23) and the symmetry condition (25) imply that the transition matrix $P(t)$ satisfies the Kolmogorov forward equation, that is

$$\frac{d}{dt}P(t) = \mathcal{A}P(t). \quad (27)$$

The strategy is to solve the above by eigendecomposition of the matrix \mathcal{A} . Assume \mathcal{A} has a complete set of eigenvectors⁴ $|Q(x)\rangle$ with eigenvalues $-x$ and $x \geq 0$: the eigenvalues must be nonpositive so that matrix elements of $P(t)$ remain bounded as $t \rightarrow \infty$. We seek an explicit form of the eigenvectors $|Q(x)\rangle$. To make this characterization, write the abstract equation $-x|Q(x)\rangle = \mathcal{A}|Q(x)\rangle$ in its component form:

$$Q_0 = 1, \quad (28)$$

$$-xQ_0 = -(\lambda_0 + \mu_0)Q_0 + \lambda_0Q_1, \quad (29)$$

$$-xQ_k = -(\lambda_k + \mu_k)Q_k + \lambda_kQ_{k+1} + \mu_kQ_{k-1}. \quad (30)$$

The components $Q_k(x)$ are defined by a three-term recurrence relation, where $\deg Q_k(x) = k$; they are a family of orthogonal polynomials in x on the real half line $x \geq 0$. Indeed, Favard's theorem states that any $Q_k(x)$ satisfying a three-term recurrence relation like the above are orthogonal polynomials with respect to some positive spectral measure $d\sigma(x)$ [44]. The polynomials $Q_k(x)$ are therefore the object of central interest. They satisfy the normalized orthogonality relation

$$\int_0^\infty Q_n(x)Q_m(x)d\sigma(x) = \frac{\delta_{nm}}{\pi_m}, \quad (31)$$

which may be derived from their recurrence relations. From these polynomials, and their associated spectral measure $d\sigma(x)$, the matrix elements of $P(t)$ are given by the integral

$$P_{nm}(t) = \pi_m \int_0^\infty e^{-xt} Q_n(x)Q_m(x)d\sigma(x). \quad (32)$$

This is the *spectral form* of the transition matrix and can easily be seen to solve the forward Eq. (27) by virtue of the recursion satisfied by the polynomials $Q_k(x)$.

However, determining the polynomials $Q_k(x)$ and their spectral measure may not be straightforward. For the process $\lambda_k = \beta(k+1)$, $\mu_k = k + \gamma + 1$, the polynomials in question can be identified—as done by Karlin and McGregor [28]—as a known set of polynomials of the Askey scheme, a classification of hypergeometric orthogonal polynomials [45]. To be precise, the polynomials of interest are of the Meixner and Laguerre families. Defined in Appendix B, each polynomial

family is associated with a different phase of the process: $\beta > 1$, $\beta = 1$, $\beta < 1$.

Given the known expressions and identities for these polynomials one can derive explicit expressions for the transition matrix elements $P_{nm}(t)$. The following subsections make use of these results to compute statistics of the suppressed birth-death process, and to analyze the effect of suppression on the population.

B. Lifetime statistics

The starting point for determining the distribution of lifetimes and hence the probability of absorption is computing the transition matrix element $P_{n0}(t)$, a result appendicized in Eq. (B27). The element P_{n0} represents the probability of transitioning from a state $j = n + 1$ to the $j = 1$ state, which physically represents a fire with a single active firelet.

In the right coordinates, the measure on lifetimes takes a particularly simple form. The random variable T , the lifetime or exact time of absorption, for processes with $j(0) = N$ is distributed as $\propto P_{N-1,0}(T)dT$. In terms of the variable $s(t) = e^{(\beta-1)t}$, the matrix element $P_{N-1,0}$ is

$$P_{N-1,0}(s) = \frac{(\gamma+2)_{N-1}}{(N-1)!} s^{\gamma+1} \left(\frac{1-\beta}{1-\beta s} \right)^{\gamma+2} \left(\frac{1-s}{1-\beta s} \right)^{N-1}. \quad (33)$$

This formula is valid for both the super- and subcritical cases. It is convenient to define yet another variable $z(t)$ as

$$z(t) = \frac{1 - e^{(\beta-1)t}}{1 - \beta e^{(\beta-1)t}} = \frac{1 - s(t)}{1 - \beta s(t)}, \quad (34)$$

$$\lim_{\beta \rightarrow 1} z(t) = z_c(t) = \frac{t}{1+t}. \quad (35)$$

The coordinate $z(t)$ is equivalent to the probability of absorption in the zero suppression, single initial firelet process, and so satisfies $0 \leq z(t) \leq 1$ always. Additionally, for any phase of the process, $z(t)$ is monotonically increasing in time. Asymptotically,

$$\lim_{t \rightarrow \infty} z(t) \equiv z_\infty = \begin{cases} 1, & \beta \leq 1, \\ \frac{1}{\beta}, & \beta > 1. \end{cases} \quad (36)$$

The measure on lifetimes $d\sigma(T) \propto P_{N-1,0}(T)dT$, after a change of variables $T \rightarrow z(T)$, may be written

$$d\sigma(T) \propto (1-z)^\gamma z^{N-1} dz(T), \quad (37)$$

where the proportionality is fixed by normalization. The normalized measure is

$$d\sigma(T) = \frac{1}{B(z_\infty; N, \gamma+1)} (1-z)^\gamma z^{N-1} dz(T). \quad (38)$$

The normalization is an incomplete β function. Defined by the integral

$$B(x; a, b) = \int_0^x dt t^{a-1} (1-t)^{b-1}, \quad (39)$$

for $x = 1$ it coincides with the classical β function of Euler:

$$B(1; a, b) = B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \quad (40)$$

⁴To be precise, one can define the symmetric operator H with matrix elements $H_{nm} = \pi_n \mathcal{A}_{nm}$ to apply a version of the spectral theorem for self-adjoint operators.

The lifetimes are therefore β -distributed, or approximately so, with shape parameters N and $\gamma + 1$. The absorption probability $p_A(t)$ is simply the cumulative distribution function, which here is given by the regularized incomplete β function [46]:

$$p_A(t) = \frac{B(z(t); N, \gamma + 1)}{B(N, \gamma + 1)}. \quad (41)$$

Finally, consider the asymptotic probability of absorption. Since $\gamma > 0$ only increases the aggregate death rate, absorption should remain asymptotically certain except in the supercritical phase. Indeed, for $\beta \leq 1$, the limit is

$$p_A(\infty) = \frac{B(1; N, \gamma + 1)}{B(N, \gamma + 1)} = 1, \quad (\beta \leq 1), \quad (42)$$

where the β function $B(1; a, b) = B(a, b)$ is “completed” for $z(\infty) = 1$. In the supercritical case the situation is more complicated as the β function remains incomplete; the exact expression is Eq. (41) with $z = 1/\beta$. In the special case $N = 1$, the expression simplifies to

$$p_A(\infty) = 1 - \left(\frac{\beta - 1}{\beta}\right)^{\gamma+1}, \quad (\beta > 1). \quad (43)$$

Note that this expression, the probability of the process ending at $j = 0$, is the complementary probability to the process ending “at infinity.” The measure at infinity and its interpretation will be discussed further in Sec. IV.

C. The effect of suppression

Having computed the distributions of lifetimes and probability of absorption, one can now ask: What is the effect of suppression $\gamma > 0$ on the population dynamics? Without suppression, the average population $\langle j(t) \rangle$ is characterized by exponential growth or decline depending on the phase of the process. With $N = 1$, the average population with suppression is

$$\langle j(t) \rangle = e^{(\beta-1)t} [1 - z(t)]^\gamma, \quad (44)$$

$$\lim_{\beta \rightarrow 1} \langle j(t) \rangle = \left(\frac{1}{1+t}\right)^\gamma. \quad (45)$$

With $\gamma \geq 0$ and $z(t) < 1$, the average population in the presence of suppression is always less than it would be with $\gamma = 0$, and the average population goes to zero in the limit of infinite suppression $\gamma \rightarrow \infty$ for any fixed value of β . Indeed, the presence of suppression greatly increases the chance that small fires absorb at zero, decreasing the time at which the absorption probabilities become reasonably close to their asymptotic values. This is demonstrated in Fig. 6, where one can see that suppression also significantly increases the asymptotic probability of absorption for supercritical processes.

To see the interplay between the initial size N of the fire and the suppression γ , consider the normalized distribution of lifetimes for $\beta \leq 1$. Here, $z(T)$ runs from 0 to 1 with the β distribution

$$d\sigma(T) = \frac{1}{B(N, \gamma + 1)} (1 - z)^\gamma z^{N-1} dz(T), \quad (46)$$

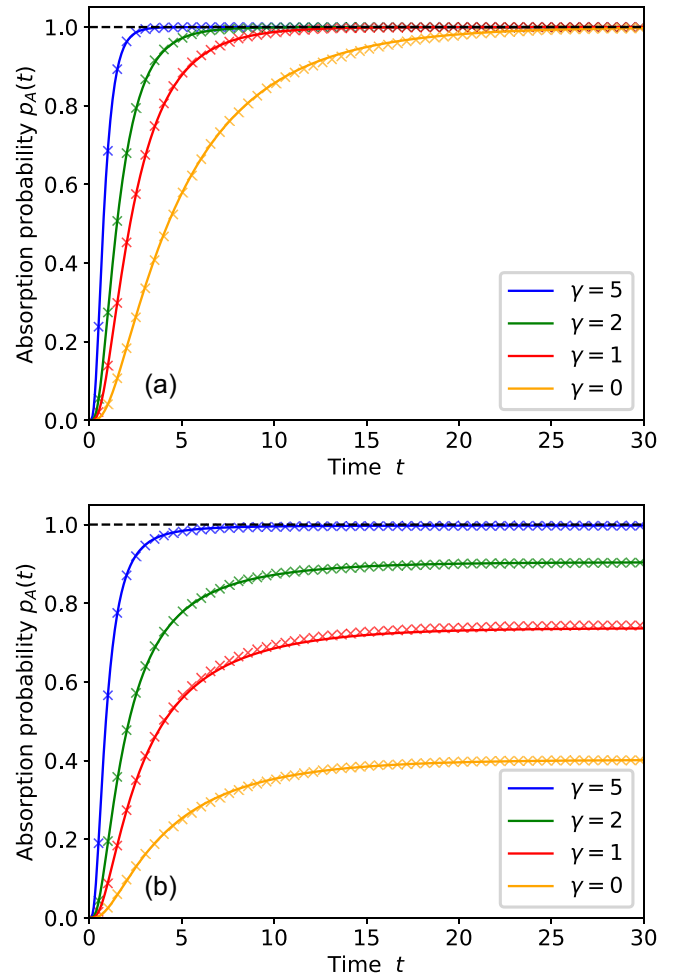


FIG. 6. Absorption probabilities versus time at nonzero suppression γ alongside simulated data (marked with an “x”) from an ensemble of 10^4 processes. Shown is the effect of suppression on the probability of absorption in a subcritical process (A, with $\beta = 0.8$) and a supercritical process (B, with $\beta = 1.2$). Both began with an initial population $N = 5$. In B, with a supercritical birth rate, suppression increases limiting value of the absorption probability and hence the overall share of absorbing processes.

where the lifetime T may be found from $z(T)$ via

$$T = \frac{1}{\beta - 1} \log \left[\frac{1 - z(T)}{1 - \beta z(T)} \right]. \quad (47)$$

When N becomes large compared to γ , the mass of the distribution (46) becomes concentrated around $z \lesssim 1$. This means that lifetimes in this regime are likely to be longer, consistent with the physical expectation that larger fires burn out more slowly. In the strict large size limit $N \gg 1$, suppression is negligible: the aggregate death rate $\mu_j = j + \gamma$ is dominated by the first term proportional to N . The limit as $\gamma \rightarrow 0$ is not singular and commutes with the limit $N \gg 1$. However, when γ is large compared to N , the distribution becomes peaked near $z \gtrsim 0$, telling us that highly suppressed processes absorb quickly. None of this is physically surprising and is demonstrated by the left-hand and right-hand distributions of Fig. 7.

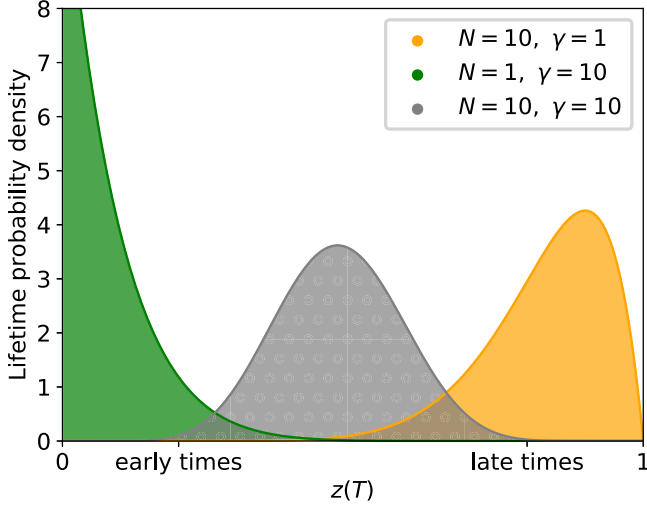


FIG. 7. The exact distribution of lifetimes for various values of the initial population N and suppression rate γ . Lifetimes are β -distributed with respect to the coordinate $z(T)$ defined in Eq. (34); here the measures are pictured with $\beta = 1$. The normal limit of the β distribution is apparent in the center distribution, where $N \gg 1$ and $\gamma \gg 1$.

However, there is another limit: simultaneously taking $N \gg 1$ and $\gamma \gg 1$. Recalling that γ is defined in units of the death rate of a single firelet, this limit corresponds to a large initial population coupled with suppression which eradicates the population much more quickly than natural deaths would occur (a physically reasonable regime for fire suppression). In this limit, the distribution on lifetimes converges to a normal distribution with mean $\langle z \rangle = N/(N + \gamma)$ and variance which scales like $\sim 1/N$. The measure (46) is shown in Fig. 7 for different values of the parameters N, γ , demonstrating the different regimes.

The median and mean coincide in the simultaneous limit $N, \gamma \gg 1$ and one has the median lifetime

$$T_m \approx \frac{1}{1 - \beta} \log \left[1 + \frac{N}{\gamma} (1 - \beta) \right], \quad (48)$$

$$\lim_{\beta \rightarrow 1} T_m \approx \frac{N}{\gamma}. \quad (49)$$

As in the zero-suppression case, the median lifetimes display a logarithmic dependence on the initial population N away from criticality which at criticality is enhanced to a stronger, linear dependence. However, the suppression rate γ now acts as a multiplicative dressing which reduces the effective initial population. Therefore, to obtain a median lifetime which is $\mathcal{O}(1)$, the suppression rate, as multiples of the natural death rate, must be of the same order as the initial size. The median lifetime may be represented exactly as an inverse regularized β function; a contour plot of the exact median lifetime T_m in the β, γ plane is shown in Fig. 8.

The statistics on lifetimes describe only those processes which eventually absorb. However, the time until absorption may be infinite: at criticality, it is on average infinite. Any process with an infinite lifetime is clearly unrealistic—it does not consider any physical bound on the population $j(t)$ or the

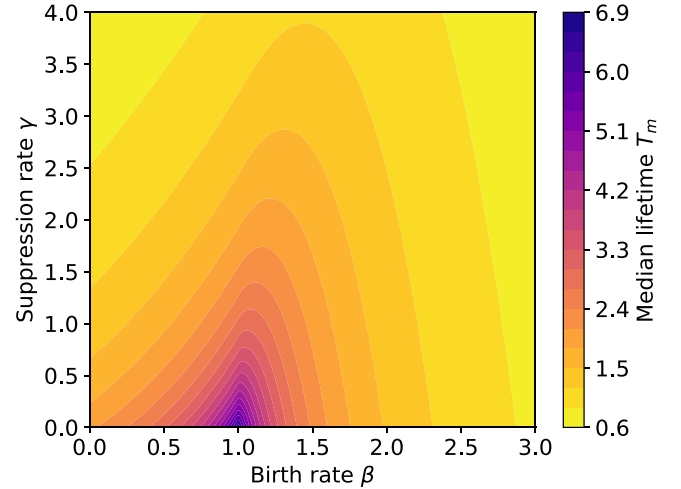


FIG. 8. Contour plot of the median lifetime in the (β, γ) plane, pictured for a fire with $N = 5$ and conditioned on eventual absorption. The longest-lived finite processes are near unsuppressed criticality.

footprint $F(t)$. In a real modeling scenario, systems are not of infinite size; there will be a finite effective size J to any system. The presence of this threshold prevents sojourns of infinite time. In the following section, we develop the theory of the footprint $F(t)$ and address these issues of infinitude.

IV. PREDICTING THE FOOTPRINT

Recall that the footprint $F(t)$ is defined to share all the positive transitions of $j(t)$. It is therefore cumulative, monotonically increasing *ad infinitum* unless absorption at $j = 0$ occurs. The notion of a cumulative population for the birth-death process was introduced by Kendall [3,25], where its interpretation was that of measuring the severity of an epidemic. In the wildfire interpretation, the cumulative population (or footprint) counts the total burned area and serves as a measure of the overall severity of the fire event.

In Appendix A, generating function methods are used to obtain predictions about the asymptotic distribution of footprints in the zero-suppression process, re-creating the results of Kendall. As before, for the population dynamics, this approach is unsuitable for the suppressed process with $\gamma > 0$. The generating function equation depends on an unknown function.

The assumption is that the goal of suppression is to extinguish a fire while keeping the footprint $F(t)$ less than some threshold J . To make this precise, define the *escape probability*, or $P_J(N)$, by

$$P_J(N) = \{\text{probability of } F(\infty) \geq J | j(0) = N\}. \quad (50)$$

This is the probability that the footprint at some point exceeds the threshold J of burnable substrate. Minimizing the escape probability is a natural metric for successful suppression.

A process which escapes, having $F(\infty) \geq J$, may or may not eventually absorb, while a process which does not escape (with $F(\infty) < J$) must end in absorption. The definition of the escape probability does not reference time (or time of absorption) but rather the asymptotic outcome of a given process.

Computing $P_j(N)$ is therefore a matter of counting *transitions* of the Markov process rather than time. However, being agnostic to the time of absorption, this quantity alone does not give any estimate of when the footprint $F(t)$ may reach some value. With no time variable, the birth-death process becomes a random walk on the state space. We turn now to analyzing this random walk.

A. Random-walk polynomials

For any continuous-time birth-death process with rates (λ_k, μ_k) with $k \geq 0$ (recall $k = j - 1$ where j is the number of firelets) there exists an embedded Markov chain: the *jump chain*. It is a random walk on the state space where the transition probabilities are

$$p_k \equiv \Pr\{k \rightarrow k + 1\} = \frac{\lambda_k}{\lambda_k + \mu_k}, \tag{51}$$

$$q_k \equiv \Pr\{k \rightarrow k - 1\} = \frac{\mu_k}{\lambda_k + \mu_k}. \tag{52}$$

The continuous-time process solved in the previous section is just this jump chain coupled with holding times in each state which are exponentially distributed $\Delta t \propto \exp(-(\lambda_k + \mu_k)t)$. This is how one simulates a continuous-time process on a computer.

Let $k(n)$ be the state of the process after n transitions: in the present context, $j = k + 1$ still represents the actively burning population, but we are no longer keeping track of time t . The state depends instead on the transition count n . Consider the aggregate birth and death rates given by

$$\lambda_k = \beta(k + 1), \quad \mu_k = k + \gamma + 1. \tag{53}$$

Recall that here, the $k = -1$ state is absorbing, with $p_{-1} = q_{-1} \equiv 0$. The jump variable absorbs after some number of transitions n_T , i.e., $k(n_T) = -1$, after which there are no further transitions. The goal is to characterize the distribution of transition counts n_T at absorption: this will also determine the distribution of footprints at absorption. The problem is analogous to determining the distribution of lifetimes T in the continuous-time process, except now one seeks a distribution over the positive integers.

In the embedded jump chain, the object of interest is the n -step transition matrix $S(n)$ with elements

$$S_{\ell, \ell'}(n) = \Pr\{k(m + n) = \ell' | k(m) = \ell\}, \quad \ell, \ell' \geq 0, \tag{54}$$

with the analogy $S(n) \sim P(t)$. It obeys a semigroup condition

$$S(n + 1) = S(1)S(n) \tag{55}$$

instead of a master equation.

In the absence of suppression, the random walk associated to the birth-death process is a simple random walk: the probabilities p_k, q_k are independent of the state k . The transition matrix $S(n)$ can therefore be determined either by the method of generating function or standard binomial arguments: it becomes equivalent to the ‘‘gambler’s ruin’’ problem; see, e.g., Ref. [47]. However, with a nonzero-suppression factor, the probabilities p_k, q_k are state-dependent and the random walk is not simple. As with the continuous-time process, the presence of a suppression term introduces an unknown function to

the generating function equation and renders a direct solution infeasible; these more standard approaches will not work.

The master equation governing the continuous-time process was solved by eigenspace decomposition. This amounted to defining and determining a set of orthogonal polynomials, an approach introduced by Karlin and McGregor [27]. In the later work [48], the same authors show that the orthogonal polynomial approach can also solve the random-walk problem.

The procedure is essentially the same as in the continuous-time process: the transition probabilities define a set of polynomials $W_k(x)$ via the relations

$$W_0(x) = 1, \tag{56}$$

$$xW_0(x) = p_0W_1(x), \tag{57}$$

$$xW_k(x) = q_kW_{k-1}(x) + p_kW_{k+1}(x). \tag{58}$$

The structure of these recurrence relations and the relation $p_k + q_k = 1$ implies that the $W_k(x)$ are orthogonal on the interval $[-1, 1]$ with respect to some even measure $d\sigma(x)$:

$$\int_{-1}^1 W_\ell(x)W_{\ell'}(x)d\sigma(x) = h_\ell\delta_{\ell\ell'}. \tag{59}$$

Given the polynomials $W_k(x)$ and their spectral measure $d\sigma(x)$, the n -step transition matrix elements have the spectral form

$$S_{\ell, \ell'}(n) = \frac{1}{h_{\ell'}} \int_{-1}^1 x^n W_\ell(x)W_{\ell'}(x)d\sigma(x). \tag{60}$$

The immediate focus is to compute this object, followed by relating properties of this matrix $S(n)$ to the distribution of footprints F .

Previously, when applying the method of orthogonal polynomials to the continuous-time process, the polynomial families of interest were known and classified: they were Meixner polynomials away from criticality and Laguerre polynomials at criticality. Both these fall into the Askey scheme of hypergeometric orthogonal polynomials [45]. The identification was made by examining the recurrence relation given by the birth and death rates and comparing it to known recurrence relations for previously classified polynomials. In the present case, the recurrence is

$$xW_n(x) = \frac{\beta(n + 1)}{(\beta + 1)(n + 1) + \gamma}W_{n+1}(x) + \frac{n + \gamma + 1}{(\beta + 1)(n + 1) + \gamma}W_{n-1}(x). \tag{61}$$

The recurrence has two parameters β, γ , and a particular (rational) dependence on n in its coefficients. An examination of all the polynomials in the Askey scheme reveals that this recurrence is not so classified. However, the $W_n(x)$ are equivalent to a previously studied set of polynomials outside the Askey scheme⁵: the Pollazcek polynomials [49,50]. The

⁵We are indebted to W. Van Assche for his correspondence, pointing out this equivalence.

Pollazcek polynomials $P_n^\lambda(x; a, b)$ have been studied in numerous physical [51,52] and mathematical contexts [53–57]. The exact correspondence is

$$W_n\left(\frac{2\sqrt{\beta}}{\beta+1}x\right) = P_n^\lambda(x; a, b)\beta^{-n/2}, \quad (62)$$

where the parameters λ , a , b are related to the birth-death-suppression parameters as

$$b = 0, \quad \lambda = 1 + \frac{\gamma}{2}, \quad \lambda + a = 1 + \frac{\gamma}{\beta + 1}. \quad (63)$$

Expressions for the Pollazcek polynomials and their measure of orthogonality have been found by various methods, but the parametrizations used are quite different from the present set. In the interest of examining the structure of these polynomials from the perspective of the birth-death-suppression process, in Appendix B 3 we explicitly construct the polynomials $W_n(x)$ and their measure analytically. The reader is directed to the Appendices for background, while in the present section we will continue to build calculational tools from those results.

The analysis required to construct the polynomials is inspired by the closely related Sec. 4 of Askey and Ismail [30] which does the same for the birth-death-immigration (BDI) process. This closely related process has aggregate birth and death rates $\lambda_j = \beta j + m$, $\mu_j = \delta j$, where m is interpreted as an immigration rate: the j -independent rate at which new units are added to the population. The suppression rate is roughly the opposite of the immigration rate, where the suppression γ is the j -independent rate at which units are removed from the population. As mentioned in the introduction, the ergodic continuous-time BDI process is dual to the continuous-time birth-death-suppression process.⁶ As a result, the phenomenology of the orthogonal polynomials for the BDI process is very similar to that of the birth-death-suppression process analyzed here. However, the duality of the two continuous-time processes does not naturally descend to a duality between the embedded jump chains of the two processes.

B. Footprint

The Pollazcek polynomials $W_n(x)$ characterize the jump chain dynamics of the birth-death-suppression process. Their explicit description and complete orthogonality relation (C55) is derived and given in Appendix B 3. The phenomenology of the polynomials changes between the phases of the process. In the subcritical and critical phase $\beta \leq 1$, they have a continuous measure of orthogonality supported only on the closed interval $[-I_\beta, I_\beta]$ where $I_\beta = 2\sqrt{\beta}/(1 + \beta)$. At criticality $\beta = 1$, they are equivalent to the Gegenbauer polynomials $C_n^\lambda(x)$ as

$$W_n^c(x; \gamma) = C_n^{1+\gamma/2}(x), \quad (64)$$

and are supported on the whole interval $[-1, 1]$. The Gegenbauer polynomials fall in the Askey scheme; they are a special

case of the Jacobi polynomials. Their measure of orthogonality is given by

$$d\sigma(x) = \frac{\Gamma(\frac{\gamma}{2} + 2)}{\sqrt{\pi}\Gamma(\frac{\gamma+3}{2})}(1-x^2)^{(\gamma+1)/2}dx. \quad (65)$$

However, in the supercritical phase $\beta < 1$, the continuous support retracts back to $|x| \leq I_\beta$. In its place a discrete measure develops at infinitely many points x_k with discrete measure weights Δ_k . In the zero-suppression limit $\gamma \rightarrow 0$, the weights of the discrete measure also disappear.

An explicit expression for $W_n(x)$ is given in Eq. (C20). The continuous measure $w(x)dx$ is given in Eq. (C39) and the discrete measure weights Δ_k in Eq. (C54). The elements of the n -step transition matrix $S(n)$ therefore have the spectral form

$$\begin{aligned} S_{\ell, \ell'}(n) &= \frac{1}{h_{\ell'}} \int_{-I_\beta}^{I_\beta} x^n W_\ell(x) W_{\ell'}(x) w(x) dx \\ &+ \frac{1}{h_{\ell'}} \sum_{k=0}^{\infty} (x_k)^n \Delta_k W_\ell(x_k) W_{\ell'}(x_k) \\ &+ \frac{1}{h_{\ell'}} \sum_{k=0}^{\infty} (-x_k)^n \Delta_k W_\ell(-x_k) W_{\ell'}(-x_k). \end{aligned} \quad (66)$$

The constants h_n , which are analogous to the (reciprocal of the) constants π_n are given by

$$h_n = \frac{(\gamma + 2)_n}{\beta^n n!} \cdot \frac{1 + \beta + \gamma}{\gamma + (\beta + 1)(n + 1)}, \quad h_0 = 1. \quad (67)$$

With all this machinery one can now explicitly (numerically) compute arbitrary matrix elements of $S(n)$. In turn these matrix elements describe aspects of the footprint dynamics.

To see this connection, consider first the analog of the lifetime distribution: the distribution of transition counts n_T at absorption. In particular, let R_N^n be the probability that, given $j(0) = N$, the process absorbs with exactly n transitions [28,48]. This quantity, analogous to the absorption probability, is given by

$$\begin{aligned} R_N^n &= q_0 S_{N-1,0}(n-1) \\ &= \frac{\gamma + 1}{\beta + \gamma + 1} \int_{-1}^1 x^{n-1} W_{N-1}(x) d\sigma(x). \end{aligned} \quad (68)$$

The R_N^n are directly related to the footprint F at absorption. Take a process which begins with an initial population N and absorbs after n_T transitions. From the perspective of the jump chain, the entire process is just a sequence of births $j \rightarrow j + 1$ and deaths $j \rightarrow j - 1$, in some particular order. If the total number of births is n_B and the total number of deaths is n_D , the process, ending in absorption, must have $n_D - n_B = N$, in addition to the basic statement $n_B + n_D = n_T$. The footprint F at absorption, by definition, is the number n_B of births plus the initial size N . One therefore has the direct constraint

$$n_T = 2F - N, \quad (70)$$

relating the number of transitions n_T of an absorbing process to the initial population N and the footprint F at absorption. Therefore, the probability that a process with $j(0) = N$ has

⁶There exists a duality mapping processes with $\mu_0 > 0$ to those with $\mu_0 = 0$; see Sec. 8.2 in Ref. [1].

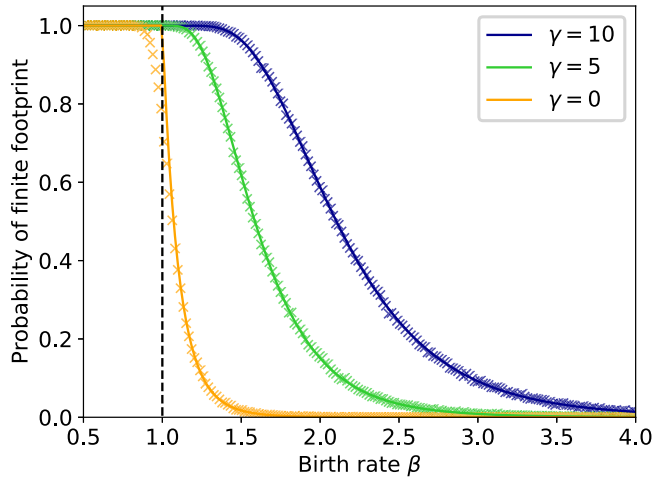


FIG. 9. The probability of a finite footprint versus birth rate. Equivalently, the probability of eventual absorption, pictured for a fire with $N = 10$ and three values of suppression γ ; simulated results (marked with an ‘x’) from an ensemble of 10^4 processes with a footprint threshold $J = 500$. The critical birth rate $\beta = 1$ is demarcated as a dashed line. The difference between the curves and 1 is the probability measure ‘at infinity.’ Suppression extends the flat portion of the curves on the left where almost all processes have finite footprint, which is true of all processes with $\beta < 1$.

footprint equal to F at absorption is equivalent to the probability that the process absorbs in exactly $2F - N$ transitions, or just R_N^{2F-N} .

While all subcritical processes eventually absorb, a large number of supercritical processes continue population growth without bound. To quantify this behavior, one can use the R_N^n to compute the probability of absorbing with a finite number of transitions, or equivalently a finite footprint. This is simply the sum over n of all the R_N^n , a quantity which is not necessarily equal to 1. It is of course equivalent to the asymptotic probability of absorption, as any process with a bounded footprint must eventually absorb. We therefore have in all phases of the process the equality

$$\Pr\{F < \infty | j(0) = N\} = \Pr\{j(\infty) = 0 | j(0) = N\}$$

$$q_0 \int_{-1}^1 \frac{W_{N-1}(x)}{1-x} d\sigma(x) = \frac{B(z_\infty; N, \gamma + 1)}{B(N, \gamma + 1)}, \quad (71)$$

where $z_\infty = \min(1, 1/\beta)$. This quantity is plotted in Fig. 9 versus birth rate for a few values of the suppression parameter; it is clear that once the supercritical regime is entered, a significant amount of measure accumulates quickly at infinity.

The escape probability of Eq. (50), that $F \geq J$ at absorption, parametrizes by J a portion of the complement to a finite process. This includes the measure at infinity. The probability $P_J(N)$ can thus be expressed in terms of a spectral integral as

$$P_J(N) = 1 - \sum_{n=0}^{2J-N} R_N^n$$

$$= 1 - q_0 \int_{-1}^1 \left(\frac{1 - x^{2J-N}}{1-x} \right) W_{N-1}(x) d\sigma(x). \quad (72)$$

As a demonstration of this machinery, we briefly revisit the asymptotic distribution of footprints. With zero suppression this was given by the power law $P_J \sim J^{-1/2}$ near criticality. Here, consider the $N = 1$ escape probability (72) at criticality $\beta = 1$. Using the Gegenbauer measure of Eq. (C4), by direct integration one finds

$$P_J(1) = \frac{\Gamma(J - 1/2)\Gamma(1 + \gamma/2)}{\sqrt{\pi}\Gamma(J + \gamma/2)}$$

$$\sim \frac{\Gamma(1 + \frac{\gamma}{2})}{\sqrt{\pi}J^{1+\gamma}} + \mathcal{O}(J^{-3/2}), \quad (73)$$

where the correct power law, with a modification due to the suppression factor, emerges in the $J \gg 1$ limit. For small initial sizes, nonzero suppression strongly affects the absorption probability and hence the distribution of footprints at large J . If N is large compared to γ , then this effect will be attenuated. However, finding simple analytic results in this regime is not straightforward.

In the critical case one can explicitly compute the average number of transitions at absorption $\langle n_T \rangle$, which is by linearity related to the average footprint at absorption $\langle F \rangle = (N + \langle n_T \rangle)/2$. For a single initial firelet $N = 1$ at critical conditions $\beta = 1$, the average number of transitions is given by the integral

$$\langle n_T \rangle = \sum_{n=1}^{\infty} n R_1^n$$

$$= \frac{\gamma + 1}{\gamma + 2} \int_{-1}^1 \frac{\Gamma(\frac{\gamma}{2} + 2)}{\sqrt{\pi}\Gamma(\frac{\gamma+3}{2})} \cdot \frac{(1-x^2)^{(\gamma+1)/2}}{(1-x)^2} dx$$

$$= \frac{\gamma + 1}{\gamma - 1}, \quad \gamma > 1. \quad (74)$$

Interestingly, the average transition count (and hence average footprint) diverges for $\gamma \leq 1$. We showed earlier that the unsuppressed critical process had an infinite average lifetime. This reiterates that result (at zero suppression) but takes into account that for $\gamma > 1$ one has $\lambda_1/\mu_1 \leq 1/2$, which causes sufficiently many processes to absorb early that the transition and hence footprint distribution has finite mean.

The transition count and thus footprint at absorption is not the only quantity over which the matrix $S(n)$ affords traction. The matrix element $S_{ij}(n)$ describes the probability of transitioning from state i to state j after n transitions. This is equivalent to the probability of transitioning from state i to state j after increasing the footprint by $n_B = (n + j - i)/2$, i.e., after n_B births $j \rightarrow j + 1$. By combining these dynamics with the continuous-time transition matrix, it could be possible to make statements about the time-dependent distribution of footprints rather than only its asymptotic behavior. This is beyond the scope of the current work.

Finally, in a modeling scenario one may need to compute the escape probability for an ensemble of processes, possibly with varying footprints. Without perfect knowledge of the time-dependent footprint distribution, this must be done approximately. If a process begins with a definite population N and threshold J , after some finite time t_0 it evolves to give probabilities $p_n(t_0)$ for each possible population state

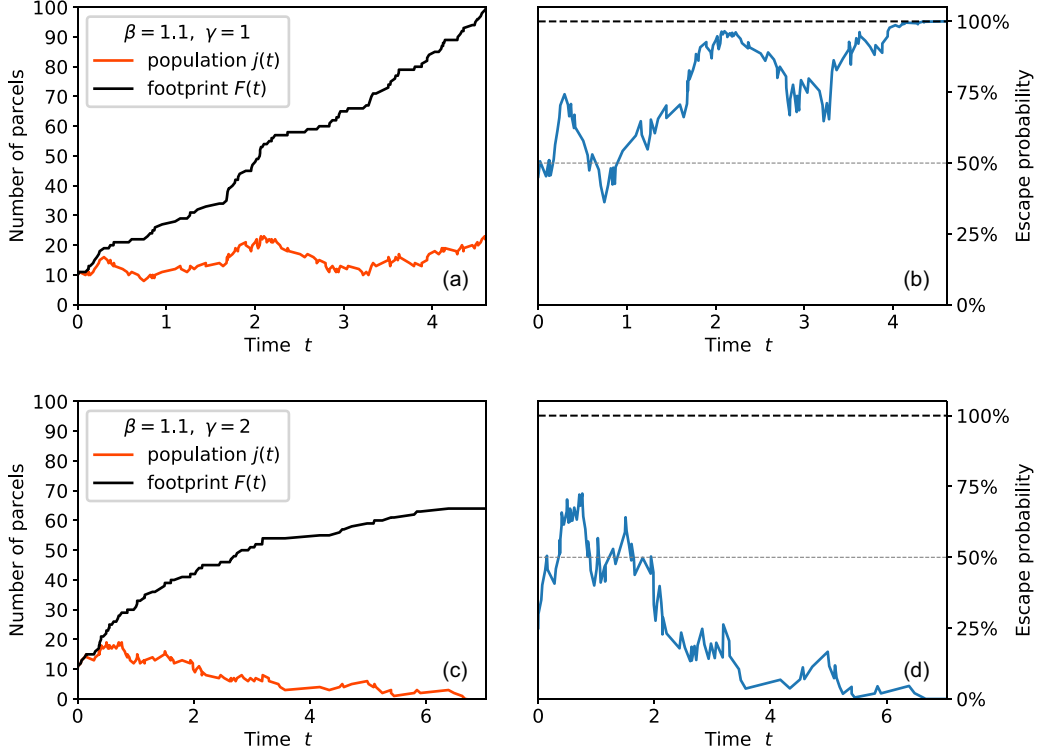


FIG. 10. Dynamic risk estimation over the lifetime of a fire event. Pictured are simulated birth-death-suppression processes (A, C) and the escape probability $P_J(N)$ recomputed after each transition, serving as a real-time risk metric (B, D). Both processes have $\beta = 1.1$, $N = 10$, $J = 100$. The process which escapes (A) has $\gamma = 1$, while the process which absorbs (C) has $\gamma = 2$.

$j(t_0) = n$.⁷ By the Markov property, one may regard this as an ensemble of new processes with different initial populations N_i , each with an ensemble probability $p_{N_i}(t_0)$. However, the footprint variable is cumulative. Therefore, the ensemble process with initial population N_i should already have footprint $F_i \geq \max(N, N_i)$. This is an underestimate of the true footprint distribution. The escape probability at time t_0 is thus (under)estimated by

$$P_J(t_0; N) \geq \sum_{n=1}^N P_{J-N}(n) p_n(t_0) + \sum_{n=N+1}^J P_{J-n}(n) p_n(t_0), \quad (75)$$

an approximation that we will make use of in the following section.

Having determined the relevant polynomials and their measures, computing essentially any probabilistic prediction of the continuous-time or discrete birth-death-suppression process is now a matter of arithmetic, or rather computing spectral integrals. This concludes the theoretical development of the solution of the process. In the final sections we show some sample applications and describe some future directions.

⁷Strictly speaking one should have $J \gg N \gg 1$ and t_0 small so that the probability of absorption is small and almost no processes have already saturated the bound $F \leq J$.

With numerical traction over the continuous-time and discrete-time processes, one can begin to ask questions about bounded, finite-lifetime birth-death-suppression processes. Developing the theory of the cumulative population or “footprint” is crucial to being able to bound these processes to some finite threshold. These tools allow one to predict the expected time until extinguishment, the distribution of footprints, and the effects of differing levels of suppression all in a dynamic framework. Having developed the theory of the population $j(t)$ and the footprint $F(t)$, we turn to applying these concepts with the wildfire interpretation at front of mind. Within the scope of the probabilistic predictions developed here the model exhibits clear trade-offs and optimal policy choices—tensions and strategies that are also reflected in decision processes related to actual fire events.

V. DYNAMIC RISK AND OPTIMIZATION

1. Dynamic risk estimation

As a first application of the technology developed in this work, we show how the “escape” probability of Eq. (72) can be dynamically computed alongside the stochastic trajectory of a birth-death process. Figures 10(b) and 10(d) show the escape probability recomputed after each transition of the stochastic trajectories in Figs. 10(a) and 10(c), which show simulated birth-death-suppression processes. The escape probability $P_J(N)$ quantifies the chance that the fire expands to have footprint above some bound J . In the wildfire interpretation, J represents the amount of burnable substrate or effective distance to built infrastructure. In an epidemic

interpretation, the bound J may represent the severity at which a certain containment protocol must be enacted, for example.

To be precise, the right-hand plots in Fig. 10 do the following: one first computes the escape probability $P_J(N)$ for a process of initial size N with a cumulative threshold J . After each transition, one updates $J \rightarrow J - j(t) + F(t)$ and $N \rightarrow j(t)$ and recomputes the escape probability. This takes advantage of the Markov property of the population dynamics and accounts for the previous growth of the footprint. As the outcome of the fire (being extinguished or escaping) becomes almost certain at later times, the predicted risk curves approach either zero or one. One can see that small changes in the actively burning population can have a large effect on the escape probability overall.

This type of calculation represents a real-time risk metric which takes into account the burnable substrate J , the cumulative footprint $F(t)$, and the size of the actively burning population $j(t)$. It is updated immediately as conditions change and new information is available. This is a simple example of using the birth-death-suppression model for real-time risk forecasting. By modulating the suppression rate γ used to compute the escape probability, one can dynamically compare the effects of policy choices by observing the effect on the escape probability.

In future work, we aim to describe in greater detail the trade-offs and dynamic optimization problems associated with a changing suppression rate over the course of a process. This elevates the dynamics from a homogeneous Markov chain to a Markov decision process over which reinforcement learning could identify optimal policies. Of course, such learning requires a cost (or reward) function. Real suppression is not free: there is a cost, and the cost could even be the removal of suppression from another, simultaneous event. The presence of a cost or reward will create optimal strategies. The nature of the cost function and hence of the optimal strategies is an important consideration in the application of this work.

2. Optimal suppression and adverse conditions

As a crude example of policy optimization, consider a policy which consists of choosing a constant suppression rate γ for the entire lifetime of a fire. A primitive measure of the severity of a fire event is given by the escape probability $P_J(N)$:

$$C_0(\gamma) = P_J(N; \gamma), \tag{76}$$

which of course depends on the applied suppression γ . To begin, let the process be in a subcritical phase with a constant birth rate; for definiteness we take $\beta = 0.9$ with $N = 20$ initial firelets in a burnable area of size $J = 100$. Physically, this represents a modestly sized ignition in mild fire conditions, where it will eventually self-extinguish. Of course, the greater the suppression, the lower the escape probability. In this case, applying more suppression always yields diminishing returns; the ‘‘cost’’ function $C_0(\gamma)$ is convex. This is the green (dotted) curve in Fig. 12.

The situation becomes more interesting if one considers an onset of dangerous fire conditions, i.e., if at some time t the birth rate increases into a supercritical phase $\beta \rightarrow \beta' > 1$, as depicted in Fig. 11. This models the impact of a wind event

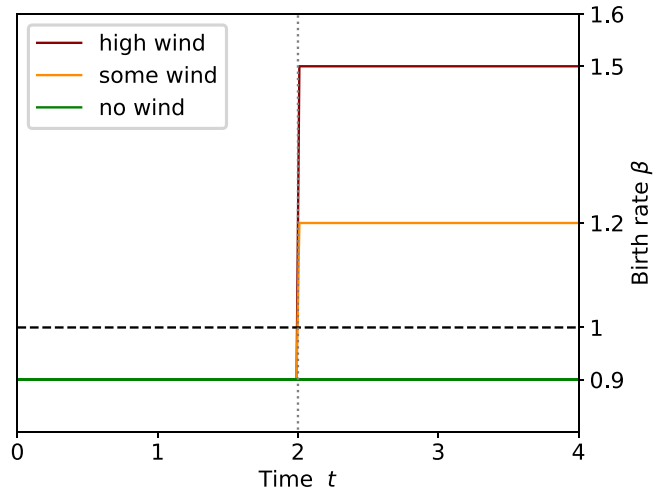


FIG. 11. The onset of dangerous conditions modeled as a step-function birth rate. We begin with $\beta = 0.9$ until $t = 2$, at which point the birth rate increases to $\beta' > 1$, reflecting, e.g., a wind event increasing the spread rate of a fire. Two different increases are shown.

on the spread of the fire. After this increase, one recomputes the escape probability, now with the higher birth rate β' . This is done in ensemble, using the expected distribution of active populations, as described at the end of Sec. IV.

Recomputing the escape probability in the face of a wind event yields new cost curves which are pictured in Fig. 12 alongside the ‘no wind’ curve. With changing conditions, the escape probability exhibits an inflection point, also marked in Fig. 12. The inflection point is the point of diminishing returns, which is now nonzero. This reflects the intuitive idea that early, prewind event suppression may pay off disproportionately in the face of dangerous conditions. By decreasing the active size of the fire early on, one decreases the impact

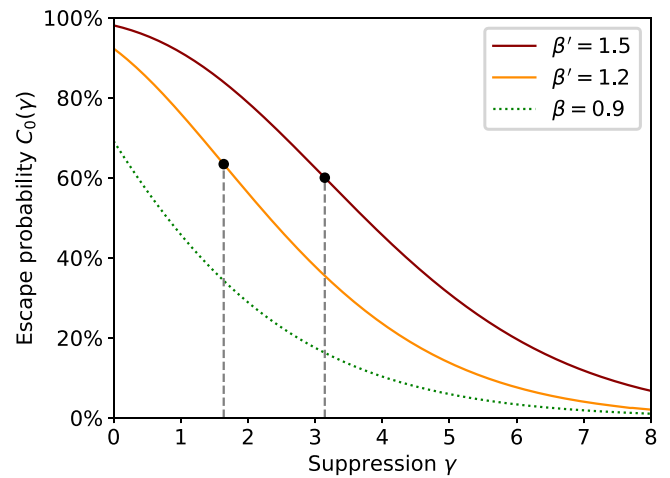


FIG. 12. Escape probability versus suppression rate. Here we have an initial population $N = 20$ and threshold $J = 100$. The green (lower, dotted) curve has a constant birth rate $\beta = 0.9$ while the orange (middle), red (upper) curves transition to a higher birth rate $\beta' > 1$ at time $t = 2$ as in Fig. 11. The point of diminishing returns is shown.

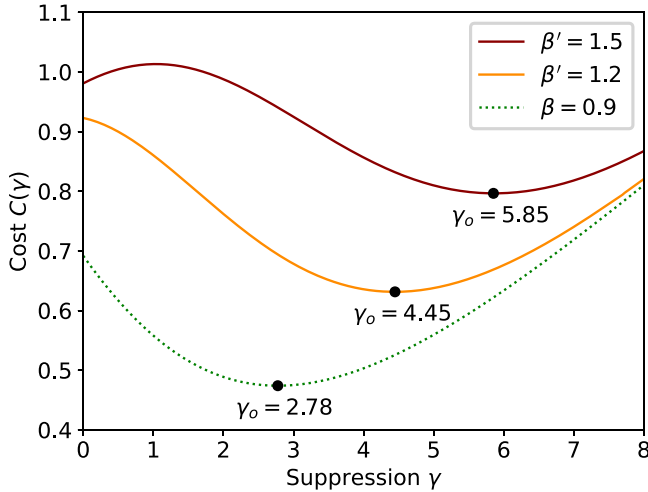


FIG. 13. Optimal suppression policies coincide with minima of the cost function. The cost is computed as in Eq. (77) with $\gamma_* = 10$ using the escape probability curves of Fig. 12. The optimal (constant) suppression strategy corresponds to the minimum γ_o in the cost versus suppression curve. The red (upper) curve also displays a local maximum.

that an increased birth rate has on the overall severity of the fire event.

With no cost to suppression, the obvious (and unrealistic) optimal strategy is to apply as much suppression as possible. A slightly more realistic cost function is obtained by adding a (linear) cost profile to suppression. If the maximum available suppression is γ_* , then a nicely scaled cost function is

$$C(\gamma) = P_j(N; \gamma) + \gamma/\gamma_*. \quad (77)$$

The addition of a cost to suppression immediately creates local minima and hence optimal (constant) suppression strategies, shown in Fig. 13. Beyond the local minima, the red (upper) curve (corresponding to an onset of high wind conditions) actually displays a local maximum: the cost of mild suppression $\gamma \approx 1, 2$ is greater than the cost of zero suppression. This happens because the effect of such low suppression is minimal on the escape probability, but does incur a cost. The cost is lower if nothing is done. Realistically, the choice of whether to apply suppression to a given event will include the choice to not apply it elsewhere. By identifying local maxima in cost functions, one can determine when removing suppression from one event in favor of another may be the best policy choice.

The cost function and constant suppression strategies outlined here are extremely coarse, and essentially the simplest possible choices: constant suppression, a linear cost profile, etc. Lots of generalizations exist. The suppression rate may become time dependent and react to conditions. A wind event as modeled in Fig. 11 could be forecasted for only a finite time, changing the eventual outlook for the fire. The cost function can be modified with a nonlinear profile with respect to suppression or made risk-averse by considering the variance in the population and penalizing high-variance scenarios.

This is all at the level of analyzing strategies for a single, isolated fire event. One may furthermore extend the

analysis to a set of multiple birth-death processes, modeling the management of resources across a multifire event. This allows resource allocation between events with dynamic risk predictions, including the time-delay associated to transferring resources between events. A multievent model of this type would provide a quantitative and dynamically updating decision support tool to agencies which manage a set of fires of varying sizes, conditions, and risk levels.

VI. CONCLUSION

In this work we review and extend the theoretical results of Karlin and MacGregor [28] and Kendall [25] on the birth-death-suppression Markov process. This process is closely related to many similar linear birth-death processes which were solved by various authors in relatively quick succession many years ago. For historical or accidental reasons, the birth-death-suppression process considered here was less studied in the present parametrization and went without the present wildfire interpretation.

We derive explicit expressions for the transition matrices for both the continuous-time and discrete-time birth-death-suppression processes. To obtain the latter, we applied a resolvent-based complex-analytic approach. This introduced a set of polynomials equivalent to the Pollaczek orthogonal polynomials; we describe their computation, asymptotics and measure of orthogonality in terms of the birth and suppression parameters. These polynomials are used to express physical predictions about the process as spectral integrals which can be evaluated numerically and in some cases exactly. In particular, the polynomials allow us to determine the asymptotic distribution of footprints (the cumulative population) as functions of the global parameters of the process. This had previously only been done in the zero-suppression process [25].

The analysis is in general closely related to and inspired by work on the birth-death-immigration (BDI) process, the process studied by Askey and Ismail [30]. Whereas the suppression rate appears as an additive increase in the aggregate death rate, in the BDI process the immigration rate appears as an additive increase to the aggregate birth rate. One could interpret an immigration rate as characterizing “spotting” or new ignition in the wildfire context, and could theoretically solve the process with both “immigration” and suppression.

The BDI process has also been extended to capture the dynamics of immigration between multiple population centers [1,58]. This same type of generalization could be made to the birth-death-suppression process, with the fire interpretation of describing the management of multiple simultaneous fire events. Already in previous work this type of resource allocation optimization was carried out for a two-fire event [5]; the n -fold generalization should be just a matter of bookkeeping. Another important aspect of the tools developed here is their Markovian property, which makes them dynamic. Not only can one describe the probabilities of certain outcomes of the process parametrically, but these probabilities can be continuously updated as new information about a given real-life stochastic process is obtained. The simplicity of the model means that these updates and recalculations can occur immediately and with minimal computational cost.

There exist a wide array of questions related to resource allocation, dynamic policy optimization, and the analysis of trade-offs that can now be quantitatively addressed. We have focused on the wildfire interpretation, but we stress that the model is general enough to describe a range of phenomena. The original motivation for the introduction of a cumulative population or “footprint,” as we have referred to it, was in the context of mathematical biology [3]. Indeed, one can draw an analogy between the wildfire interpretation and, for example, the susceptible-infectious-recovered (SIR) model of disease dynamics. The firelets $j(t)$ are the “infectious” states, the burnable substrate $J - F(t)$ is the “susceptible” population, and the footprint F represents the “recovered” group. One could apply the same formulas used in this work in the context of wildfire to ask questions about SIR dynamics: what is the probability of infecting all susceptible individuals in a finite group? What is the optimal level of treatment (suppression) to maximize the probability of eradication (absorption at $j = 0$) with some cost profile to the treatment? There exist a wide range of applications of which this type of stochastic model affords quantitative analysis.

The focus of the work has been to develop the theory of the birth-death-suppression process, but the motivation and guiding context for the authors remains wildfire modeling. As a model of fire, our process is temporal in nature: its main credence comes from its universality as a model of any continuously fluctuating population, not from its accurate representation of the spatial dynamics of fire. There of course exist far more accurate and widely used high-fidelity physics-based fire models [59–66] which far outperform the present model in the simulation of physical fire. Some of the analysis herein could reasonably be augmented by such high-fidelity models, and they could be applied to gain a quantitative understanding of the relationship between the Markov process parameters and the measurable data of real fires, fuels, topography, and weather conditions. The practical reality of firefighting is far removed from the decision analysis presented herein. However, it is our hope that the dynamic risk forecasting and quantitative control we have developed here lead to tools which support streamlined, quantitatively informed decisions that help us live in the modern climactic reality.

ACKNOWLEDGMENTS

We thank W. Van Assche for his correspondence and expertise on orthogonal polynomials. This work was supported in part by the Office of Secretary of Defense Strategic Environmental Research and Development Program (SERDP) Project No. RC21-1233.

APPENDIX A: ZERO-SUPPRESSION PROCESS

The zero-suppression process is the linear birth-death process with aggregate birth and death rates

$$\lambda_j = \beta j, \quad \mu_j = \delta j. \quad (\text{A1})$$

This process has been solved by a variety of methods [1]. For the solution as the limit of a finite system, see Ref. [38]. Here, we focus on the generating function approach of Ref. [25],

and show how results cited in Sec. II B may be derived such that the work is self-contained.

The present focus is to determine the population probabilities $p_n(t) \equiv \Pr\{j(t) = n\}$ given definite initial configurations $p_n(0) = \delta_{n,N}$ [equivalently $j(0) = N$]. The dynamics are governed by

$$\frac{d}{dt} p_n(t) = \lambda_{n-1} p_{n-1}(t) + \mu_{n+1} p_{n+1}(t) - (\lambda_n + \mu_n) p_n(t). \quad (\text{A2})$$

This differential-difference equation may be solved by employing a generating function; define

$$G(t, z) = \sum_{n=0}^{\infty} p_n(t) z^n \quad (\text{A3})$$

in terms of the population probabilities, where $p_0(t) \equiv p_A(t)$ is the absorption probability. By ansatz this generating function is analytic as $z \rightarrow 0$. For an initial population $j(0) = N$, the generating function satisfies the boundary condition $G(0, z) = z^N$. The function $G(t, z)$ allows computation of many quantities of interest:

$$\langle j^k(t) \rangle = \lim_{z \rightarrow 1} \left(z \frac{\partial}{\partial z} \right)^k G(t, z), \quad (\text{A4})$$

$$p_n(t) = \frac{1}{n!} \lim_{z \rightarrow 0} \frac{\partial^n}{\partial z^n} G(t, z). \quad (\text{A5})$$

Equation (A2) defines relations between the series coefficients of $G(t, z)$. For the moment, keep $\gamma \neq 0$ and use the birth-death rates $\lambda_n = \beta n$, $\mu_n = n + \gamma$. Taking the time derivative of $G(t, z)$ and applying the recurrence relation descends to a partial differential equation for the generating function:

$$\begin{aligned} \frac{\partial G}{\partial t} &= (\beta z - 1)(z - 1) \frac{\partial G}{\partial z} \\ &+ \gamma \left(\frac{1 - z}{z} \right) (G(t, z) - p_A(t)). \end{aligned} \quad (\text{A6})$$

This is a linear, nonhomogeneous first order equation; in theory, it may be solved by the method of characteristics. The term proportional to γ appears to display a singularity as $z \rightarrow 0$. This nonanalytic divergence is removed by the subtraction of the absorption probability $p_A(t)$ from $G(t, z)$. However, without an explicit form of the absorption probability $p_A(t)$, one cannot solve Eq. (A6) explicitly. In the presence of nonzero suppression different techniques are needed. However, setting $\gamma = 0$, the simple linear process of Eq. (A1) gives the homogeneous PDE

$$\frac{\partial G}{\partial t} = (\beta z - 1)(z - 1) \frac{\partial G}{\partial z}. \quad (\text{A7})$$

For an initial population $j(0) = N$, and hence $G(0, z) = z^N$, this equation is solved by

$$G(t, z) = \left[\frac{(z - 1)e^{(\beta - 1)t} - \beta z + 1}{\beta(z - 1)e^{(\beta - 1)t} - \beta z + 1} \right]^N. \quad (\text{A8})$$

Taking the limit as $\beta \rightarrow 1$ gives the critical generating function

$$G^c(t, z) = \left[\frac{z + t(1 - z)}{1 + t(1 - z)} \right]^N. \quad (\text{A9})$$

1. Population statistics

The average population is given by

$$\langle j(t) \rangle = \lim_{z \rightarrow 1} \partial_z G(z, t) = N e^{(\beta-1)t}, \quad (\text{A10})$$

which defines the supercritical $\beta > 1$ and subcritical $\beta < 1$ phases of exponential growth and extinction, respectively. In the critical case $\beta = 1$, the mean population is constant in time. The population variance is given by

$$\begin{aligned} \Delta j^2(t) &= \lim_{z \rightarrow 1} (\partial_z^2 G(z, t) + \partial_z G(z, t) - [\partial_z G(z, t)]^2) \\ &= \frac{\beta + 1}{\beta - 1} N e^{(\beta-1)t} (e^{(\beta-1)t} - 1). \end{aligned} \quad (\text{A11})$$

For $\beta > 1$, the variance grows exponentially at late times, and is monotonically increasing. The critical variance is linear in time $\Delta j_c^2 = 2Nt$ and similarly increases without bound. However, the subcritical variance peaks at some finite time.

The time-dependent probability of absorption $p_A(t)$ is trivial to calculate by just plugging in $z = 0$ to both generating functions:

$$p_A(t) = \left(\frac{1 - e^{(\beta-1)t}}{1 - \beta e^{(\beta-1)t}} \right)^N, \quad (\text{A12})$$

$$\lim_{\beta \rightarrow 1} p_A(t) = \left(\frac{t}{t+1} \right)^N. \quad (\text{A13})$$

By taking the limit $t \rightarrow \infty$, one obtains the asymptotic absorption probability:

$$p_A(\infty) = \lim_{t \rightarrow \infty} G_0(0, t) = \begin{cases} \beta^{-N}, & \beta > 1, \\ 1, & \beta \leq 1, \end{cases} \quad (\text{A14})$$

which states that absorption (extinguishing) is all but certain except in the supercritical case. Finally, consider the distribution of lifetimes until absorption $\propto p_1(T)dT$. With an initial population to $j(0) = N$, the normalized distribution of lifetimes T is

$$\begin{aligned} d\sigma(T) &= N \max(\beta, 1)^N (\beta - 1)^2 e^{(\beta-1)T} \\ &\quad \times \frac{(1 - e^{(\beta-1)T})^{N-1}}{(1 - \beta e^{(\beta-1)T})^{N+1}} dT \end{aligned} \quad (\text{A15})$$

for $\beta \neq 1$, reducing in the critical case to

$$d\sigma(T) = N \frac{T^{N-1}}{(1+T)^{N+1}} dT. \quad (\text{A16})$$

First, let $N = 1$ to analyze the small fire limit. It is convenient to make the change of variables $s = e^{(\beta-1)T}$; note that the limit $\lim_{T \rightarrow \infty} s$ depends on the value of β . In particular, as $T \rightarrow \infty$ one has $s \rightarrow 0$ in a subcritical phase ($\beta < 1$) and $s \rightarrow \infty$ in a supercritical phase ($\beta > 1$). The single initial population measure for lifetimes becomes

$$d\sigma(T) = \max(1, \beta) |\beta - 1| \frac{1}{(1 - \beta s)^2} ds. \quad (\text{A17})$$

From this one can write down expressions for the mean and median lifetimes of the fire. Note that the median lifetime T_m is the time such that $p_A(\infty) = 2p_A(T_m)$: the time after which half of all processes in ensemble will have absorbed.

The average lifetime is infinite in the critical case, and otherwise is given by

$$\langle T \rangle = \begin{cases} -\frac{1}{\beta} \log(1 - \beta), & \beta < 1, \\ -\log(1 - 1/\beta), & \beta > 1. \end{cases} \quad (\text{A18})$$

Recall that this represents the average lifetime of fires which end in absorption; it does not count the fires which formally diverge in population size. One can also explicitly compute the median lifetime for a fire with $j(0) = 1$:

$$T_m = \begin{cases} \frac{1}{1-\beta} \log(2 - \beta), & \beta < 1, \\ \frac{1}{\beta-1} \log(2 - 1/\beta), & \beta > 1, \end{cases} \quad (\text{A19})$$

where, in the above, the limit as $\beta \rightarrow 1$ exists and is equal to 1. In fact, one can solve $p_A(\infty) = 2p_A(T_m)$ exactly to obtain the median lifetime. In the limit of large initial size $N \gg 1$, the leading order contributions are

$$T_m \approx_{N \gg 1} \begin{cases} \frac{1}{1-\beta} \log N - \frac{1}{1-\beta} \log\left(\frac{1-\beta}{\log 2}\right), & \beta < 1, \\ \frac{N}{\log 2} - 1/2, & \beta = 1, \\ \frac{1}{\beta-1} \log N + \frac{1}{\beta-1} \log\left(\frac{1-1/\beta}{\log 2}\right), & \beta > 1, \end{cases}$$

discarding terms of order $\mathcal{O}(N^{-1})$. In the large population limit the difference between sub- and supercritical phases is washed out, and the lifetimes increase only logarithmically in the size of the initial population. The next-to-leading order terms differ between the sub- and supercritical phases and become relevant near criticality, where the dependence on the initial population is much stronger.

This completes discussion of the dynamics of the suppression-free population. In Sec. II B, these results are summarized along with some visualizations. Here, we move on to constructing the suppression-free dynamics of the cumulative population or footprint $F(t)$.

2. Footprint statistics

The generating function approach may also be employed to determine at least the low moments of the footprint distribution. Recall that the footprint $F(t)$ shares all the births of the population $j(t)$ but none of the deaths. It is natural to define the joint probability matrix \mathcal{P} with elements $\mathcal{P}_{j,F}(t)$ which are the probability of having population j and footprint F at time t . These matrix elements satisfy a differential-difference relation as before:

$$\begin{aligned} \frac{d}{dt} \mathcal{P}_{j,F}(t) &= \lambda_{j-1} \mathcal{P}_{j-1,F-1}(t) + \mu_{j+1} \mathcal{P}_{j+1,F}(t) \\ &\quad - (\lambda_j + \mu_j) \mathcal{P}_{j,F}(t). \end{aligned} \quad (\text{A20})$$

To solve this by generating function, define a function of three variables

$$\Psi(t, z, w) = \sum_{j,F \geq 0} \mathcal{P}_{j,F}(t) z^j w^F. \quad (\text{A21})$$

Restricting to the case $\gamma = 0$, this joint generating function solves the differential equation

$$\frac{\partial \Psi}{\partial t} = (\beta z^2 w - z(\beta + 1) + 1) \frac{\partial \Psi}{\partial z}, \quad (\text{A22})$$

where, for an initial population $j(0) = F(0) = N$, there is the initial condition $\Psi(0, z, w) = (zw)^N$. This equation has an exact solution, but it is difficult to determine closed form expressions for its expansion coefficients [25]. Instead, one can extract a few important predictions about the footprint by solving for the cumulant generating function $K(t, u, v)$:

$$e^{K(t, u, v)} = \Psi(t, e^u, e^v). \quad (\text{A23})$$

Substituting this into Eq. (A22) shows that K satisfies the equation

$$\frac{\partial K}{\partial t} = [\beta(e^{u+v} - 1) - (1 - e^{-u})] \frac{\partial K}{\partial u} \quad (\text{A24})$$

along with the initial condition $K(0, u, v) = N(u + v)$. The cumulant generating function, by definition, has series coefficients given by the various cumulants of the joint distribution of j, F . To quadratic order these are

$$K = u\langle j(t) \rangle + v\langle F(t) \rangle + \frac{1}{2}u^2\Delta j^2(t) + uv \text{Cov}(j, F) + \frac{1}{2}v^2\Delta F^2(t) + \dots \quad (\text{A25})$$

The strategy is then to solve Eq. (A24) order-by-order to determine the low cumulants listed above. One finds

$$\langle F(t) \rangle = \frac{N}{\beta - 1} (\beta e^{(\beta-1)t} - 1), \quad (\text{A26})$$

$$\lim_{\beta \rightarrow 1} \langle F(t) \rangle = N(t + 1). \quad (\text{A27})$$

Only in the subcritical case does the asymptotic average footprint have a finite value. One may obtain explicit forms for the other cumulants, but they are not especially enlightening.

Finally, consider the asymptotic distribution of footprints. To find this, one should solve Eq. (A22) and then take the limit $t \rightarrow \infty$; the solution is known and the limit may be taken explicitly. Setting $z = 1, N = 1$ and entering the asymptotic time regime, one finds

$$\begin{aligned} \Psi(\infty, 1, w) &= \sum_{F=1}^{\infty} \Pr\{F(\infty) = F\} w^F \\ &= \frac{1}{2\beta} [1 + \beta - \sqrt{(\beta + 1)^2 - 4\beta w}]. \end{aligned} \quad (\text{A28})$$

For an absorbing process with $\beta \leq 1$, one can explicitly evaluate the series coefficients of the above. In the limit of large footprint $F \gg 1$, they are approximately given by

$$\Pr\{F(\infty) = F\} \sim \left(\frac{4\beta}{(1 + \beta)^2} \right)^F F^{-3/2}. \quad (\text{A29})$$

The cumulative distribution of footprints, which we call the ‘‘escape probability’’ $P(F \geq J)$, is then given by the integral of the above. The quantity $P(F \geq J)$ expresses the probability that the footprint of a given process will end up larger than some value J . This is the distribution which is empirically expected to follow a power law $\sim J^{-\alpha}$ with exponent close to $\alpha \approx 1/2$.

In the eventually absorbing regime $\beta \leq 1$, write $\beta = 1 - a$ and consider small, positive a , expanding away from criticality into the subcritical regime. The escape probability can then

be expanded to leading order in a as

$$P(F(\infty) \geq J) \approx \frac{J^{-1/2}}{\sqrt{\pi}} + \frac{a}{2} \left(\frac{J^{-1/2}}{\sqrt{\pi}} - 1 \right), \quad (\text{A30})$$

which reproduces the desired power-law scaling exactly in the critical $\beta = 1$ phase, and approximately for $\beta \lesssim 1$. This characterizes the asymptotic distribution of footprints, but not the timescales associated with saturation. Since the solution to Eq. (A22) is known, one could in principle compute the exact coefficients $P_{j,F}(t)$. This is beyond the scope of this work, and would likely be unreasonably difficult.

APPENDIX B: EXPLICIT TRANSITION MATRICES

In this section we review the method of Karlin and MacGregor for solving the population dynamics in the birth-death-suppression process [26,28], starting by defining some of the families of orthogonal polynomials associated to each phase of the process. These polynomials fall into the Askey scheme classification of Ref. [45]. The first are the Meixner polynomials $\phi_n(x)$, which are defined in terms of the Gauss hypergeometric function:

$$\phi_n(x) \equiv \phi_n(x; b, g) = {}_2F_1(-n, -x; b; 1 - 1/g). \quad (\text{B1})$$

Here, $b > 0$ and $0 < g < 1$. Recall that

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \quad (\text{B2})$$

$$(a)_n = \frac{\Gamma(a + n)}{\Gamma(a)}, \quad (\text{B3})$$

so that when a is a negative integer, the Pochhammer symbol $(a)_n$ vanishes at some order and the function ${}_2F_1$ is a polynomial in z . These polynomials are orthogonal with respect to a discrete measure with atoms at $k = 0, 1, 2, \dots$ and inner product

$$\sum_{k=0}^{\infty} \phi_n(k) \phi_m(k) \rho_k = \delta_{nm} \frac{n!}{(b)_n g^n}, \quad (\text{B4})$$

where the spectral measure ρ_k is

$$\rho_k = (1 - g)^b \frac{(b)_k}{k!} g^k. \quad (\text{B5})$$

Most importantly, the polynomials $\phi_n(x)$ satisfy the following recurrence relation:

$$-x \frac{(1 - g)}{g} \phi_n(x) = \frac{n}{g} \phi_{n-1} - \left(n + \frac{n}{g} + b \right) \phi_n + (n + b) \phi_{n+1}. \quad (\text{B6})$$

The other set of polynomials needed are the associated Laguerre polynomials, defined as

$$L_n^\alpha(x) = \frac{e^x x^{-\alpha}}{n!} \frac{d^n}{dx^n} (e^{-x} x^{n+\alpha}) \quad (\text{B7})$$

$$= \sum_{m=0}^n (-1)^m \frac{(n + \alpha)!}{(n - m)! (\alpha + m)! m!} x^m. \quad (\text{B8})$$

Their spectral measure is continuous, leading to the orthogonality relation

$$\int_0^\infty e^{-x} x^\alpha L_n^\alpha(x) L_m^\alpha(x) dx = \frac{(n+\alpha)!}{n!} \delta_{mn}. \quad (\text{B9})$$

These functions satisfy the three-term recurrence relation

$$\begin{aligned} -xL_n^\alpha(x) &= (n+\alpha)L_{n-1}^\alpha(x) + (n+1)L_n^\alpha(x) \\ &\quad - (2n+\alpha+1)L_n^\alpha(x). \end{aligned} \quad (\text{B10})$$

It so happens that in general, the measures of orthogonality develop discrete parts when the process moves away from criticality.

1. Polynomials for the suppression process

Here, we identify the set of orthogonal polynomials for each phase of the process and present their properly normalized measures of orthogonality, which should satisfy

$$\int_0^\infty Q_n(x) Q_m(x) d\sigma(x) = \frac{\delta_{nm}}{\pi_n}. \quad (\text{B11})$$

With $\lambda_n = \beta(n+1)$ and $\mu_n = n + \gamma + 1$, the constants π_n are given by

$$\pi_0 = 1; \quad \pi_n = \frac{\beta^n \cdot n!}{(\gamma+2)_n}, \quad (\text{B12})$$

where the index n runs over $0, 1, 2, \dots$, with $n = j - 1$ where j is the number of firelets. The polynomials $Q_n(x)$ satisfy the three-term recurrence relation

$$\begin{aligned} -xQ_n(x) &= (n+\gamma+1)Q_{n-1}(x) + \beta(n+1)Q_{n+1}(x) \\ &\quad - (n+\beta n + \beta + \gamma + 1)Q_n(x). \end{aligned} \quad (\text{B13})$$

a. Critical phase

In the critical phase $\beta = 1$, the recursion reduces to

$$\begin{aligned} -xQ_n(x) &= (n+\gamma+1)Q_{n-1}(x) + (n+1)Q_n(x) \\ &\quad - (2n+\gamma+2)Q_n(x). \end{aligned} \quad (\text{B14})$$

Comparing to the Laguerre recurrence in Eq. (B10), one can easily identify $Q_n(x) = L_n^\alpha(x)$, where $\alpha = \gamma + 1$. Thus, for the critical process, the desired family of orthogonal polynomials is

$$Q_n(x) = L_n^{\gamma+1}(x). \quad (\text{B15})$$

The normalized measure of orthogonality is

$$d\sigma(x) = \frac{1}{\Gamma(\gamma+2)} e^{-x} x^{\gamma+1} dx, \quad (\text{B16})$$

which is continuous over the real half-line.

b. Subcritical phase

Now consider the case $\beta < 1$. Define a modified form of the Meixner polynomials

$$\xi_n(x) = \frac{(b)_n}{n!} \phi_n\left(\frac{x}{1-\beta} - b + 1; b, \beta\right), \quad (\text{B17})$$

which satisfy a recurrence relation slightly modified from (B6), working out as

$$\begin{aligned} -x\xi_n(x) &= (n+b-1)\xi_{n-1}(x) + \beta(n+1)\xi_{n+1}(x) \\ &\quad - (\beta n + \beta + n + b - 1)\xi_n(x). \end{aligned} \quad (\text{B18})$$

This matches the recursion for the $Q_n(x)$ provided $b = \gamma + 2$. So, in the $\beta < 1$ case, the polynomials are

$$Q_n(x) = \frac{(\gamma+2)_n}{n!} \phi_n\left(\frac{x}{1-\beta} - \gamma - 1; \gamma + 2, \beta\right). \quad (\text{B19})$$

Their normalized measure of orthogonality is discrete with support at $\rho_k \equiv (1-\beta)(k+\gamma+1)$ for $k \in \mathbb{Z}_+$; it may be written

$$d\sigma(x) = (1-\beta)^{\gamma+2} \sum_{k=0}^{\infty} \frac{(\gamma+2)_k}{k!} \beta^k \delta(x - \rho_k) dx. \quad (\text{B20})$$

c. Supercritical case

For $\beta > 1$, define an altered set of Meixner polynomials by

$$T_n(x) = \frac{(b)_n}{n! \beta^n} \phi_n\left(\frac{x}{\beta-1} - 1; b, \frac{1}{\beta}\right), \quad (\text{B21})$$

which satisfy the same recursion as the $\xi_n(x)$, again identifying $b = \gamma + 2$. The difference is the arguments of the ϕ_n . The conclusion is that, for the supercritical $\beta > 1$ case, one has the orthogonal polynomials

$$Q_n(x) = \frac{(\gamma+2)_n}{n! \beta^n} \phi_n\left(\frac{x}{\beta-1} - 1; \gamma + 2, \frac{1}{\beta}\right). \quad (\text{B22})$$

The normalized measure of orthogonality has atoms at $r_k \equiv (\beta-1)(k+1)$ for $k \in \mathbb{Z}_+$ and is given by

$$d\sigma(x) = \left(\frac{\beta-1}{\beta}\right)^{\gamma+2} \sum_{k=0}^{\infty} \frac{(\gamma+2)_k}{\beta^k k!} \delta(x - r_k) dx. \quad (\text{B23})$$

2. Explicit transition matrices

With explicit forms of orthogonal polynomials for the process $(\lambda_n, \mu_n) = (\beta(n+1), n + \gamma + 1)$, all that is left is to apply the spectral formula for the transition matrix:

$$P_{nm}(t) = \pi_m \int_0^\infty e^{-xt} Q_n(x) Q_m(x) d\sigma(x). \quad (\text{B24})$$

We leave out the computation and simply state the results of Ref. [28], correcting various errata and writing in our parametrization. All of the expressions given here are valid only for $0 \leq k \leq \ell$; by using the transposition formula $P_{k\ell} = (\pi_\ell/\pi_k) P_{\ell k}$ one can easily generate all other elements. While these expressions are cumbersome, they can be easily evaluated numerically.

a. Noncritical cases

Recall the auxiliary variable $s(t) = \exp((\beta-1)t)$. When $\beta < 1$, we have $s(\infty) = 0$; when $\beta > 1$, instead one has $s(\infty) = \infty$. Define also the auxiliary function

$$X(s) = \frac{(\beta s - 1)(s - \beta)}{\beta(s-1)^2}, \quad (\text{B25})$$

which has the limits $X(0) = X(\infty) = 1$. This becomes the argument of the following hypergeometric function:

$$F_{k\ell}(t) = \frac{\Gamma(k + \ell + \gamma + 2)}{\Gamma(\ell + 1)\Gamma(k + \gamma + 2)} \times {}_2F_1(-k, -\ell; -1 - k - \ell - \gamma; X[s(t)]). \quad (\text{B26})$$

Numerically, this expression is poorly scaled and troublesome: one should take logarithms for stability. The full transition matrix is the expression

$$P_{k\ell}(t) = \frac{\pi_\ell}{\pi_k} \beta^k s^{\gamma+1} \left(\frac{1-\beta}{1-\beta s} \right)^{\gamma+2} \left(\frac{1-s}{1-\beta s} \right)^{k+\ell} F_{k\ell}(t), \quad (\text{B27})$$

which is valid for both cases of $\beta \neq 1$ and in the above form for $k \leq \ell$.

b. Critical case

To find the critical transition matrix elements, one may simply take the limit as $\beta \rightarrow 1$ of the above expressions. One finds

$$\lim_{\beta \rightarrow 1} X(t) = 1 - \frac{1}{t^2}; \quad \lim_{\beta \rightarrow 1} \frac{1-s}{1-\beta s} = \frac{t}{1+t}. \quad (\text{B28})$$

Defining the auxiliary function

$$F_{k\ell}^c(t) = \frac{\Gamma(k + \ell + \gamma + 2)}{\Gamma(\ell + 1)\Gamma(k + \gamma + 2)} \times {}_2F_1(-k, -\ell; -1 - k - \ell - \gamma; 1 - 1/t^2), \quad (\text{B29})$$

the critical transition matrix elements are

$$P_{k\ell}^c(t) = \frac{\pi_\ell}{\pi_k} \left(\frac{1}{1+t} \right)^{\gamma+2} \left(\frac{t}{1+t} \right)^{k+\ell} F_{k\ell}^c(t). \quad (\text{B30})$$

Python code which implements these functions for easy evaluation is available on request.

3. Low-population moments

The generating function $G(t, z)$ for the occupation probabilities $p_n(t)$ satisfies the inhomogeneous equation

$$\frac{\partial G}{\partial t} = (\beta z - 1)(z - 1) \frac{\partial G}{\partial z} + \gamma \left(\frac{1-z}{z} \right) (G(t, z) - p_A(t)). \quad (\text{B31})$$

The cumulant generating function, defined as $K(t, u) = \log G(t, e^u)$, therefore satisfies the related equation

$$\frac{\partial K}{\partial t} = (\beta e^u - (\beta + 1) + e^{-u}) \frac{\partial K}{\partial u} + \gamma (e^{-u} - 1) [1 - e^{-K} p_A(t)]. \quad (\text{B32})$$

Expanding the function $K(t, u)$ to quadratic order

$$K(t, u) = \langle j(t) \rangle u + \frac{1}{2} \Delta j(t)^2 u^2 + \mathcal{O}(u^3), \quad (\text{B33})$$

and solving order-by-order yields coupled equations for the average population $\langle j(t) \rangle$ and the variance $\Delta j(t)^2$. The first of these, governing the average population $\langle j(t) \rangle$, is

$$\frac{\partial}{\partial t} \langle j(t) \rangle = (\beta - 1) \langle j(t) \rangle + \gamma (p_A(t) - 1), \quad (\text{B34})$$

which is solved by

$$\langle j(t) \rangle = N e^{(\beta-1)t} + \gamma \int_0^t d\tau e^{(\beta-1)(t-\tau)} [p_A(\tau) - 1]. \quad (\text{B35})$$

In the simple case $N = 1$, we have

$$p_A(t) = 1 - [1 - z(t)]^{\gamma+1}; \quad z(t) = \frac{1 - e^{(\beta-1)t}}{1 - \beta e^{(\beta-1)t}}, \quad (\text{B36})$$

which leads to the average population

$$\langle j(t) \rangle = e^{(\beta-1)t} [1 - z(t)]^\gamma. \quad (\text{B37})$$

APPENDIX C: RANDOM-WALK POLYNOMIALS

In this Appendix we construct analytically the random-walk polynomials for the birth-death-suppression process. These polynomials are equivalent under reparametrization to the Pollaczek polynomials, as discussed in Sec. IV. We use the method of Stieltjes inversion to construct the measure of orthogonality given expressions for the polynomials $W_n(x)$.

1. The critical polynomials

To begin, consider the critical phase: when $\beta = 1$, the recurrence relation Eq. (61) simplifies to

$$x W_n^c(x) = \frac{n+1}{2(n+1)+\gamma} W_{n+1}^c(x) + \frac{n+\gamma+1}{2(n+1)+\gamma} W_{n-1}^c(x). \quad (\text{C1})$$

The critical polynomials $W_n^c(x)$ are equivalent to a known set of orthogonal polynomials: the Gegenbauer or ultraspherical polynomials $C_n^\lambda(x)$. The exact correspondence is

$$W_n^c(x; \gamma) = C_n^{1+\gamma/2}(x). \quad (\text{C2})$$

A generalization of the spherical harmonics, the Gegenbauer polynomials fall in the Askey scheme and many of their properties are well-known (for reference, see Sec. 1.8.1 in Ref. [45]). They are a special case of the Jacobi polynomials. Their generating function is

$$\frac{1}{(1-2xz+z^2)^{1+\gamma/2}} = \sum_{n=0}^{\infty} W_n^c(x; \gamma) z^n. \quad (\text{C3})$$

These critical random-walk polynomials are orthogonal with respect to the normalized, continuous measure

$$d\sigma(x) = \frac{\Gamma(\frac{\gamma}{2} + 2)}{\sqrt{\pi} \Gamma(\frac{\gamma+3}{2})} (1-x^2)^{(\gamma+1)/2} dx. \quad (\text{C4})$$

supported on the interval $[-1, 1]$. One could proceed to immediately calculate with these expressions. Instead, their introduction here will be used to verify the noncritical results by showing that they agree in the critical limit.

a. The noncritical polynomials

Away from criticality the $W_n(x)$ do not admit a simple hypergeometric formula. For convenience, we now refer to the general, noncritical polynomials $W_n(x; \beta, \gamma) \equiv W_n(x)$ as the *firewalk* polynomials, recalling of course that they are equivalent to the Pollaczek family. One can find an expression

for these firewalk polynomials $W_n(x; \beta, \gamma)$ by the method of generating function: define

$$\Phi(z, x; \beta, \gamma) = \sum_{n=0}^{\infty} W_n(x; \beta, \gamma) z^n. \quad (C5)$$

The recurrence relation (61) satisfied by the W_n descends to a differential equation for the generating function:

$$\frac{\partial \Phi}{\partial z} = \frac{x(\beta + \gamma + 1) - z(\gamma + 2)}{z^2 + \beta - xz(\beta + 1)} \Phi, \quad (C6)$$

along with the boundary condition $\Phi(0, x; \beta, \gamma) = W_0(x) = 1$. First, consider the singular points of the equation above. These occur at the roots of the quadratic $z^2 - xz(\beta + 1) + \beta$. The discriminant $D(x)$ of this quadratic is

$$D(x) = \sqrt{x^2(\beta + 1)^2 - 4\beta}, \quad (C7)$$

where for consistency we take the positive root always. The discriminant vanishes at $x = I_\beta$, where

$$I_\beta = \frac{2\sqrt{\beta}}{1 + \beta}, \quad I_\beta \leq 1. \quad (C8)$$

Specifically, $D(x)$ is real for $|x| \geq I_\beta$ and pure imaginary otherwise. The bound $I_\beta = 1$ only at criticality when $\beta = 1$. To solve Eq. (C6), one computes the partial fraction decomposition

$$\frac{x(\beta + \gamma + 1) - z(\gamma + 2)}{z^2 - xz(\beta + 1) + \beta} = \frac{A}{z - u} + \frac{B}{z - v}. \quad (C9)$$

The roots u, v of the quadratic $z^2 - xz(\beta + 1) + \beta$ are

$$u = \frac{x(\beta + 1)}{2} + \frac{D(x)}{2}, \quad v = \frac{x(\beta + 1)}{2} - \frac{D(x)}{2}, \quad (C10)$$

noting the transformation rule $u(-x) = -v(x)$. The relative magnitude of these roots depends on x : if $x \in [-I_\beta, I_\beta]$ the discriminant D is pure imaginary and $|u| = |v|$. If $x > I_\beta$, then $|v| < |u|$; by symmetry for $x < -I_\beta$ one has $|u| < |v|$. The dominant root (that of smallest magnitude) is relevant when constructing the measure of orthogonality. The coefficients A, B of the partial fraction decomposition (C9) are given by

$$A = -\frac{\gamma + 2}{2} - \frac{x\gamma(\beta - 1)}{2D(x)}, \quad (C11)$$

$$B = -\frac{\gamma + 2}{2} + \frac{x\gamma(\beta - 1)}{2D(x)}, \quad (C12)$$

where under $x \mapsto -x$ the coefficients $A \leftrightarrow B$ are interchanged. The equation to be solved is thus rewritten as

$$\frac{\partial \Phi}{\partial z} = \left(\frac{A}{z - u} + \frac{B}{z - v} \right) \Phi, \quad (C13)$$

which, along with the boundary condition $\Phi(0, x) = 1$, determines

$$\Phi(z, x; \beta, \gamma) = (1 - z/u)^A (1 - z/v)^B. \quad (C14)$$

A first check is that this function reproduces the generating function (C3) in the limit $\beta \rightarrow 1$. In the critical limit, the

parameters reduce to

$$\lim_{\beta \rightarrow 1} A = \lim_{\beta \rightarrow 1} B = -(1 + \gamma/2), \quad (C15)$$

$$\lim_{\beta \rightarrow 1} u = x + i\sqrt{1 - x^2}, \quad (C16)$$

$$\lim_{\beta \rightarrow 1} v = x - i\sqrt{1 - x^2}, \quad (C17)$$

which allows one to verify that, indeed, as $\beta \rightarrow 1$ the generating function Φ simplifies to

$$\lim_{\beta \rightarrow 1} \Phi(z, x; \beta, \gamma) = \frac{1}{(1 - 2xz + z^2)^{1+\gamma/2}}, \quad (C18)$$

agreeing with the Gegenbauer generating function in Eq. (C3). The firewalk or Pollazcek polynomials $W_n(x; \beta, \gamma)$ can therefore be considered a deformation of the Gegenbauer polynomials parametrized by the birth rate β . To find a closed form, one must determine the n -th series coefficient of Eq. (C14). An expression is

$$W_n(x) = (-B)_n \frac{v^{-n}}{n!} {}_2F_1(-n, -A; -n + B + 1; v/u). \quad (C19)$$

By the use of some hypergeometric identities [46,49], this is equivalent to the form

$$W_n(x) = \frac{(\gamma + 2)_n}{n!} u^{-n} {}_2F_1(-n, -B, \gamma + 2; -uD(x)/\beta). \quad (C20)$$

While this expression does not obviously give polynomials in x , direct calculation shows

$$W_1(x) = \frac{x(\beta + \gamma + 1)}{\beta}, \quad (C21)$$

$$W_2(x) = \frac{x^2(\beta + \gamma + 1)(2\beta + \gamma + 2) - \beta(\gamma + 2)}{2\beta^2}, \quad (C22)$$

noting that the W_n are odd for odd n and even for even n , consistent with their measure of orthogonality being even. But determining this expression for the firewalk polynomials $W_n(x)$ is only the first step. To evaluate matrix elements of the jump chain transition matrix $S(n)$, we also must determine the correct measure of orthogonality.

2. Measure by Stieltjes inversion

By Favard's theorem there exists a real, even, positive measure of orthogonality $d\sigma(x)$ on $[-1, 1]$ associated to the firewalk polynomials $W_n(x)$. From such a measure, the Stieltjes transformation defines a function $\chi(z)$, the *resolvent* via

$$\chi(z) = \int_{-1}^1 \frac{d\sigma(x)}{z - x}. \quad (C23)$$

This function is analytic in the complex z plane away from the interval $[-1, 1]$, where certain singularities arise exactly where the measure of orthogonality is supported. Generically, the measure may be a mix of continuous and discrete parts:

$$d\sigma(x) = w(x)dx + \sum_k \Delta_k \delta(x - x_k)dx, \quad (C24)$$

$$\chi(z) = \int \frac{w(x)}{z - x} dx + \sum_k \frac{\Delta_k}{z - x_k}. \quad (C25)$$

At points x_k where the measure has some discrete atomic weight Δ_k the resolvent $\chi(z)$ therefore has a simple pole with residue Δ_k . However, where the measure has continuous support the function $\chi(z)$ has a branch cut. The transformation from measure to resolvent can thus be inverted by finding the discontinuity in $\chi(z)$ across the cut, or the residue of $\chi(z)$ at its poles. In Ref. [30], the authors carry out this calculus for the birth, death, and immigration process. For a more thorough review of the analytic methods used here, see also Refs. [49,50,52,67].

a. Computing the resolvent

The starting point for computing the resolvent $\chi(z)$ is to compute the firewalk polynomials “of the second kind,” $W_n^*(x)$. They satisfy the same recurrence relations (61) but with the altered initial conditions

$$W_0^*(x) = 0; \quad W_1^*(x) = \frac{1}{p_0} = \frac{\beta + \gamma + 1}{\beta}. \quad (C26)$$

Any set of orthogonal polynomials has an partner set ‘of the second kind’ defined in this manner. A key theorem due to Markov [49] is that the resolvent $\chi(z)$ may be written in terms of the limit

$$\chi(z) = \lim_{n \rightarrow \infty} \frac{W_n^*(z)}{W_n(z)}, \quad (C27)$$

which allows computation of $\chi(z)$ without knowledge of the measure. Computing these asymptotics for each set of polynomials W_n, W_n^* involves analyzing their respective generating functions and applying the method of Darboux. This lemma for estimating the asymptotic series coefficients of a complex function is reviewed in Appendix D. As the $W_n^*(x)$ satisfy the same recurrence as the $W_n(x)$, their generating function Φ^* satisfies a very similar differential equation. The difference is in the initial conditions for the recurrence which is reflected by the introduction of an inhomogeneous term. In particular, the $W_n^*(x)$ are generated by $\Phi^*(z)$ which solves

$$\begin{aligned} \frac{\partial \Phi^*}{\partial z} &= \frac{x(\beta + \gamma + 1) - z(\gamma + 2)}{z^2 - xz(\beta + 1) + \beta} \Phi^* \\ &+ \frac{\beta + \gamma + 1}{z^2 - xz(\beta + 1) + \beta}. \end{aligned} \quad (C28)$$

Using the same partial fraction decomposition as before one can write

$$\frac{\partial \Phi^*}{\partial z} = \left[\frac{A}{z - u} + \frac{B}{z - v} \right] \Phi^* + \frac{\beta + \gamma + 1}{(z - u)(z - v)}. \quad (C29)$$

The solution is $\Phi^* = f(z)\Phi$ where Φ solves the homogeneous equation. With the boundary condition $\Phi^*(0) = 0$ one finds

$$\begin{aligned} \Phi^*(x, z) &= (\beta + \gamma + 1)(u - z)^A (v - z)^B \\ &\times \int_0^z dt (u - t)^{-A-1} (v - t)^{-B-1}. \end{aligned} \quad (C30)$$

With the generating functions Φ, Φ^* the asymptotics of both the $W_n(x), W_n^*(x)$ can now be determined by an application of Darboux’s method. This is done in Appendix D. For $\text{Re } x > 0$

and off the cut $[-I_\beta, I_\beta]$, the results are

$$\lim_{n \rightarrow \infty} W_n(x) \sim (1 - v/u)^A \frac{n^{-B-1}}{\Gamma(-B)} v^{-n}, \quad (C31)$$

$$\lim_{n \rightarrow \infty} W_n^*(x) \sim (\beta + \gamma + 1) \frac{D^A}{\Gamma(-B)} v^{B-n} n^{-B-1} \quad (C32)$$

$$\times \int_0^v dt (u - t)^{-A-1} (v - t)^{-B-1}. \quad (C33)$$

The resolvent $\chi(z)$ is constructed by taking the ratio of these, recalling that u, v, A, B, D should all be considered as functions of a complex variable z :

$$\begin{aligned} \chi(z) &= \lim_{n \rightarrow \infty} \frac{W_n^*(z)}{W_n(z)} = (\beta + \gamma + 1) u^A v^B \int_0^v dt \\ &\times (u - t)^{-A-1} (v - t)^{-B-1}, \quad \text{Re } z \geq 0. \end{aligned} \quad (C34)$$

This expression, valid for $\text{Re } z > 0, \text{Im } z \neq 0$, is analytic for z away from the cut $[-1, 1]$ on the real line. It is precisely at this singular locus that the resolvent gives the data of the measure.

b. Stieltjes inversion

With the resolvent computed, one can now proceed to compute the elements of the measure. It so happens that there are both continuous and discrete parts, sourced by the branch cuts and poles of the resolvent $\chi(z)$, respectively. Formally, the Stieltjes transformation $d\sigma(x) \mapsto \chi(z)$ may be inverted by

$$\sigma(x_2) - \sigma(x_1) = \lim_{\varepsilon \rightarrow 0^+} \int_{x_1}^{x_2} dx \frac{1}{\pi} \text{Im} [\chi(x + i\varepsilon)]. \quad (C35)$$

This inversion formula is physically analogous to the optical theorem, where $\chi(z)$ represents the scattering amplitude, and the measure $d\sigma(x)$ is analogous to the cross-section. First consider the continuous part of the measure which is supported on the real interval $[-I_\beta, I_\beta]$. Taking the limit $x_1 \rightarrow x_2$ in the above yields

$$w(x) \equiv \frac{d\sigma(x)}{dx} = \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\pi} \text{Im} [\chi(x + i\varepsilon)]; \quad |x| \leq I_\beta, \quad (C36)$$

where x lies on the branch cut. As one approaches the cut $\varepsilon \rightarrow 0$, we have $u \rightarrow \bar{v}$ and $A \rightarrow \bar{B}$. This means that the resolvent prefactor $u^A v^B$ as well as the integrand are real as the cut is approached—they are symmetric in $u \leftrightarrow v, A \leftrightarrow B$. The integral bounds, however, induce a nonzero imaginary part;

$$\begin{aligned} 2i \text{Im} \int_0^v dt f(t) &= \int_0^v dt f(t) - \overline{\int_0^v dt f(t)} \\ &= \int_0^v dt f(t) + \int_{\bar{v}}^0 dt f(t) \\ &= \int_u^v dt f(t), \end{aligned} \quad (C37)$$

where the overbar represents complex conjugation. As a result, the continuous part of the measure of orthogonality is

$$w(x) = \frac{\beta + \gamma + 1}{2\pi i} u^A v^B \int_v^u dt (u - t)^{-A-1} (v - t)^{-B-1}, \quad (C38)$$

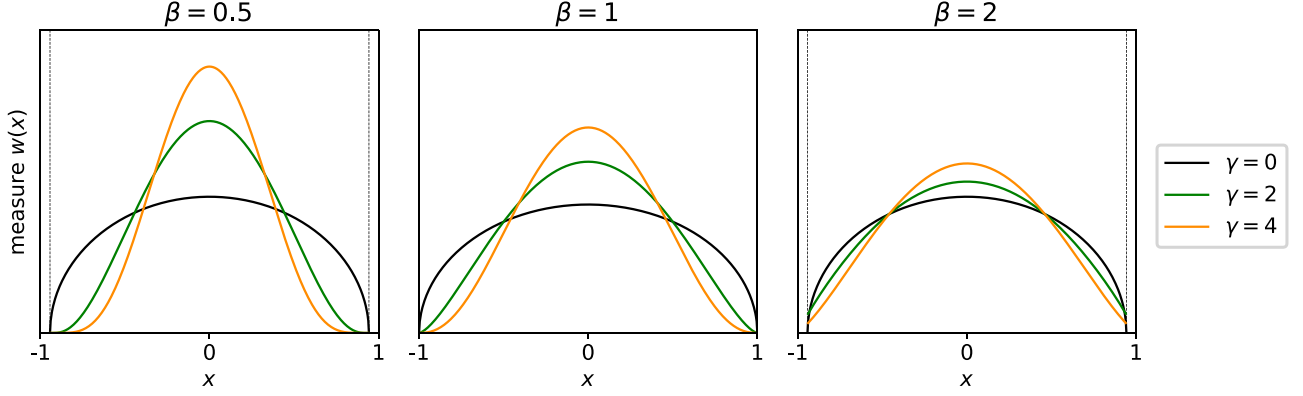


FIG. 14. The continuous measure of orthogonality $w(x) = d\sigma/dx$ of the firewalk polynomials $W_n(x)$ for various birth rates β and suppression rates γ . The dashed lines at $x = \pm I_\beta$ show that the measure is not supported on the entire interval $x \in [-1, 1]$. Larger suppression values can be seen to sharpen the peak of the distribution near zero.

for x in the interval $[-I_\beta, I_\beta]$. The definite integral, after some rescaling, is an Euler β function. In particular, one finds the expression

$$w(x) = \frac{\beta + \gamma + 1}{2\pi i} u^A v^B \times \frac{(u-v)^{-A} \Gamma(-A)\Gamma(-B)}{(v-u)^{B+1} \Gamma(\gamma+2)}, \quad (\text{C39})$$

noting that the formula is singular for $A, B \in \mathbb{Z}_+$. While obtuse, this measure, supported only on $[-I_\beta, I_\beta]$ defines a smooth weight function plotted in Fig. 14 for various values of the parameters. In the limit $\beta \rightarrow 1$ it should reduce to the Gegenbauer polynomial measure (C4). One may check that in the critical limit $\beta \rightarrow 1$,

$$u^A v^B \rightarrow 1, \quad (\text{C40})$$

$$\frac{(u-v)^{-A}}{(v-u)^{B+1}} \rightarrow 2^{\gamma+1} i (1-x^2)^{(\gamma+1)/2}, \quad (\text{C41})$$

$$\frac{\Gamma(-A)\Gamma(-B)}{\Gamma(\gamma+2)} \rightarrow \frac{\sqrt{\pi} 2^{-(\gamma+1)} \Gamma(\gamma/2+1)}{\Gamma(\gamma/2+3/2)}, \quad (\text{C42})$$

after applying some γ function identities. Therefore, one finds

$$\begin{aligned} \lim_{\beta \rightarrow 1} w(x) &= \frac{\gamma+2}{2\pi i} \cdot i (1-x^2)^{(\gamma+1)/2} \cdot \frac{\sqrt{\pi} \Gamma(\gamma/2+1)}{\Gamma(\frac{\gamma+3}{2})} \\ &= \frac{\Gamma(\frac{\gamma}{2}+2)}{\sqrt{\pi} \Gamma(\frac{\gamma+3}{2})} (1-x^2)^{(\gamma+1)/2}, \end{aligned} \quad (\text{C43})$$

which agrees perfectly with the known Gegenbauer measure. However, the work is not yet done. The continuous measure just constructed is not supported for $I_\beta < |x| < 1$. The full measure of orthogonality for the firewalk polynomials does have support in this region, but only at a discrete set of points. These points occur are precisely the poles of the resolvent.

c. Discrete measure

The continuous measure found in Eq. (C39) is even, as one can explicitly check by considering the transformation of u, v, A, B under $x \mapsto -x$. It is undefined at the points where A, B are nonnegative integers, which suggests that at these loci

the full measure has discrete weight and thus discontinuous derivative. From the resolvent formula

$$\begin{aligned} \chi(z) &= (\beta + \gamma + 1) u^A v^B \\ &\times \int_0^v dt (u-t)^{-A-1} (v-t)^{-B-1}, \quad \text{Re } z \geq 0, \end{aligned} \quad (\text{C44})$$

one can see that a singularity arises in the integrand as $t \rightarrow v$. This occurs in particular when $B = k$ is a nonnegative integer, and only where this form is valid in the right half-plane. Transforming $z \mapsto -z$, one finds

$$\begin{aligned} \chi(z) &= -(\beta + \gamma + 1) u^A v^B \\ &\times \int_0^u dt (u-t)^{-A-1} (v-t)^{-B-1}, \quad \text{Re } z \leq 0, \end{aligned} \quad (\text{C45})$$

so that in the left half-plane, a singularity arises as $t \rightarrow u$ and when $A = k \in \mathbb{Z}_+$. These singularities therefore occur at $\pm x_k$ satisfying

$$k = -\frac{\gamma+2}{2} + \frac{|x_k| \gamma (\beta-1)}{2D(x_k)}, \quad k \in \mathbb{Z}_+. \quad (\text{C46})$$

Since $\gamma \geq 0$ always, this equation only has solutions when $\beta > 1$. Therefore the discrete measure develops only for $\beta > 1$, $\gamma > 0$, noting that $A = B = -1$ uniformly in the case of zero suppression. Since the measure is even, we restrict to considering $x > 0$, $\beta > 1$, and the singularities associated with $B = k \in \mathbb{Z}_+$. These occur at the values $\pm x_k$ satisfying

$$x_k^2 = \frac{\beta[\gamma+2(k+1)]^2}{[\gamma+(k+1)(\beta+1)][\beta\gamma+(k+1)(\beta+1)]}. \quad (\text{C47})$$

For all k one has $I_\beta < x_k < 1$, with $x_k \rightarrow I_\beta$ as $k \rightarrow \infty$. In fact, the x_k approach I_β quite quickly as k increases. One now should compute the residue of the resolvent at the points x_k . By examining the resolvent expression in Eq. (C44), it is clear that with $B = k$ the relevant singularity arises when $t \rightarrow v$ in

the integral. Let $s = v - t$; then one has

$$\chi(z) = \frac{\beta + \gamma + 1}{\beta} \left(\frac{u}{u-v}\right)^{A+1} v^{B+1} \times \int_0^v ds \left(1 + \frac{s}{u-v}\right)^{-A-1} s^{-B-1}, \quad (C48)$$

with the goal being to isolate the singularity and extract its residue. From here, one can expand the first part of the integrand as an infinite sum, writing

$$\int_0^v ds \left(1 + \frac{s}{u-v}\right)^{-A-1} s^{-B-1} = \sum_{n=0}^{\infty} \frac{(A+1)_n}{n!} \left(\frac{1}{v-u}\right)^n \left[\int_0^v ds s^{n-B-1}\right]. \quad (C49)$$

Here, one should regard the definite integral as its analytic continuation to all $B \notin \mathbb{Z}_+$ in the sense of Hadamard [30,52]. The quantities u, v, A, B all take finite, real values at the $|x_k| \geq I_\beta$. The measure weight is the residue of the resolvent as $z \rightarrow x_k$; by definition this is the coefficient of $(z - x_k)^{-1}$ in the expansion of the function around $z = x_k$. Near this point, one has

$$\lim_{z \rightarrow x_k} B = k + \alpha(z - x_k) + \dots, \quad (C50)$$

$$\alpha = \left. \frac{dB}{dz} \right|_{z=x_k} = -2\beta\gamma(\beta - 1)D(x_k)^{-3}. \quad (C51)$$

Formally evaluating the integral near $B = k$ yields the expansion

$$\int ds s^{n-B-1} = \frac{1}{n-B} s^{n-B} = \frac{1}{n-k-\alpha(z-x_k)} \times [s^{n-k} - \alpha s^{n-k} \log(s)(z-x_k) + \dots]. \quad (C52)$$

For any $n \neq k$, this function is formally analytic as $z \rightarrow x_k$. A nonzero residue at x_k is therefore generated only by the term with $n = k$. Setting $n = k$ in Eqs. (C48) and (C49) and evaluating the definite integral one finds that the residue Δ_k at x_k is

$$\Delta_k = \frac{\beta + \gamma + 1}{\beta} \left(\frac{u}{u-v}\right)^{A+1} \times \frac{v^{k+1}}{(v-u)^k} \frac{(A+1)_k}{k!} \left(-\frac{dx}{dB}\right) \Big|_{x=x_k}, \quad (C53)$$

which by even-ness is also the weight of the measure at $-x_k$. Writing $u_k \equiv u(x_k)$ and evaluating everything at x_k one finds that the weight Δ_k of the discrete measure at $\pm x_k$ is

$$\Delta_k = \frac{\beta + \gamma + 1}{\beta^{\gamma+3}} (u_k)^{-k} (v_k)^{k+\gamma+2} \times \frac{(\gamma+2)_k}{k!} \cdot \frac{D_k^{\gamma+4}}{2\gamma(\beta-1)}, \quad \beta > 1, \quad (C54)$$

with the understanding that $\Delta_k \equiv 0$ for $\beta \leq 1$. Along with the continuous measure in Eq. (C39), these discrete weights fully

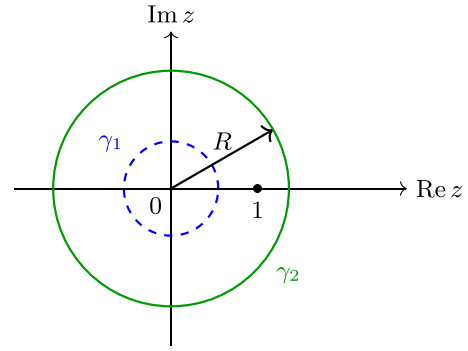


FIG. 15. Contours for Darboux’s method. By choosing a comparison function $g(z)$ to subtract off the singularity at $z = 1$, the function $f(z) - g(z)$ can be extended to the contour γ_2 , where large n asymptotics may be taken.

characterize the orthogonality relation for the $W_n(x)$:

$$\int_{-I_\beta}^{I_\beta} W_n(x)W_m(x)w(x)dx + \sum_{k=0}^{\infty} \Delta_k W_n(x_k)W_m(x_k) + \sum_{k=0}^{\infty} \Delta_k W_n(-x_k)W_m(-x_k) = h_n \delta_{n,m}. \quad (C55)$$

In principle, one can now calculate arbitrary matrix elements of the n -step transition matrix $S(n)$. However, these expressions rarely, if ever, have a reasonable analytical form. They are better suited to numerical analysis.

APPENDIX D: METHOD OF DARBOUX

In this Appendix we review the method of Darboux for estimating the asymptotic coefficients of a meromorphic function $f(z)$. For a thorough reference, see Refs. [67–69]. Here, we develop only the tools necessary to compute the measure in Sec. IV, for completeness and pedagogical purposes. Begin with a meromorphic function $f(z)$ analytic at the origin: the goal is to estimate its asymptotic series coefficients. For simplicity, let the dominant singularity (that of smallest magnitude) of $f(z)$ be at $z = 1$, so that the series

$$f(z) = \sum_{n=0}^{\infty} c_n z^n \quad (D1)$$

converges inside the unit disk. Of course, any meromorphic function may be holomorphically mapped to this form. It turns out that the asymptotic behavior of the c_n depends strongly on the nature of the singularities of the function $f(z)$. To see why this is the case, recall that the constants $c_n \equiv [z^n]f(z)$ may be computed by the following contour integral:

$$c_n = [z^n]f(z) = \frac{1}{2\pi i} \oint_{\gamma_1} f(z) \frac{dz}{z^{n+1}}, \quad (D2)$$

where the contour γ_1 encircles the origin, as shown in Fig. 15. Define another function $g(z)$ which is analytic for $|z| < 1$ and such that $f(z) - g(z)$ is continuous on $|z| = 1$. This defines a sort of ‘minimal subtraction’ of the singularity at $z = 1$. Now, by construction, the function $h(z) = f(z) - g(z)$ is analytic in the disk $|z| \leq R$ for some $R > 1$. The series coefficients of

$f(z) - g(z)$ are therefore

$$f_n - g_n = [z^n]h(z) = \oint_{\gamma_1} h(z) \frac{dz}{z^{n+1}} = \oint_{\gamma_2} h(z) \frac{dz}{z^{n+1}},$$

where the contour of integration can be radially extended to γ_2 , located at a radius $R > 1$. By parametrizing the contour in polar coordinates, one bounds the difference $f_n - g_n$ as

$$|f_n - g_n| = \left| \oint_{\gamma_2} h(z) \frac{dz}{z^{n+1}} \right| \tag{D3}$$

$$= R^{-n} \left| \int_0^{2\pi} h(z(\phi)) e^{-in\phi} d\phi \right| \tag{D4}$$

$$\leq R^{-n} \int_0^{2\pi} |h(z)| d\phi, \tag{D5}$$

where, in the final step, the integral does not depend on n . An asymptotic statement can now be made: as $n \rightarrow \infty$, since $R > 1$, we have

$$|f_n - g_n| \lesssim \mathcal{O}(R^{-n}) \Rightarrow f_n \sim g_n. \tag{D6}$$

This lemma is often referred to as Darboux’s method [30]. Informally, it states that a comparison function $g(z)$ which shares the same dominant singularity as $f(z)$ has asymptotically identical series coefficients. To make use of this, one can compute the asymptotic series coefficients for a singularity of a standard form, then use this theorem to transfer the results to a more complex function. Consider the basic singularity

$$(1 - z/z_0)^\alpha = \sum_{n=0}^{\infty} (-1)^n \binom{\alpha}{n} \left(\frac{z}{z_0}\right)^n. \tag{D7}$$

Here, the explicit form of the series coefficients is known, so one can easily extract the asymptotics by applying the Stirling approximation:

$$(-1)^n \binom{\alpha}{n} = \frac{(-\alpha)_n}{n!} \tag{D8}$$

$$\sim \frac{n^{-\alpha-1}}{\Gamma(-\alpha)} \left[1 + \frac{\alpha(\alpha+1)}{2n} + \dots \right]. \tag{D9}$$

This gives the general result

$$[z^n](1 - z/z_0)^\alpha \sim z_0^{-n} \frac{n^{-\alpha-1}}{\Gamma(-\alpha)}, \quad \alpha \notin \mathbb{Z}_+. \tag{D10}$$

For a derivation of this formula using only explicit contour integration, see Ref. [69]. Note that this vanishes when the exponent α is a nonnegative integer. The reason is straightforward: the left hand side of Eq. (D7) is a polynomial in z of degree α , so its asymptotic series coefficients are identically zero [67]. In this special case, the leading term in the limit $n \rightarrow \infty$ is simply the term of highest degree.

Firewalk polynomial asymptotics

Having established the asymptotic approximation for the standard singularity, we can turn to the generating function

(C14) for the firewalk polynomials $W_n(x)$:

$$\Phi(z, x) = (1 - z/u)^A (1 - z/v)^B = \sum_{n=0}^{\infty} W_n(x) z^n. \tag{D11}$$

This function has two singularities which may be coincident in norm depending on the value of x ; one has

$$u = \frac{x(\beta + 1)}{2} + \frac{D(x)}{2}, \quad v = \frac{x(\beta + 1)}{2} - \frac{D(x)}{2}, \tag{D12}$$

$$D(x) = \sqrt{x^2(\beta + 1)^2 - 4\beta}. \tag{D13}$$

Recall that for x real and $|x| \leq I_\beta$, the roots u, v are coincident in norm with $u = \bar{v}$. For positive $x > I_\beta$, one has $|v| < |u|$, whereas for negative $x < -I_\beta$ the inequality is reversed. Take $x \notin \mathbb{R}$ and $\text{Re } x > 0$; this means that v is the dominant singularity of the generating function (D11). One should construct a comparison function $g(z)$ which shares the singularity at $z = v$. To do so, expand the generating function $\Phi(z, x)$ around the point $z = v$, taking only the leading term as $z \rightarrow v$:

$$g(z) = (1 - v/u)^A (1 - z/v)^B. \tag{D14}$$

One can now simply apply the result (D10) to determine the asymptotic coefficients of $g(z)$; Darboux’s lemma ensures that these are also the asymptotics of the $W_n(x)$. Therefore, one finds

$$\lim_{n \rightarrow \infty} W_n(x) \sim (1 - v/u)^A \frac{n^{-B-1}}{\Gamma(-B)} v^{-n}, \tag{D15}$$

with the caveat that this formula vanishes for $B \in \mathbb{Z}_+$. For the firewalk polynomials of the second kind $W_n^*(x)$, one has the generating function

$$\begin{aligned} \Phi^*(x, z) &= (\beta + \gamma + 1)(u - z)^A (v - z)^B \\ &\times \int_0^z dt (u - t)^{-A-1} (v - t)^{-B-1}. \end{aligned} \tag{D16}$$

By the same arguments as above, for positive x , the root v is the dominant singularity of this function. The procedure is therefore essentially the same. To define a comparison function, expand Φ^* around the point $z = v$:

$$\begin{aligned} g(z) &= (\beta + \gamma + 1) D^A v^B (1 - z/v)^B \\ &\times \int_0^v dt (u - t)^{-A-1} (v - t)^{-B-1}, \end{aligned} \tag{D17}$$

to which one can apply the result for the standard singularity. This gives the asymptotic result

$$\begin{aligned} \lim_{n \rightarrow \infty} W_n^*(x) &\sim (\beta + \gamma + 1) \frac{D^A}{\Gamma(-B)} v^{B-n} n^{-B-1} \\ &\times \int_0^v dt (u - t)^{-A-1} (v - t)^{-B-1}. \end{aligned} \tag{D18}$$

This completes the asymptotics needed to compute the resolvent for the firewalk polynomial measure.

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