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Developing an Optimal Class of Generic Sixteenth-Order Simple-Root Finders and Investigating Their Dynamics

Young Hee Geum [1], Young Ik Kim [1,*] and Beny Neta [2]

[1] Department of Applied Mathematics, Dankook University, Cheonan 330-714, Korea; conpana@empas.com
[2] Naval Postgraduate School, Department of Applied Mathematics, Monterey, CA 93943, USA; bneta@nps.edu
* Correspondence: yikbell@yahoo.co.kr; Tel.: +82-41-550-3415

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Abstract: Developed here are sixteenth-order simple-root-finding optimal methods with generic weight functions. Their numerical and dynamical aspects are investigated with the establishment of a main theorem describing the desired optimal convergence. Special cases with polynomial and rational weight functions have been extensively studied for applications to real-world problems. A number of computational experiments clearly support the underlying theory on the local convergence of the proposed methods. In addition, to investigate the relevant global convergence, we focus on the dynamics of the developed methods, as well as other known methods through the visual description of attraction basins. Finally, we summarized the results, discussion, conclusion, and future work.

Keywords: sixteenth-order optimal convergence; weight function; asymptotic error constant; global convergence; purely imaginary extraneous fixed point; attractor basin

MSC: 65H05; 65H99

1. Introduction

The governing equations of real-world natural phenomena are often described by nonlinear equations whose exact solutions are infeasible due to their inherent complexities. The attainment of precise numerical approximations to the roots of such complicated nonlinear functions is important for many scientific fields. The classical second-order Newton's method is best known as the numerical root-finder for the governing equations. For several decades, many authors [1–11] have developed higher-order multipoint methods. If an iterative root-finding method satisfies Kung-Traub's conjecture [12], then it is said to be optimal. A few authors [12–14] have recently established optimal sixteenth-order methods, despite the lack of applicability to real-life nonlinear governing equations due to their algebraic complexities, not only to emphasize the theoretical importance of developing extremely high-order methods, but also to apply them to root-finding of real-world nonlinear problems, we strongly desire to establish a new optimal family of sixteenth-order simple-root finders that are comparable to or competitive against the existing methods.

For the sake of comparison with the new optimal family of methods to be proposed in this paper, we introduce existing three optimal sixteenth-order Equations [12–14] respectively given by Equation (1), (2), and (4) below.

- Kung-Traub method (KT16):

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \frac{f(x_n)^2 f(y_n)}{f'(x_n)[f(x_n)-f(y_n)]^2}, \\ w_n = z_n - \frac{f(x_n)^2 f(y_n)}{f'(x_n)[f(x_n)-f(y_n)]^2} \cdot \frac{[f(x_n)^2+f(y_n)^2-f(y_n)f(z_n)]f(z_n)}{[f(x_n)-f(z_n)]^2[f(y_n)-f(z_n)]}, \\ x_{n+1} = w_n - \frac{f(x_n)^2 f(y_n)}{f'(x_n)[f(x_n)-f(y_n)]^2} \cdot \frac{f(w_n)f(z_n)\{h_0 f(x_n)^2+h_1 f(y_n)f(z_n)\}}{h_1[f(x_n)-f(w_n)]^2[f(x_n)-f(z_n)]^2}, \end{cases} \tag{1}$$

where $h_0 = f(y_n)[f(x_n)^2 - f(w_n)f(y_n) + f(y_n)^2] + f(z_n)[(f(w_n) - f(z_n))(f(w_n) - 2f(x_n) + f(z_n)]$ and $h_1 = [f(y_n) - f(w_n)][f(y_n) - f(z_n)][f(z_n) - f(w_n)]$.

- Maroju-Behl-Motsa method (MBM):

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \frac{f(x_n)+\beta f(y_n)}{f(x_n)+(\beta-2)f(y_n)} \cdot \frac{f(y_n)}{f'(x_n)}, \ \beta \in \mathbb{R}, \\ w_n = z_n - G(u,s) \cdot \frac{f(z_n)}{f'(x_n)}, s = \cdot \frac{f(y_n)}{f(x_n)}, u = \cdot \frac{f(z_n)}{f(y_n)}, \\ x_{n+1} = x_n - \theta_5 f(x_n), \end{cases} \tag{2}$$

where $G : \mathbb{C}^2 \to \mathbb{C}$ is an analytic function in a neighborhood of $(0,0)$ satisfying $G_{00} = 1, G_{01} = 2, G_{10} = 1, G_{02} = 10 - 4\beta, G_{11} = 4, G_{03} = 12(\beta^2 - 6\beta + 6)$, with $G_{ij} = \frac{\partial^{i+j}}{\partial u^i \partial s^j} G(u,s)|_{(u=0,s=0)}$ for $i, j = 0, 1, 2, 3$, and θ_5 is given by the following:

$$\theta_5 = \frac{a_n b_n [u_1 f(x_n)^2 f(y_n) + u_2 f'(x_n) f(w_n) f(z_n)]}{v_1 f(x_n)^3 + v_2 f'(x_n) f(w_n) f(z_n)}, \tag{3}$$

with
$u_1 = f(w_n)[b_n^2 f'(x_n) + b_n f(x_n) - c_n f(z_n)] + a_n[f(x_n) - a_n f'(x_n)]f(z_n)$,
$u_2 = a_n b_n c_n f'(x_n)[f(y_n) - f(x_n)] + c_n f(y_n) f(x_n)(a_n - b_n)$,
$v_1 = f(y_n)[b_n f(w_n)\{b_n^2 f'(x_n) + b_n f(x_n) - c_n f(z_n)\} + \{a_n^3 f'(x_n) + c_n a_n f(w_n) - a_n^2 f(x_n)\}f(z_n)]$,
$v_2 = a_n^2 b_n^2 c_n f'(x_n)^2\{2f(y_n) - f(x_n)\} + a_n b_n c_n (2a_n - c_n) f'(x_n) f(y_n) f(x_n)$
$+ c_n\{a_n b_n - a_n c_n - b_n^2\} f(y_n) f(x_n)^2, \ a_n = x_n - z_n, b_n = w_n - x_n, c_n = w_n - z_n$.

- Sharma-Argyros-Kumar method (SAK):

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = x_n - \frac{f(y_n)-f(x_n)}{f'(x_n)f(y_n)-f[x_n,y_n]f(x_n)} \cdot f(x_n), \\ w_n = x_n - \frac{D_3}{\Delta_3} \cdot f(x_n) \\ x_{n+1} = x_n - \frac{D_4}{\Delta_4} \cdot f(x_n) \end{cases} \tag{4}$$

where $f[r,t] \equiv \frac{f(r)-f(t)}{r-t}$, $D_3 = \begin{vmatrix} 1 & f(x_n) & x_n f(x_n) \\ 1 & f(y_n) & y_n f(y_n) \\ 1 & f(z_n) & z_n f(z_n) \end{vmatrix}$, $\Delta_3 = \begin{vmatrix} f'(x_n) & f(x_n) & x_n f(x_n) \\ f[x_n,y_n] & f(y_n) & y_n f(y_n) \\ f[x_n,z_n] & f(z_n) & z_n f(z_n) \end{vmatrix}$,

$$D_4 = \begin{vmatrix} 1 & f(x_n) & x_n f(x_n) & x_n^2 f(x_n) \\ 1 & f(y_n) & y_n f(y_n) & y_n^2 f(y_n) \\ 1 & f(z_n) & z_n f(z_n) & z_n^2 f(z_n) \\ 1 & f(w_n) & w_n f(w_n) & w_n^2 f(w_n) \end{vmatrix}, \quad \Delta_4 = \begin{vmatrix} f'(x_n) & f(x_n) & x_n f(x_n) & x_n^2 f(x_n) \\ f[x_n,y_n] & f(y_n) & y_n f(y_n) & y_n^2 f(y_n) \\ f[x_n,z_n] & f(z_n) & z_n f(z_n) & z_n^2 f(z_n) \\ f[x_n,w_n] & f(w_n) & w_n f(w_n) & w_n^2 f(w_n) \end{vmatrix},$$

with $|\bullet|$ denoting the determinant of $\bullet$.

In order to develop the desired competitive optimal sixteenth-order simple-root finders, we seek a class of iterative methods with generic weight functions:

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - Q_f(s)\frac{f(y_n)}{f'(x_n)}, \\ w_n = z_n - K_f(s,u)\frac{f(z_n)}{f'(x_n)} = x_n - \left[1 + sQ_f(s) + suK_f(s,u)\right]\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = w_n - J_f(s,u,v)\frac{f(w_n)}{f'(x_n)} = x_n - \left[1 + sQ_f(s) + suK_f(s,u) + suvJ_f(s,u,v)\right]\frac{f(x_n)}{f'(x_n)}, \end{cases} \tag{5}$$

where $s = \frac{f(y_n)}{f(x_n)}$, $u = \frac{f(z_n)}{f(y_n)}$, $v = \frac{f(w_n)}{f(z_n)}$; $Q_f : \mathbb{C} \to \mathbb{C}$ is analytic [15] in a neighborhood of 0, $K_f : \mathbb{C}^2 \to \mathbb{C}$ holomorphic [16,17] in a neighborhood of $(0,0)$, and $J_f : \mathbb{C}^3 \to \mathbb{C}$ holomorphic in a neighborhood of $(0,0,0)$.

One should observe that Systems (1), (2) and (4) are special cases of (5) with appropriate forms of weight functions Q_f, K_f, and J_f, as respectively shown by Systems (6), (7), and (10):

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \frac{1}{(1-s)^2}\cdot\frac{f(y_n)}{f'(x_n)}, \\ w_n = z_n - \frac{1+(1-u)s^2}{(1-s)^2(1-u)(1-su)^2}\cdot\frac{f(z_n)}{f'(x_n)}, \\ x_{n+1} = w_n - \frac{-1+2su^2(v-1)+s^4(u-1)u^2(v-1)(uv-1)+s^2[uv-1-u^3(v^2-1)]}{(1-s)^2(u-1)(su-1)^2(v-1)(uv-1)(suv-1)^2}\cdot\frac{f(w_n)}{f'(x_n)}. \end{cases} \tag{6}$$

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - Q_f(s)\cdot\frac{f(y_n)}{f'(x_n)}, \\ w_n = z_n - K_f(s,u)\cdot\frac{f(z_n)}{f'(x_n)}, \\ x_{n+1} = w_n - J_f(s,u,v)\cdot\frac{f(w_n)}{f'(x_n)}, \end{cases} \tag{7}$$

where $Q_f(s) = \frac{1+\beta s}{1+(\beta-2)s}$, $K_f(s,u) = G(u,s)$ and

$$J_f(s,u,v) = -\frac{K_f(s,u)[Q_f(s) + uK_f(s,u)][suK_f(s,u) + 1 + sQ_f(s)]^2\lambda_0}{Q_f(s)(v-1)(1+sQ_f(s))^2 + uvK_f(s,u)[\psi_0 + (1+sQ_f(s))\lambda_1 + suK_f(s,u)\lambda_2]}, \tag{8}$$

where $\psi_0 = s^2u^2K_f(s,u)^2\lambda_0$, $\lambda_0 = 1 - u[1 + (2-3s)sQ_f(s) + \gamma s^2Q_f(s)^2]$, $\lambda_1 = 1 + 3sQ_f(s) - u[1 + 3sQ_f(s) + (3-4s)s^2Q_f(s)^2 + \gamma s^3Q_f(s)^3]$, $\lambda_2 = 2 + 3sQ_f(s) - u[2 + (6-4s)sQ_f(s) + (6-9s)s^2Q_f(s)^2 + 2\gamma s^3Q_f(s)^3]$, and

$$\gamma = 1 - 2s. \tag{9}$$

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \frac{1}{\gamma}\frac{f(y_n)}{f'(x_n)}, \\ w_n = z_n - \frac{(s-1)^2}{\gamma\delta}\frac{f(z_n)}{f'(x_n)}, \\ x_{n+1} = w_n + \frac{(s-1)^2(\delta+(s-1)^2u)((s^2+s-1)u+\gamma)^2}{\gamma\delta\cdot[v\{\psi_1u^3-\gamma^3+u^2\gamma(\gamma^2+2s^2(2\gamma+s^2))+u\gamma^2(1-3s^2)\}+\gamma\delta^2]}\cdot\frac{f(w_n)}{f'(x_n)}, \end{cases} \tag{10}$$

where $\psi_1 = (s-1)(1-3s+4s^2-4s^3-2s^4+6s^5)$, γ is given by (9) and $\delta = 1-2s-u+2s^2u$.

Besides the above-mentioned recent sixteenth-order methods, we found a classical work developed by Neta [18] in 1981, which is a one-parameter family of optimal sixteenth-order methods:

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \frac{f(x_n)+Af(y_n)}{f(x_n)+(A-2)f(y_n)}\frac{f(y_n)}{f'(x_n)}, \ A \in \mathbb{R}, \\ s_n = y_n + \delta_1 f^2(x_n) + \delta_2 f^3(x_n), \\ x_{n+1} = y_n + \theta_1 f^2(x_n) + \theta_2 f^3(x_n) + \theta_3 f^4(x_n), \end{cases} \tag{11}$$

where $\delta_2 = -\frac{\phi_y-\phi_z}{F_y-F_z}$, $\delta_1 = \phi_y + \delta_2 F_y$, $\theta_3 = \frac{\Delta_1-\Delta_2}{F_s-F_y}, \theta_2 = -\Delta_1 + \theta_3(F_s + F_z)$, $\theta_1 = \phi_s + \theta_2 F_s - \theta_3 F_s^2$ with $\Delta_1 = \frac{\phi_s-\phi_z}{F_s-F_z}$, $\Delta_2 = \frac{\phi_y-\phi_z}{F_y-F_z}, \phi_s = \frac{1}{F_s}(\frac{s_n-x_n}{F_s} - \frac{1}{f'(x_n)}), \phi_y = \frac{1}{F_y}(\frac{y_n-x_n}{F_y} - \frac{1}{f'(x_n)}), \phi_z = \frac{1}{F_z}(\frac{z_n-x_n}{F_z} - \frac{1}{f'(x_n)}), F_s = f(s_n) - f(x_n), F_y = f(y_n) - f(x_n)$ and $F_z = f(z_n) - f(x_n)$.

Evidently, the form of Equation (11) shows an example that is not a member of (1.5).

Our main aim is to devise an optimal class of sixteenth-order methods by characterizing the algebraic structure of weight functions $Q_f(s)$, $K_f(s,u)$, and $J_f(s,u,v)$, as well as to explore their dynamics through basins of attractions [19] behind the extraneous fixed points [20] with applications to $f(z) = (z-a)^m(z-b)^m$. The right side of final substep of (5) conveniently locates extraneous fixed points from the roots of the combined weight function $1 + sQ_f(s) + suK_f(s,u) + suvJ_f(s,u,v)$.

It is important that we seek appropriate parameters for which the attractor basin contains larger regions of convergence. A motivation undertaking this research was to extensively study the dynamics behind the extraneous fixed points, which would impact on the relevant dynamics of the iterative methods by producing attractive, indifferent, repulsive, and other chaotic orbits. The entire complex plane is composed of two symmetrical half-planes whose boundary is the imaginary axis. We display the convergence behavior in the dynamical planes through the attractor basins within a square region centered at the origin. We also want to make the dynamics behind the extraneous fixed points on the imaginary axis less influenced by the possible periodic or chaotic attractors. Therefore, in addition to general cases with free parameters leading us to simple weight functions, we preferably include some interesting cases with free parameters, chosen for the purely imaginary extraneous fixed points.

In Section 2, the main theorem is presented with three weight functions, Q_f, K_f, and J_f, containing free parameters. Section 3 considers special cases of weight functions. Section 4 discusses the extraneous fixed points, including purely imaginary ones and investigates their stabilities. Section 5 presents numerical experiments as well as illustrates the relevant dynamics and summarizes the overall work together with future work.

2. Methods and Materials

The main theorem is established by describing the error equation and the asymptotic error constant with relationships among generic weight functions $Q_f(s)$, $K_f(s,u)$, and $J_f(s,u,v)$:

Theorem 1. *Suppose that $f : \mathbb{C} \to \mathbb{C}$ has simple root α and is analytic in a neighborhood of α. Let $c_j = \frac{f^{(j)}(\alpha)}{j!f'(\alpha)}$ for $j = 2, 3, \cdots$. Let x_0 be an initial guess selected in a sufficiently small region containing α. Assume $L_f : \mathbb{C} \to \mathbb{C}$ is analytic in a neighborhood of 0. Let $Q_i = \frac{1}{i!}\frac{d^i}{ds^i}Q_f(s)\big|_{(s=0)}$ for $0 \le i \le 6$. Let $K_f : \mathbb{C}^2 \to \mathbb{C}$ be holomorphic in a neighborhood of $(0,0)$. Let $J_f : \mathbb{C}^3 \to \mathbb{C}$ be holomorphic in a neighborhood of $(0,0,0)$. Let $K_{ij} = \frac{1}{i!j!}\frac{\partial^{i+j}}{\partial s^i \partial u^j}K_f(s,u)\big|_{(s=0,u=0)}$ for $0 \le i \le 12$ and $0 \le j \le 6$. Let $J_{ijk} = \frac{1}{i!j!k!}\frac{\partial^{i+j+k}}{\partial s^i \partial u^j \partial v^j}J_f(s,u,v)\big|_{(s=0,u=0,v=0)}$ for $0 \le i \le 8$, $0 \le j \le 4$ and $0 \le k \le 2$. If $Q_0 = 1, Q_1 = 2, K_{00} = 1, K_{10} = 2, K_{01} = 1, K_{20} = 1 + Q_2, K_{11} = 4, K_{30} = -4 + 2Q_2 + Q_3, J_{000} = 1, J_{100} = 2, J_{200} = 1 + Q_2, J_{010} = 1, J_{110} = 4, J_{300} = -4 + 2Q_2 + Q_3, J_{001} = 1, J_{020} = K_{02}, J_{210} = 1 + K_{21}, J_{400} = K_{40}, J_{101} = 2, J_{120} = 2 + K_{12}, J_{310} = -4 + K_{31} + 2Q_2, J_{500} = K_{50}, J_{011} = 2, J_{201} = 1 + Q_2, J_{030} = -1 + K_{02} + K_{03}, J_{220} = 1 + K_{21} + K_{22} - Q_2, J_{410} = -3 + K_{40} + K_{41} + Q_2 - Q_4, J_{600} = K_{60}, J_{111} = 8, J_{301} = -4 + 2Q_2 + Q_3, J_{130} = -4 + 2K_{02} + K_{12} + K_{13}, J_{320} = -6 + 2K_{21} + K_{31} + K_{32} - 2Q_2 - Q_3, J_{510} = 6 + 2K_{40} + K_{50} + K_{51} - 3Q_3 - 2Q_4 - Q_5, J_{700} = K_{70}$ are fulfilled, then scheme (5) leads to an optimal class of sixteenth-order root-finders possessing the following error equation: with $e_n = x_n - \alpha$ for $n = 0, 1, 2, \cdots$,*

$$e_{n+1} = -c_2(\tau c_2^2 + c_3)\left[\eta_0 c_2^4 + \eta_1 c_2^2 c_3 + (K_{02}-1)c_3^2 - c_2 c_4\right]\Psi\, e_n^{16} + O(e_n^{17}), \tag{12}$$

where $\tau = Q_2 - 5$, $\nu = K_{40} - 2Q_3 - Q_4$, $\mu = K_{21} - 9$, $\eta_0 = \nu - \mu\tau + \tau^2 K_{02}, \eta_1 = 2\tau K_{02} - \tau - \mu + 5$, $\Psi = \Delta_1 c_2^8 + \Delta_2 c_2^6 c_3 + \Delta_3 c_2^5 c_4 + (J_{040} + J_{021}(1-K_{02}) + J_{002}(1-K_{02})^2 + 2K_{02} - K_{03} - K_{04} - 1)c_3^4 + \Delta_4 c_2^3 c_3 c_4 + (J_{021} - 3 + 2J_{002}(1 - K_{02}) + K_{02})c_2 c_3^2 c_4 + \Delta_5 c_2^4 + \Delta_6 c_2^2$, $\Delta_1 = J_{800} - K_{80} + \nu(2Q_3 + Q_4 - J_{401} + J_{002}\nu) + (21 - J_{610} - 19K_{40} + 2K_{50} + K_{60} + K_{61} + 16Q_4 - 2Q_5 - Q_6 - 2Q_3(\mu - 19) + J_{401}\mu + J_{211}\nu - \mu(Q_4 + 2J_{002}\nu))\tau + (18 + J_{420} - 2K_{31} - K_{41} - K_{42} + 2Q_3 + Q_4 + \mu(19 - J_{211} +$

$J_{002}\mu) - J_{021}\nu + K_{02}(2Q_3 + Q_4 - J_{401} + 2\nu J_{002}))\tau^2 + (2K_{12} + K_{22} + K_{23} - 9 - J_{230} + J_{021}\mu + K_{02}(J_{211} - 19 - 2J_{002}\mu))\tau^3 + (J_{040} - J_{021}K_{02} + J_{002}K_{02}^2 - K_{03} - K_{04})\tau^4$, $\Delta_2 = 21 - J_{610} - 34K_{40} + 2K_{50} + K_{60} + K_{61} + Q_2^3 - 2Q_5 - Q_6 - 265\mu - 2Q_2^2(13+\mu) + Q_2(105 + 2K_{40} + 63\mu) + 2(J_{420} - 2K_{31} - K_{41} - K_{42})\tau - 3(J_{230} - 2K_{12} - K_{22} - K_{23})\tau^2 + 4(J_{040} - K_{03} - K_{04})\tau^3 - 2Q_3(\mu + \tau - 29) - Q_4(\mu + \tau - 26) + J_{401}(\mu + \tau - 5 - 2K_{02}\tau) + 2K_{02}\tau(2Q_3 + Q_4 - 31\tau + \tau^2) + 2J_{002}(\mu + \tau - 5 - 2K_{02}\tau)(\mu\tau - \nu - K_{02}\tau^2) + J_{021}\tau(-2\nu + \tau(-5 + 3\mu + \tau - 4K_{02}\tau)) + J_{211}(\nu + \tau(5 - 2\mu - \tau + 3K_{02}\tau))$, $\Delta_3 = J_{401} + K_{40} - 4Q_3 - 2Q_4 + (25 - J_{211} - \mu)\tau + (J_{021} + K_{02})\tau^2 - 2J_{002}(\nu - \mu\tau + K_{02}\tau^2)$, $\Delta_4 = 35 - J_{211} - \mu + (2J_{021} + 2K_{02} - 3)\tau + 2J_{002}(\mu + \tau - 5 - 2K_{02}\tau)$, $\Delta_5 = -c_5\tau + c_3^2(J_{420} - 123 + J_{401}(1 - K_{02}) - 2K_{31} - K_{41} - K_{42} + K_{02}(2Q_3 + Q_4) + 24\mu + 2\nu + (23 - 3J_{230} - 67K_{02} + 6K_{12} + 3K_{22} + 3K_{23} - 4\mu)\tau + (1 + 6J_{040} + 6K_{02} - 6K_{03} - 6K_{04})\tau^2 + J_{211}(5 - \mu + (3K_{02} - 2)\tau) + J_{021}(-\nu + (3\mu - 10)\tau + 3(1 - 2K_{02})\tau^2) + J_{002}((\mu - 5)^2 - 2\nu + 2K_{02}\nu + 2\tau(K_{02}(10 - 3\mu) + 2\mu - 5) + (1 - 6K_{02} + 6K_{02}^2)\tau^2))$, $\Delta_6 = (J_{002} - 1)c_4^2 - c_3c_5 + c_3^3(25 - J_{230} + J_{211}(K_{02} - 1) - 24K_{02} + 2K_{12} + K_{22} + K_{23} - 2\mu + (4J_{040} - 1 + 6K_{02} - 4K_{03} - 4K_{04})\tau + J_{021}(\mu + 3\tau - 5 - 4K_{02}\tau) + 2J_{002}(K_{02} - 1)(5 - \mu - \tau + 2K_{02}\tau))$.

Proof. Since Scheme (5) employs five functional evaluations, namely, $f'(x_n), f(x_n), f(y_n), f(z_n), and f(w_n)$, optimality can be achieved if the corresponding convergence order is 16. In order to induce the desired order of convergence, we begin by the 16th-order Taylor series expansion of $f(x_n)$ about α:

$$f(x_n) = f'(\alpha)\{e_n + \sum_{i=2}^{16} c_i\, e_n^i + O(e_n^{17})\}. \tag{13}$$

It follows that

$$f'(x_n) = f'(\alpha)\{1 + \sum_{i=2}^{16} i\, c_i\, e_n^{i-1} + O(e_n^{16})\}. \tag{14}$$

For brevity of notation, we abbreviate e_n with e. Using Mathematica [21], we find:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} = \alpha + c_2e^2 - 2(c_2^2 - c_3)e^3 + Y_4e^4 + Y_5e^5 + Y_6e_n^6 + Y_7e_n^7 + \sum_{i=8}^{16} Y_i\, e_n^i + O(e^{17}), \tag{15}$$

where $Y_4 = 4c_2^3 - 7c_2c_3 + 3c_4$, $Y_5 = -2(4c_2^4 - 10c_2^2c_3 + 3c_3^2 + 5c_2c_4 - 2c_5)$, $Y_6 = 16c_2^5 - 52c_2^3c_3 + 33c_2c_3^2 + 28c_2^2c_4 - 17c_3c_4 - 13c_2c_5 + 5c_6$, $Y_7 = -2\big[16c_2^6 - 64c_2^4c_3 - 9c_3^3 + 36c_2^3c_4 + 6c_4^2 + 9c_2^2(7c_3^2 - 2c_5) + 11c_3c_5 + c_2(-46c_3c_4 + 8c_6) - 3c_7\big]$ and $Y_i = Y_i(c_2, c_3, \cdots, c_{16})$ for $8 \le i \le 16$.

Since $f(y_n) = f(x_n)|_{e_n \to (y_n - \alpha)}$, we are led to an expression:

$$f(y_n) = f'(\alpha)[c_2e^2 - 2(c_2^2 - c_3)e^3 + (5c_2^3 - 7c_2c_3 + 3c_4)e^4 + \sum_{i=5}^{16} D_i\, e^i + O(e^{17})], \tag{16}$$

where $D_i = D_i(c_2, c_3, \cdots, c_{16})$ for $5 \le i \le 16$. Hence, we have:

$$s = \frac{f(y_n)}{f(x_n)} = c_2e + (-3c_2^2 + 2c_3)e^2 + (8c_2^3 - 10c_2c_3 + 3c_4)e^3 + \sum_{i=4}^{15} E_i\, e^i + O(e^{16}), \tag{17}$$

where $E_i = E_i(c_2, c_3, \cdots, c_{16})$ for $4 \le i \le 15$.

In the third substep of Scheme (5), $w_n = O(e^8)$ can be achieved based on Kung-Traub's conjecture. To reflect the effect on w_n from z_n in the second substep, we need to expand z_n up to eighth-order terms; hence, we carry out a sixth-order Taylor expansion of $Q_f(s)$ about 0 by noting that $s = O(e)$ and $\frac{f(y_n)}{f'(x_n)} = O(e^2)$:

$$Q_f(s) = Q_0 + Q_1s + Q_2s^2 + Q_3s^3 + Q_4s^4 + Q_5s^5 + Q_6s^6 + O(e^7), \tag{18}$$

where $Q_j = \frac{1}{j!}\frac{d^j}{ds^j}Q_f(s)$ for $0 \le j \le 6$. As a result, we come up with:

$$z_n = x_n - Q_f(s)\frac{f(y_n)}{f'(x_n)} = \alpha + (1-Q_0)e^2 + [c_2^2(4Q_0 - Q_1 - 2) - 2c_3(Q_0-1)]e^3 + \sum_{i=4}^{16} W_i\, e^i + O(e^{17}),$$

where $W_i = W_i(c_2, c_3, \cdots, c_{16}, Q_0, \cdots, Q_6)$ for $4 \le i \le 16$. Selecting $Q_0 = 1$ and $Q_1 = 2$ leads us to an expression:

$$z_n = \alpha - c_2[c_2^2(Q_2-5) + c_3]e^4 + \Sigma_{i=5}^{16} W_i\, e^i + O(e^{17}). \tag{19}$$

On account of the fact that $f(z_n) = f(x_n)|_{e_n \to (z_n - \alpha)}$, we deduce:

$$f(z_n) = f'(\alpha)[-c_2[c_2^2(Q_2-5) + c_3]e^4 + \Sigma_{i=5}^{16} F_i\, e^i + O(e^{17})], \tag{20}$$

where $F_i = F_i(c_2, c_3, \cdots, c_{16}, Q_2, \cdots, Q_6)$ for $5 \le i \le 16$. Consequently, we find:

$$u = \frac{f(z_n)}{f(y_n)} = [-c_3 - c_2^2(Q_2-5)]e^2 + [-2c_4 - 4c_2c_3(Q_2-5) + \beta_0 c_2^3]e^3 + \Sigma_{i=4}^{16} G_i\, e^i + O(e^{17}), \tag{21}$$

where $\beta_0 = 8Q_2 - Q_3 - 26$ and $G_i = G_i(c_2, c_3, \cdots, c_{16}, Q_2, \cdots, Q_6)$ for $4 \le i \le 16$.

In the last substep of Scheme (5), $x_{n+1} = O(e^{16})$ can be achieved based on Kung-Traub's conjecture. To reflect the effect on x_{n+1} from w_n in the third substep, we need to expand w_n up to sixteenth-order terms; hence, we carry out a 12th-order Taylor expansion of $K_f(s,u)$ about $(0,0)$ by noting that: $s = O(e)$, $u = O(e^2)$ and $\frac{f(z_n)}{f'(x_n)} = O(e^4)$ with $K_{ij} = 0$ satisfying $i + 2j > 12$ for all $0 \le i \le 12, 0 \le j \le 6$:

$$\begin{aligned} K_f(s,u) = {} & K_{00} + K_{10}s + K_{20}s^2 + K_{30}s^3 + K_{40}s^4 + K_{50}s^5 + K_{60}s^6 + K_{70}s^7 + K_{80}s^8 + K_{90}s^9 + K_{100}s^{10} + K_{110}s^{11} + \\ & K_{120}s^{12} + (K_{01} + K_{11}s + K_{21}s^2 + K_{31}s^3 + K_{41}s^4 + K_{51}s^5 + K_{61}s^6 + K_{71}s^7 + K_{81}s^8 + K_{91}s^9 + K_{101}s^{10})u + \\ & (K_{02} + K_{12}s + K_{22}s^2 + K_{32}s^3 + K_{42}s^4 + K_{52}s^5 + K_{62}s^6 + K_{72}s^7 + K_{82}s^8)u^2 + \\ & (K_{03} + K_{13}s + K_{23}s^2 + K_{33}s^3 + K_{43}s^4 + K_{53}s^5 + K_{63}s^6)u^3 + \\ & (K_{04} + K_{14}s + K_{24}s^2 + K_{34}s^3 + K_{44}s^4)u^4 + (K_{05} + K_{15}s + K_{25}s^2)u^5 + K_{06}u^6 + O(e^{13}). \end{aligned} \tag{22}$$

Substituting $z_n, f(x_n), f(y_n), f(z_n), f'(x_n)$, and $K_f(s,u)$ into the third substep of (5) leads us to:

$$w_n = z_n - K_f(s,u)\cdot\frac{f(z_n)}{f'(x_n)} = \alpha + (K_{00}-1)c_2[(Q_2-5)c_2^2 + c_3]e^4 + \sum_{i=5}^{16}\Gamma_i\, e^i + O(e^{17}), \tag{23}$$

where $\Gamma_i = \Gamma_i(c_2, c_3, \cdots, c_{16}, Q_2, \cdots, Q_6, K_{j\ell})$, for $5 \le i \le 16, 0 \le j \le 12$ and $0 \le \ell \le 6$. Thus $K_{00} = 1$ immediately annihilates the fourth-order term. Substituting $K_{00} = 1$ into $\Gamma_5 = 0$ and solving for K_{10}, we find:

$$K_{10} = 2. \tag{24}$$

Continuing the algebraic operations in this manner at the i-th ($6 \le i \le 7$) stage with known values of $K_{j\ell}$, we solve $\Gamma_i = 0$ for remaining $K_{j\ell}$ to find:

$$K_{20} = 1 + Q_2,\ K_{01} = 1. \tag{25}$$

Using $K_{00} = 1, K_{10} = 2, K_{20} = 1 + Q_2, K_{01} = 1$ yields

$$w_n = -c_2(\tau c_2^2 + c_3)\big[(\nu - \mu\tau + \tau^2 K_{02})c_2^4 + (2\tau K_{02} - \tau - \mu + 5)c_2^2 c_3 + (K_{02}-1)c_3^2 - c_2c_4\big], \tag{26}$$

where τ, μ, ν are described in (12). Consequently, we obtain:

$$v = \frac{f(w_n)}{f(z_n)} = -\big[\eta_0 c_2^4 + \eta_1 c_2^2 c_3 + (K_{02}-1)c_3^2 - c_2c_4\big]e^4 + \sum_{i=5}^{16} T_i\, e^i + O(e^{17}), \tag{27}$$

where η_0 and η_1 are described in (12) and $T_i = T_i(c_2, c_3, \cdots, c_{16}, Q_2, \cdots, Q_6)$ for $5 \le i \le 16$.

To compute the last substep of Scheme (5), it is necessary to have an eighth-order Taylor expansion of $J_f(s, u, v)$ about $(0, 0, 0)$ due to the fact that $\frac{f(w_n)}{f'(x_n)} = O(e^8)$. It suffices to expand J_f up to eighth-, fourth-, and second-order terms in s, u, v in order, by noting that $s = O(e),\ u = O(e^2), v = O(e^4)$ with $J_{ijk} = 0$ satisfying $i + 2j + 4k > 8$ for all $0 \le i \le 8, 0 \le j \le 4, 0 \le k \le 2$:

$$\begin{aligned} J_f(s,u,v) = {} & J_{000} + J_{100}s + J_{200}s^2 + J_{300}s^3 + J_{400}s^4 + J_{500}s^5 + J_{600}s^6 + J_{700}s^7 + J_{800}s^8 + (J_{010} + J_{110}s + J_{210}s^2 + \\ & J_{310}s^3 + J_{410}s^4 + J_{510}s^5 + J_{610}s^6)u + (J_{020} + J_{120}s + J_{220}s^2 + J_{320}s^3 + J_{420}s^4)u^2 + (J_{030} + J_{130}s + J_{230}s^2)u^3 + \\ & J_{040}u^4 + (J_{001} + J_{101}s + J_{201}s^2 + J_{301}s^3 + J_{401}s^4 + (J_{011} + J_{111}s + J_{211}s^2)u + J_{021}u^2)v + J_{002}v^2. \end{aligned} \tag{28}$$

Substituting $w_n, f(x_n), f(y_n), f(z_n), f(w_n), f'(x_n)$ and $J_f(s, u, v)$ in System (5), we arrive at:

$$x_{n+1} = w_n - J_f(s,u,v) \cdot \frac{f(w_n)}{f'(x_n)} = \alpha + \phi e^8 + \sum_{i=9}^{16} \Omega_i\, e^i + O(e^{17}), \tag{29}$$

where $\phi = (1 - J_{000})c_2(\tau c_2^2 + c_3)\left[\eta_0 c_2^4 + \eta_1 c_2^2 c_3 + (K_{02} - 1)c_3^2 - c_2 c_4\right]$, $\Omega_i = \Omega_i(c_2, c_3, \cdots, c_{16}, Q_2, \cdots, Q_6, K_{\rho\gamma}, J_{jk\ell})$, for $9 \le i \le 16, 0 \le \rho \le 12, 0 \le \gamma \le 6, 0 \le j \le 8, 0 \le k \le 4, 0 \le \ell \le 2$.

Substituting $J_{000} = 1$ into $\Omega_9 = 0$ and solving for J_{100}, we find:

$$J_{100} = 2. \tag{30}$$

Continuing the algebraic operations in the same manner at the i-th ($10 \le i \le 15$) stage with known values of $J_{jk\ell}$, we solve $\Omega_i = 0$ for remaining $J_{jk\ell}$ to find:

$$\begin{cases} J_{200} = 1 + Q_2, J_{010} = 1, J_{110} = 4, J_{300} = -4 + 2Q_2 + Q_3, J_{001} = 1, J_{020} = K_{02}, J_{210} = 1 + K_{21}, \\ J_{400} = K_{40}, J_{101} = 2, J_{120} = 2 + K_{12}, J_{310} = -4 + K_{31} + 2Q_2, J_{500} = K_{50}, J_{011} = 2, J_{201} = 1 + Q_2, \\ J_{111} = 8, J_{030} = -1 + K_{02} + K_{03}, J_{220} = 1 + K_{21} + K_{22} - Q_2, J_{410} = -3 + K_{40} + K_{41} + Q_2 - Q_4, \\ J_{301} = -4 + 2Q_2 + Q_3, J_{130} = -4 + 2K_{02} + K_{12} + K_{13}, J_{320} = -6 + 2K_{21} + K_{31} + K_{32} - 2Q_2 - Q_3, \\ J_{600} = K_{60}, J_{510} = 6 + 2K_{40} + K_{50} + K_{51} - 3Q_3 - 2Q_4 - Q_5,\ J_{700} = K_{70}. \end{cases} \tag{31}$$

Upon substituting Relation (31) into Ω_{16}, we finally obtain:

$$\Omega_{16} = -c_2(\tau c_2^2 + c_3)\left[(\nu - \mu\tau + \tau^2 K_{02})c_2^4 + (2\tau K_{02} - \tau - \mu + 5)c_2^2 c_3 + (K_{02} - 1)c_3^2 - c_2 c_4\right]\Psi \tag{32}$$

where τ, ν, μ, and Ψ as described in (12). This completes the proof. □

Special Cases of Weight Functions

Theorem 1 enables us to obtain $Q_f(s)$, $K_f(s, u)$, and $J_f(s, u, v)$ by means of Taylor polynomials:

$$\begin{cases} Q_f(s) = 1 + 2s + Q_2 s^2 + Q_3 s^3 + Q_4 s^4 + Q_5 s^5 + Q_6 s^6 + O(e^7), \\ K_f(s,u) = 1 + 2s + (1 + Q_2)s^2 + (2Q_2 + Q_3 - 4)s^3 + K_{40}s^4 + K_{50}s^5 + K_{60}s^6 + K_{70}s^7 + K_{80}s^8 \\ +K_{90}s^9 + K_{100}s^{10} + K_{110}s^{11} + K_{120}s^{12} + (1 + 4s + K_{21}s^2 + K_{31}s^3 + K_{41}s^4 + K_{51}s^5 + K_{61}s^6 \\ +K_{71}s^7 + K_{81}s^8 + K_{91}s^9 + K_{101}s^{10})u + (K_{02} + K_{12}s + K_{22}s^2 + K_{32}s^3 + K_{42}s^4 + K_{52}s^5 \\ +K_6 s^6 + K_{72}s^7 + K_{82}s^8)u^2 + (K_{03} + K_{13}s + K_{23}s^2 + K_{33}s^3 + K_{43}s^4 + K_{53}s^5 + K_{63}s^6)u^3 \\ +(K_{04} + K_{14}s + K_{24}s^2 + K_{34}s^3 + K_{44}s^4)u^4 + (K_{05} + K_{15}s + K_{25}s^2)u^5 + K_{06}u^6 + O(e^{13}), \\ J_f(s,u,v) = 1 + 2s + (1 + Q_2)s^2 + (2Q_2 + Q_3 - 4)s^3 + K_{40}s^4 + K_{50}s^5 + K_{60}s^6 + K_{70}s^7 + J_{800}s^8 \\ +(1 + 4s + (1 + K_{21})s^2 + (K_{31} + 2Q_2 - 4)s^3 + (K_{40} + K_{41} - 3 + Q_2 - Q_4)s^4 + (2K_{40} + K_{50} + K_{51} + 6 \\ -3Q_3 - 2Q_4 - Q_5)s^5 + J_{610}s^6)u + (K_{02} + (2 + K_{12})s + (K_{21} + K_{22} - Q_2 + 1)s^2 + (2K_{21} + K_{31} + K_{32} - 6 \\ -2Q_2 - Q_3)s^3 + J_{420}s^4)u^2 + (K_{02} + K_{03} - 1 + (2K_{02} + K_{12} + K_{13} - 4)s + J_{230}s^2)u^3 + J_{040}u^4 \\ +(1 + 2s + (1 + Q_2)s^2 + (2Q_2 + Q_3 - 4)s^3 + J_{401}s^4 + (2 + 8s + J_{211}s^2)u + J_{021}u^2)v + J_{002}v^2 + O(e^9), \end{cases} \tag{33}$$

where parameters Q_2–Q_6, $K_{40}, K_{50}, K_{60}, K_{70}, K_{80}, K_{90}, K_{100}, K_{110}, K_{120}, K_{21}, K_{31}, K_{41}, K_{51}, K_{61}, K_{71}, K_{81}, K_{91}, K_{101}, K_{02}, K_{12}, K_{22}, K_{32}, K_{42}, K_{52}, K_{62}, K_{72}, K_{82}, K_{03}, K_{13}, K_{23}, K_{33}, K_{43}, K_{53}, K_{63}, K_{04}, K_{14}, K_{24}, K_{34}, K_{44}, K_{05}, K_{15}, K_{25}, K_{06}$ and $J_{040}, J_{002}, J_{021}, J_{211}, J_{230}, J_{401}, J_{420}, J_{610}, J_{800}$ may be free.

Among various possible weight functions $Q_f(s)$, $K_f(s,u)$, and $J_f(s,u,v)$, we restrict the current study to simple ones employing polynomials, as well as low-order rational functions. We consider the first case with simple polynomial weight functions by setting all available free parameters to the desired values, as follows:

Case 1: Polynomial weight functions

After setting all free parameters to zero, we obtain:

$$\textbf{Case 1A}: \begin{cases} Q_f(s) = 1 + 2s, \\ K_f(s,u) = 1 + s^2 - 4s^3 + u + 2s(1+2u), \\ J_f(s,u,v) = 1 + 2s + s^2 - 4s^3 + (1 + 4s + s^2 - 4s^3 - 3s^4 + 6s^5)u \\ \quad +(2s + s^2 - 6s^3)u^2 - (1+4s)u^3 + [1 + 2s + s^2 - 4s^3 + 2(1+4s)u]v. \end{cases} \tag{34}$$

After setting $K_{21} = -1, K_{12} = -2, K_{03} = 1, K_{31} = 4, K_{13} = 6, K_{32} = 4, K_{41} = 3, K_{51} = -6$, and all other free parameters to zero, we obtain:

$$\textbf{Case 1B}: \begin{cases} Q_f(s) = 1 + 2s, \\ K_f(s,u) = 1 + 2s + s^2 - 4s^3 + (1 + 4s - s^2 + 4s^3 + 3s^4 - 6s^5)u \\ \quad +2s(2s^2 - 1)u^2 + (1+6s)u^3, \\ J_f(s,u,v) = (1-s)(1 + 3s + 4s^2)(1+v) + (1+4s)u(1+2v). \end{cases} \tag{35}$$

After setting $Q_2 = -1$ and all other free parameters to zero, we obtain:

$$\textbf{Case 1C}: \begin{cases} Q_f(s) = 1 + 2s - s^2, \\ K_f(s,u) = 1 - 6s^3 + u + 2s(1+2u), \\ J_f(s,u,v) = (1 + 2s - 6s^3)(1+v) + (1 + 4s + s^2 - 6s^3 - 4s^4 + 6s^5)u \\ \quad +2(1-s)s(1+2s)u^2 + (1+4s)u(2v - u^2). \end{cases} \tag{36}$$

After setting $Q_2 = -1$, $Q_3 = 6$ and all other free parameters to zero, we obtain:

$$\textbf{Case 1D}: \begin{cases} Q_f(s) = 1 + 2s - s^2 + 6s^3, \\ K_f(s,u) = 1 + 2s + (1+4s)u, \\ J_f(s,u,v) = (1+2s)(1+v) + (1 + 4s + s^2 - 6s^3 - 4s^4 - 12s^5)u \\ \quad +(2s + 2s^2 - 10s^3)u^2 + (1+4s)u(2v - u^2). \end{cases} \tag{37}$$

As a second case, we restrict ourselves to considering all three rational-type weight functions with real coefficients:

Case 2: Rational weight functions of Type 1

$$\begin{cases} Q_f(s) &= \frac{1}{1-2s}, \\ K_f(s,u) &= \frac{1+(2+b_1)s + a_2 s^2 + (2+2a_2+b_1+b_3)s^3 + \frac{1}{2}(a_5 - 2 - b_1 - b_5 + 2a_5 s)u}{1 + b_1 s + (a_2 - 5 - 2b_1)s^2 + b_3 s^3 + \frac{1}{2}(a_5 - 4 - b_1 - b_5 + 2b_5 s)u}, \\ J_f(s,u,v) &= \frac{1 + \sum_{i=1}^{8} q_i s^i + u \sum_{i=9}^{15} q_i s^{i-9} + u^2 \sum_{i=16}^{19} q_i s^{i-16} + u^3 (q_{20} + q_{21} s) + v[\sum_{i=22}^{26} q_i s^{i-22} + (q_{27} + q_{28} s)u]}{1 + \sum_{i=1}^{8} r_i s^i + u \sum_{i=9}^{15} r_i s^{i-9} + u^2 \sum_{i=16}^{19} r_i s^{i-16} + u^3 (r_{20} + r_{21} s) + v[\sum_{i=22}^{26} r_i s^{i-22} + (r_{27} + r_{28} s)u]}, \end{cases} \tag{38}$$

where $a_i, b_i, r_i, q_i \in \mathbb{R}$ are to be determined for optimal sixteenth-order convergence; the coefficients of Q_f and K_f are already selected to satisfy the constraints stated in Theorem 1, while the coefficients of J_f should satisfy the constraints $J_{100} = 2$ and affine relations described by Relation (31). These 25 constraints determine 25 relations among the 56 coefficients r_i, q_i, $(1 \le i \le 28)$, from which 25 out of 56 coefficients may be solved as an appropriate affine combination of the remaining 31 coefficients.

For ease of analysis, we employ some simple forms of K_f by appropriate choices of the free parameters as follows:

$$K_f(s,u) = \begin{cases} \frac{1}{1-2s-s^2-u}, & \text{if } a_2 = a_5 = b_3 = b_5 = 0, b_1 = -2, & \text{— (A)} \\ \frac{1+s^2}{1-2s-2s^3-u}, & \text{if } a_5 = b_5 = 0, a_2 = 1, b_1 = -2, b_3 = -2, & \text{— (B)} \\ \frac{1+2s-s^2}{1-6s^2-(1+2s)u}, & \text{if } a_5 = b_1 = b_3 = 0, a_2 = -1, b_5 = -2, & \text{— (C)} \\ \frac{(s-1)^2}{(1-2s)(1-2s-u)}, & \text{if } a_5 = b_3 = 0, b_1 = -4, b_5 = 2, a_2 = 1, & \text{— (D)} \\ \frac{(1+s)(1-s+2s^2)}{1-2s-u}, & \text{if } a_5 = b_5 = b_3 = 0, b_1 = -2, a_2 = 1, & \text{— (E)} \\ \frac{(1-s)(2+s+s^2)}{2-5s+(s-2)u}, & \text{if } a_5 = a_2 = b_3 = 0, b_1 = -\frac{5}{2}, b_5 = \frac{1}{2}, & \text{— (F)} \\ \frac{5-2s+s^2}{5-12s+(2s-5)u}, & \text{if } a_5 = b_3 = 0, b_1 = -\frac{12}{5}, a_2 = \frac{1}{5}, b_5 = \frac{2}{5}. & \text{— (G)} \end{cases} \tag{39}$$

To consider simple forms of J_f connected with K_f via Relation (31), we first conveniently set all 31 free parameters q_8, q_{12}–q_{15}, q_{17}–q_{21}, q_{24}–q_{28}, r_6–r_8, r_{11}, r_{13}–r_{15}, r_{17}, r_{19}–r_{21}, r_{23}–r_{26}, r_{28} to zero. Then, we get seven forms of J_f matching with seven forms, (A)–(G), of (39) in order as follows:

$$J_f(s,u,v) = \begin{cases} \frac{-2-2s+5s^2+12s^3-16s^4-8s^5-32s^6-72s^7+(4s+11s^2)u-(3+4s)v}{-2+2s+11s^2+4s^3-45s^4+12s^5+(2+6s-36s^3)u-5s^2u^2+(3u-1)v}, & \text{— (A)}, \\ \frac{-5-6s+10s^2+24s^3-30s^4+10s^5-25s^6-70s^7+(10s+30s^2)u-(7+10s)v}{-5+4s+27s^2+10s^3-93s^4+40s^5+(5+16s-94s^3)u-18s^2u^2+(7u-2)v}, & \text{— (B)}, \\ \frac{7+8s-15s^2-36s^3+33s^4+78s^5+193s^6+414s^7-(14s+29s^2)u+(10+14s)v}{7-6s-38s^2-14s^3+113s^4+54s^5+(-7-22s+106s^3)u-s^2u^2+(3-10u)v}, & \text{— (C)}, \\ \frac{-5-8s+5s^2+12s^3-61s^4-42s^5-119s^6-258s^7+(10s+25s^2)u-(6+10s)v}{-5+2s+26s^2+10s^3-95s^4+50s^5+(5+18s-82s^3)u-9s^2u^2+(6u-1)v}, & \text{— (D)}, \\ \frac{-5-8s+5s^2+12s^3-61s^4-2s^5-99s^6-370s^7+(10s+25s^2)u-(6+10s)v}{-5+2s+26s^2+10s^3-115s^4+58s^5+(5+18s-82s^3)u-9s^2u^2+(6u-1)v}, & \text{— (E)}, \\ \frac{-24-22s+65s^2+156s^3-161s^4-86s^5-302s^6-498s^7+(48s+137s^2)u-(37+48s)v}{-24+26s+133s^2+48s^3-514s^4+126s^5+(24+70s-444s^3)u-69s^2u^2+(37u-13)v}, & \text{— (F)}, \\ \frac{A_0+(1450s+4045s^2)u-(1070+1450s)v}{B_0+(725+2210s-13130s^3)u-1945s^2u^2+(1070u-345)v}, & \text{— (G)}, \end{cases} \tag{40}$$

where $A_0 = -725 - 760s + 1725s^2 + 4140s^3 - 5625s^4 - 1914s^5 - 9743s^6 - 22{,}386s^7$ and $B_0 = -725 + 690s + 3970s^2 + 1450s^3 - 15{,}775s^4 + 4986s^5$.

We denote these seven subcases described by weight functions $Q_f(s) = \frac{1}{1-2s}$, $K_f(s,u)$ in (39) and $J_f(s,u,v)$ in (40) by **Cases 2A–2G** in order. For example, **Case 2A** takes the form of:

$$\textbf{Case 2A}: \begin{cases} Q_f(s) & = \frac{1}{1-2s}, \\ K_f(s,u) & = \frac{1}{1-2s-s^2-u}, \\ J_f(s,u,v) & = \frac{-2-2s+5s^2+12s^3-16s^4-8s^5-32s^6-72s^7+(4s+11s^2)u-(3+4s)v}{-2+2s+11s^2+4s^3-45s^4+12s^5+(2+6s-36s^3)u-5s^2u^2+(3u-1)v}. \end{cases} \tag{41}$$

As the final case, with four second-order rational weight functions K_f in (39), i.e., with K_f given by (39)-(A), (39)-(C), (39)-(D), (39)-(G), we will pursue possible forms of $J_f(s,u,v)$ whose all extraneous fixed points are purely imaginary, when prototype polynomial $f(z) = z^2 - 1$ is applied. According to our previous studies [22,23] on the dynamics of root finders for nonlinear equations behind the purely imaginary extraneous fixed points, the relevant convergence behavior is improved compared to the usual root finders. This convergence advantage inspires us to investigate Case 3 below underlying the presence of purely imaginary extraneous fixed points.

Case 3: Rational weight functions of Type 2

$$\begin{cases} Q_f(s) & = \frac{1}{1-2s}, \\ K_f(s,u) & = Q_f(s) \cdot \frac{(s-1)^2}{1-2s-u+\beta s^2 u}, \beta \in \mathbb{R}, \\ J_f(s,u,v) & = K_f(s,u) \cdot \frac{1+\sum_{i=1}^{3} q_i s^i + u \sum_{i=4}^{8} q_i s^{i-4} + u^2 \sum_{i=9}^{14} q_i s^{i-9} + u^3 \sum_{i=15}^{21} q_i s^{i-15}}{\mathcal{J}(s,u)+v\cdot(\sum_{i=22}^{25} r_i s^{i-22} + u \sum_{i=26}^{30} r_i s^{i-26} + u^2 \sum_{i=31}^{36} r_i s^{i-31} + u^3 \sum_{i=37}^{43} q_i s^{i-37})}, \end{cases} \tag{42}$$

where $\mathcal{J}(s,u) = 1 + \sum_{i=1}^{3} r_i s^i + u\sum_{i=4}^{8} r_i s^{i-4} + u^2\sum_{i=9}^{14} r_i s^{i-9} + u^3\sum_{i=15}^{21} r_i s^{i-15}$ and the specific choice of parameter β for K_f and determination of the 64 coefficients q_i, r_i of J_f are described below. Relationships were sought among all free parameters of $J_f(s,u,v)$, giving us a simple governing equation for extraneous fixed points of the proposed Family of Methods (5).

To this end, we first express s and u for $f(z) = z^2 - 1$ as follows:

$$s = \frac{1}{4}(1 - \frac{1}{t}),\ u = \frac{1}{4} \cdot \frac{(t-1)^2}{(t+1)^2},\ \text{with } t = z^2. \tag{43}$$

In order to obtain a simple form of $J_f(s,u,v)$, we needed to closely inspect how it is connected with $K_f(s,u)$. In view of Relation (31), it is appropriate to select a form of $K_f(s,u)$ that reduces to a lower-order rational function in t. Such a lower-order rational weight function $K_f(s,u)$ would eventually lead us to obtaining a simplified $J_f(s,u,v)$. When applying to $f(z) = z^2 - 1$, we find $K_f(s,u)$ with $t = z^2$ as shown below:

$$K_f(s,u) = \frac{(t-1)^4[t^4(\beta+7)^2 - 4t^3(\beta^2-34\beta+37) + 2t^2(3\beta^2+34\beta-53) - 4t(\beta^2-10\beta+13) + (\beta-1)^2]}{[t^4(\beta+16) - 4t^3(\beta-32) + t^2(6\beta+80) - 4t(\beta-8) + \beta]^2}, \tag{44}$$

where β should be selected in such a way that the order of rational function $K_f(s,u)$ is minimized. For such minimization, we first conveniently let

$$K_1(t) = t^4(\beta+7)^2 - 4t^3(\beta^2 - 34\beta + 37) + 2t^2(3\beta^2 + 34\beta - 53) - 4t(\beta^2 - 10\beta + 13) + (\beta-1)^2,$$
$$K_2(t) = t^4(\beta+16) - 4t^3(\beta-32) + t^2(6\beta+80) - 4t(\beta-8) + \beta.$$

Since $K_2(1) = 256$ guarantees that K_2 does not have a factor $(t-1)^4$ for any value of β, we need to check if K_1 and K_2 have common factors for some values of β reducing the order of rational function K_f. By eliminating β between K_1 and K_2, we find $(1+3t)^4 = 0$, from which $(1+3t)^j = 0$ with some $j \in \{1,2,3,4\}$ is found to be a common factor yielding a unique $\beta = 2$ in view of the fact that $K_1(-\frac{1}{3}) = \frac{256(\beta-2)^2}{81}$, $K_2(-\frac{1}{3}) = \frac{256(\beta-2)}{81}$. Hence, with this $\beta = 2$ employed, (44) reduces to a desired simple rational weight function:

$$K_f(s,u) = \frac{4t(1+t)}{t^2+6t+1}. \tag{45}$$

Using the two selected weight functions Q_f, K_f(with $\beta = 2$), we continue to determine coefficients q_i, r_i of J_f yielding a simple governing equation for extraneous fixed points of the proposed methods when $f(z) = z^2 - 1$ is applied. As a result of tedious algebraic operations reflecting the 25 constraints (with possible rank deficiency) given by (30) and (31), we find only 21 effective relations, as follows:

$$\begin{cases} q_1 = \frac{1}{2}(-4 - q_{12} + q_{16} + r_{12} - r_{16} + r_{25} - r_{27}), q_2 = q_{12} - q_{16} - r_{12} + r_{16} - \frac{r_{25}}{2} + r_{27}, \\ q_3 = -r_{25}, q_4 = \frac{1}{2}(-2 + q_{16} - r_{16}), q_5 = -2 - q_1 - r_{27}, q_{10} = 2 + r_{10}, q_{15} = r_{15}, \\ r_1 = q_1, r_2 = q_2, r_3 = q_3, r_4 = q_4, r_5 = q_5, r_6 = q_6 - 1, r_7 = q_7 - q_1 - 2, r_8 = q_8 - \frac{r_{25}}{2}, \\ r_9 = q_9, r_{11} = 2 + q_{11} + q_{12} - r_{12} + \frac{3}{2}(r_{16} - q_{16}) - r_{25} + r_{27}, r_{22} = -1, \\ r_{23} = -q_1, r_{24} = -q_2, r_{26} = -1 - q_4. \end{cases} \tag{46}$$

The two relations $J_{300} = -4 + 2Q_2 + Q_3$ and $J_{400} = K_{40}$ equivalently represent $r_3 = q_3$, while three relations, $J_{500} = K_{50}, J_{600} = K_{60}, and J_{700} = K_{70}$ are identically satisfied and give us no valuable information at all.

In what follows, we classify various subcases based on interesting selections of the 43 remaining free parameters. For each subcase, the concrete forms of J_f are shown without displaying weight functions Q_f and K_f since they remain the same as in (42) with $\beta = 2$.

Case 3A: All 43 free parameters set to zero:

$$\begin{cases} q_6 = q_7 = q_8 = q_9 = q_{11} = q_{12} = q_{13} = q_{14} = q_{16} = q_{17} = q_{18} = q_{19} = q_{20} = q_{21} = r_{10} = \\ r_{12} = r_{13} = r_{14} = r_{15} = r_{16} = r_{17} = r_{18} = r_{19} = r_{20} = r_{21} = r_{25} = r_{27} = r_{28} = r_{29} = r_{30} = \\ r_{31} = r_{32} = r_{33} = r_{34} = r_{35} = r_{36} = r_{37} = r_{38} = r_{39} = r_{40} = r_{41} = r_{42} = r_{43} = 0. \end{cases}$$

$$J_f(s,u,v) = K_f(s,u) \cdot \frac{(1 - 2s - u + 2su^2)}{1 - 2s + (-1 - s^2)u + 2s^2u^2 + (-1 + 2s)v} \tag{47}$$

Case 3B: One free parameter set to nonzero, while 42 remaining ones set to zero:

$$\begin{cases} q_6 = 1, q_7 = 0, r_{25} = 0,\ q_8 = q_9 = q_{11} = q_{12} = q_{13} = q_{14} = q_{16} = q_{17} = q_{18} = q_{19} = q_{20} = \\ q_{21} = r_{10} = r_{12} = r_{13} = r_{14} = r_{15} = r_{16} = r_{17} = r_{18} = r_{19} = r_{20} = r_{21} = r_{27} = r_{28} = r_{29} = \\ r_{30} = r_{31} = r_{32} = r_{33} = r_{34} = r_{35} = r_{36} = r_{37} = r_{38} = r_{39} = r_{40} = r_{41} = r_{42} = r_{43} = 0. \end{cases}$$

$$J_f(s,u,v) = K_f(s,u) \cdot \frac{1 - 2s + (-1 + s^2)u + 2su^2}{1 - 2s - u + 2s^2u^2 + (-1 + 2s)v} \tag{48}$$

Case 3C: Two free parameters set to nonzero, while 41 remaining ones set to zero:

$$\begin{cases} q_6 = 1, q_7 = 0, r_{25} = 2,\ q_8 = q_9 = q_{11} = q_{12} = q_{13} = q_{14} = q_{16} = q_{17} = q_{18} = q_{19} = q_{20} = \\ q_{21} = r_{10} = r_{12} = r_{13} = r_{14} = r_{15} = r_{16} = r_{17} = r_{18} = r_{19} = r_{20} = r_{21} = r_{27} = r_{28} = r_{29} = \\ r_{30} = r_{31} = r_{32} = r_{33} = r_{34} = r_{35} = r_{36} = r_{37} = r_{38} = r_{39} = r_{40} = r_{41} = r_{42} = r_{43} = 0. \end{cases}$$

$$J_f(s,u,v) = K_f(s,u) \cdot \frac{1 - s - s^2 - 2s^3 + (-1 - s + s^2)u + 2su^2}{1 - s - s^2 - 2s^3 + (-1 - s - s^3 - s^4)u + (-1 + s + s^2 + 2s^3)v} \tag{49}$$

Case 3D: Two free parameters set to nonzero, while 41 remaining ones set to zero:

$$\begin{cases} q_6 = 1, q_7 = 0, r_{25} = 4,\ q_8 = q_9 = q_{11} = q_{12} = q_{13} = q_{14} = q_{16} = q_{17} = q_{18} = q_{19} = q_{20} = \\ q_{21} = r_{10} = r_{12} = r_{13} = r_{14} = r_{15} = r_{16} = r_{17} = r_{18} = r_{19} = r_{20} = r_{21} = r_{27} = r_{28} = r_{29} = \\ r_{30} = r_{31} = r_{32} = r_{33} = r_{34} = r_{35} = r_{36} = r_{37} = r_{38} = r_{39} = r_{40} = r_{41} = r_{42} = r_{43} = 0. \end{cases}$$

$$J_f(s,u,v) = K_f(s,u) \cdot \frac{1 - 2s^2 - 4s^3 + (-1 - 2s + s^2)u + 2su^2}{1 - 2s^2 - 4s^3 + (-1 - 2s - 2s^3 - 2s^4)u - 2s^2u^2 + (-1 + 2s^2 + 4s^3)v} \tag{50}$$

Case 3E: Seven free parameters set to zero, while 36 remaining ones set to nonzero:

$$\begin{cases} r_{15} = 0, r_{16} = 0, r_{17} = 0, r_{18} = 0, r_{19} = 0, r_{20} = 0, r_{21} = 0, q_6 = -3, q_7 = -20, q_8 = 20, \\ q_9 = 1, q_{11} = -15, q_{12} = 26, q_{13} = -1, q_{14} = -14, q_{16} = -2, q_{17} = 7, q_{18} = -4, q_{19} = -7, \\ q_{20} = 4, q_{21} = 3, r_{10} = -2, r_{12} = 8, r_{13} = 4, r_{14} = -8, r_{25} = 8, r_{27} = -4, r_{28} = 1, r_{29} = 12, \\ r_{30} = -12, r_{31} = 1, r_{32} = -6, r_{33} = 16, r_{34} = -24, r_{35} = 18, r_{36} = -4, r_{37} = -1, r_{38} = 4, \\ r_{39} = -7, r_{40} = 8, r_{41} = -2, r_{42} = -8, r_{43} = 6. \end{cases}$$

$$J_f(s,u,v) = K_f(s,u) \cdot \frac{1-6s+12s^2-8s^3+(-2+8s-3s^2-20s^3+20s^4)u+A_0(s,u)}{1-6s+12s^2-8s^3+(-2+8s-4s^2-16s^3+16s^4)u+(1-2s-4s^2+8s^3+4s^4-8s^5)u^2+A_1(s,u)v}, \tag{51}$$

where $A_0(s,u) = (1 - 15s^2 + 26s^3 - s^4 - 14s^5)u^2 + (-2s + 7s^2 - 4s^3 - 7s^4 + 4s^5 + 3s^6)u^3$ and $A_1(s,u) = -1 + 6s - 12s^2 + 8s^3 + (1 - 4s + s^2 + 12s^3 - 12s^4)u + (1 - 6s + 16s^2 - 24s^3 + 18s^4 - 4s^5)u^2 + (-1 + 4s - 7s^2 + 8s^3 - 2s^4 - 8s^5 + 6s^6)u^3$.

Remark 1. *The above Case 3E represents a method obtained with a different approach by Sharma et. al [14].*

For ease of analysis for interesting subcases of Case 3F, we first impose further constraints:

$$r_{32} = r_{33} = r_{34} = r_{35} = r_{36} = r_{37} = r_{38} = r_{39} = r_{40} = r_{41} = r_{42} = r_{43} = 0. \tag{52}$$

and then we seek parameter relationships yielding purely imaginary extraneous fixed points of the proposed family of methods when $f(z) = z^2 - 1$ is applied.

To this end, after substituting the 21 effective relations given by (46) into J_f in (42) and by applying to $f(z) = z^2 - 1$, we first find v:

$$v = \frac{(t-1)^4}{4(t^2+6t+1)^2}, \text{ with } t = z^2, \tag{53}$$

and construct governing equation $H(z) = 1 + sQ_f(s) + suK_f(s,u) + suvJ_f(s,u,v) = 0$ for extraneous fixed points:

$$H(z) = \frac{\mathcal{A} \cdot G(t)}{(1+t)(1+6t+t^2) \cdot W(t)}, \text{ with } t = z^2, \tag{54}$$

where $\mathcal{A}$ is a constant factor, $G(t) = \sum_{i=0}^{19} g_i t^i$, with $g_0 = -q_{21} + 4r_{21} + r_{43}, g_1 = (16q_{14} + 4q_{20} + 19q_{21} - 64r_{14} - 16r_{20} + 84r_{21} - 16r_{36} - 4r_{42} + 5r_{43})$, $g_i = g_i(q_1, q_6, \cdots r_{43})$, for $2 \le i \le 19$ and $W(t) = \sum_{i=0}^{16} w_i t^i$, with $w_0 = 4r_{21} + r_{43}$, $w_1 = -4(16r_{14} + 4r_{20} + 4r_{36} + r_{42} + 4r_{43})$, $w_i = w_i(q_1, q_6, \cdots r_{43})$, for $2 \le i \le 16$. The coefficients of both polynomials, $G(t)$ and $W(t)$, contain at most 43 free parameters.

We first observe that two partial expressions of $H(z)$, namely, $1 + sQ_f(s) = \frac{1+3t}{2(1+t)}$, $1 + sQ_f(s) + suK_f(s,u) = \frac{1+21t+35t^2+7t^3}{4(1+t)(1+6t+t^2)}$ hold with $t = z^2$ when $f(z) = z^2 - 1$ is applied. With such an observation of presence of factors $(1+3t), (1+t), (1+6t+t^2), (1+21t+35t^2+7t^3)$, we seek a special subcase in which $G(t)$ contains all the interested factors as follows:

$$\begin{cases} G(t) = (1+3t)(1+t)(1+6t+t^2)(1+21t+35t^2+7t^3) \cdot \Phi(t), \\ \text{where } \Phi(t) \text{ is a twelfth-degree polynomial.} \end{cases} \tag{55}$$

The degree of $\Phi(t)$ is decreased from 12 to 3 by gradually annihilating the relevant coefficients containing free parameters. Similarly, by doing so, the degree of $W(t)$ is decreased from 16 to 12. This lengthy algebraic process of factorization and annihilation eventually leads us to Case 3F whose coefficients are given below with 10 additional free parameters set to zero:

Case 3F:

$$\begin{cases} q_{11} = q_{12} = q_{13} = q_{17} = q_{18} = q_{19} = q_{20} = r_{19} = r_{20} = r_{29} = 0, \\ q_1 = \frac{-614{,}733{,}185+60{,}736\lambda}{22{,}680{,}993}, q_2 = \frac{100{,}706{,}911{,}309-994{,}851{,}864\lambda}{1{,}209{,}652{,}96}, q_3 = \frac{-283{,}900{,}855{,}559+2{,}982{,}612{,}040\lambda}{181{,}447{,}944}, \\ q_4 = \frac{11(-17{,}383{,}391{,}837+269{,}152{,}984\lambda)}{362{,}895{,}888}, q_5 = \frac{111{,}931{,}468{,}151-2{,}928{,}482{,}376\lambda}{60{,}482{,}648}, \\ q_6 = \frac{4{,}933{,}965{,}273{,}065+1{,}837{,}8410{,}208\lambda}{241{,}930{,}592}, q_7 = -\frac{4(29{,}481{,}072{,}883+418{,}276{,}453\lambda)}{22{,}680{,}993}, \\ q_8 = \frac{2{,}489{,}044{,}195{,}457-2{,}289{,}580{,}384\lambda}{22{,}680{,}993}, q_9 = \frac{-3{,}757{,}975{,}357{,}457+4{,}002{,}738{,}016\lambda}{241{,}930{,}592}, \\ q_{10} = \frac{6{,}010{,}745{,}286{,}103-6{,}261{,}215{,}648\lambda}{181{,}447{,}944}, q_{14} = \frac{32(-177{,}550{,}689{,}713+203{,}887{,}264\lambda)}{22{,}680{,}993}, \\ q_{15} = \frac{549{,}794{,}099{,}671-669{,}060{,}000\lambda}{60{,}482{,}648}, q_{16} = \frac{-989{,}650{,}155{,}895+1{,}204{,}838{,}624\lambda}{45{,}361{,}986}, \\ q_{21} = -\frac{128(-20{,}581{,}176{,}851+25{,}197{,}088\lambda)}{22{,}680{,}993}, r_1 = q_1, r_2 = q_2, r_3 = q_3, r_4 = q_4, r_5 = q_5, r_6 = q_6 - 1, \\ r_7 = q_7 - q_1 - 2, r_8 = q_8 + \frac{q_3}{2}, r_9 = q_9, r_{10} = q_{10} - 2, r_{11} = \frac{209{,}800{,}084{,}463-296{,}2626{,}376\lambda}{362{,}895{,}888}, \\ r_{12} = \frac{-43{,}339{,}5815{,}285+880{,}7862{,}232\lambda}{181{,}447{,}944}, r_{13} = \frac{102{,}682{,}7316{,}571-6{,}282{,}476{,}696\lambda}{90{,}723{,}972}, \\ r_{14} = \frac{8(-55{,}140{,}7240{,}435+666{,}226{,}016\lambda)}{22{,}680{,}993}, r_{15} = q_{15}, r_{16} = \frac{-125{,}591{,}5403{,}087+619{,}557{,}224\lambda}{60{,}482{,}648}, \\ r_{17} = \frac{-4{,}288{,}671{,}068{,}273+17{,}244{,}727{,}552\lambda}{181{,}447{,}944}, r_{18} = \frac{2{,}539{,}703{,}624{,}807-6{,}562{,}173{,}832\lambda}{45{,}361{,}986}, \\ r_{21} = \frac{32(9{,}162{,}991{,}123+11{,}896{,}288\lambda)}{22{,}680{,}993}, r_{22} = -1, r_{23} = -q_1, r_{24} = -q_2, r_{25} = -q_3, r_{26} = -1 - q_4, \\ r_{27} = -2 - q_1 - q_5, r_{28} = \frac{-94{,}292{,}600{,}453-15{,}125{,}589{,}368\lambda}{181{,}447{,}944}, r_{30} = \frac{4(26{,}628{,}387{,}546+11{,}700{,}527\lambda)}{7{,}560{,}331}, \end{cases} \tag{56}$$

where $\lambda = r_{31}$ is a free parameter to be determined for the purely imaginary extraneous fixed points. In addition, the corresponding J_f is given by:

$$J_f(s,u,v) = K_f(s,u) \cdot \frac{1+\sum_{i=1}^{3} q_i s^i + u\sum_{i=4}^{8} q_i s^{i-4} + u^2 \sum_{i=9}^{14} q_i s^{i-9} + u^3 \sum_{i=15}^{21} q_i s^{i-15}}{\mathcal{J}(s,u) + v \cdot (\sum_{i=22}^{25} r_i s^{i-22} + u \sum_{i=26}^{30} r_i s^{i-26} + \lambda u^2)}, \tag{57}$$

where $\mathcal{J}(s,u)$ is given by (42).

With the coefficients given by (56), we also find $H(z)$, $\Phi(t)$, and $W(t)$ as follows:

$$\begin{cases} H(z) = -\frac{64(1+3t)(1+21t+35t^2+7t^3)\Phi(t)}{W(t)}, \text{ with } t = z^2, \\ \Phi(t) = 9056\lambda - 2{,}787{,}643 + t(593{,}958{,}773 - 788{,}128\lambda) + t^2(1{,}798{,}696{,}455 - 394{,}464\lambda) + a_0 t^3, \\ W(t) = 9{,}162{,}991{,}123 + 11{,}896{,}288\lambda + t(2{,}205{,}628{,}961{,}740 - 2{,}664{,}904{,}064\lambda) + \cdots + a_1 t^{12}, \\ a_0 = 31(16{,}558{,}049 + 37{,}856\lambda),\ a_1 = (-46{,}242{,}701 + 120{,}544\lambda). \end{cases} \tag{58}$$

It still remains for us to select values of λ for the desired purely imaginary extraneous fixed points. To this end, we need to determine the conditions for negative roots of the cubic equation $\Phi(t) = 0$ whose discriminant should be nonnegative and coefficients should have the same sign. (See Lemma 4.1 in Reference [23]). As a result, the values of λ should satisfy the following constraints:

$$307.823\cdots < \lambda \le 594.545\cdots. \tag{59}$$

Three values of $\lambda \in \{310, \frac{46{,}242{,}701}{120{,}544} \simeq 383.617, 594\}$ are selected for sub-cases **Cases 3F1, 3F2, 3F3** in order.

As a final case, we consider subcase Case 3G, leading us to purely imaginary extraneous fixed points. To easily proceed with our investigation, we initially impose three more constraints than Case 3F as follows:

$$r_{29} = r_{30} = r_{31} = r_{32} = r_{33} = r_{34} = r_{35} = r_{36} = r_{37} = r_{38} = r_{39} = r_{40} = r_{41} = r_{42} = r_{43} = 0. \tag{60}$$

We simply let $G(t)$ have a factor $(1+3t)$ by taking the effect of $1 + sQ_f(s) = \frac{1+3t}{2(1+t)}$ into account, when $f(z) = z^2 - 1$ is applied. This case will give us the governing equation $H(z)$ on the extraneous fixed points as follows:

$$H(z) = \frac{\mathcal{A}(1+3t)\cdot\mathcal{G}(t)}{(1+t)(1+6t+t^2)\cdot\mathcal{W}(t)}, \text{ with } t = z^2, \tag{61}$$

where $\mathcal{A}$ is a constant factor; $\mathcal{G}(t)$ and $\mathcal{W}(t)$ are 18-degree and 16-degree polynomials respectively with their coefficients containing free parameters. The two polynomials $\mathcal{G}(t)$ and $\mathcal{W}(t)$ shall reduce to 9-degree polynomial $\mathcal{G}_9(t)$ and 7-degree polynomial $\mathcal{W}_7(t)$, respectively, after lowering their degrees by annihilating their relevant coefficients gradually. As a result of this process of factorization and annihilation, we obtain a set of relations among the desired coefficients with 9 additional free parameters set to zero as follows:

Case 3G:

$$\begin{cases} q_{12} = q_{13} = q_{17} = q_{18} = q_{19} = q_{20} = r_{10} = r_{19} = r_{20} = 0, \\ q_1 = \frac{-3{,}055{,}820{,}263{,}252-76{,}497{,}245\lambda}{142{,}682{,}111{,}242}, q_2 = \frac{56{,}884{,}034{,}112{,}404+44{,}614{,}515{,}451\lambda}{285{,}364{,}222{,}484}, \\ q_3 = \frac{-45{,}802{,}209{,}949{,}332-44{,}308{,}526{,}471\lambda}{142{,}682{,}111{,}242}, q_4 = -\frac{3(17{,}778{,}426{,}888{,}128+67{,}929{,}066{,}997\lambda)}{1426821112420}, \\ q_5 = \frac{2(21{,}034{,}820{,}227{,}211+132{,}665{,}343{,}294\lambda)}{356{,}705{,}278{,}105}, q_6 = \frac{-1{,}589{,}080{,}655{,}012{,}451+134{,}087{,}681{,}464\lambda}{142{,}682{,}111{,}242}, \\ q_7 = \frac{2(-780{,}300{,}304{,}419{,}180+71{,}852{,}971{,}399\lambda)}{71341055621}, q_8 = \frac{12{,}288(-727{,}219{,}117{,}761+128{,}167{,}952\lambda)}{71341055621}, \\ q_9 = \frac{1{,}353{,}974{,}063{,}793{,}787-212{,}746{,}858{,}830\lambda}{142{,}682{,}111{,}242}, q_{10} = 2, q_{11} = \frac{2(-741{,}727{,}036{,}224{,}277+126{,}275{,}739{,}062\lambda)}{71{,}341{,}055{,}621}, \\ q_{14} = -\frac{8192(-3{,}964{,}538{,}065{,}856+615{,}849{,}113\lambda)}{71{,}341{,}055{,}621}, q_{15} = \frac{8(-226{,}231{,}159{,}891{,}830+34{,}083{,}208{,}621\lambda)}{71341055621}, \\ q_{16} = -\frac{24(-908{,}116{,}719{,}056{,}544+136{,}634{,}733{,}499\lambda)}{356705278105}, q_{21} = \frac{131{,}072(-918{,}470{,}889{,}768+13{,}6352{,}293\lambda)}{356{,}705{,}278{,}105}, \\ r_1 = q_1, r_2 = q_2, r_3 = q_3, r_4 = q_4, r_5 = q_5, r_6 = q_6 - 1, r_7 = q_7 - q_1 - 2, r_8 = q_8 + \frac{q_3}{2}, \\ r_9 = q_9, r_{11} = \frac{-29{,}558{,}910{,}226{,}378{,}916+5{,}256{,}346{,}708{,}371\lambda}{1{,}426{,}821{,}112{,}420}, r_{12} = \frac{-55{,}018{,}830{,}261{,}476-109{,}759{,}858{,}153\lambda}{142{,}682{,}111{,}242}, \\ r_{13} = \frac{25(-75{,}694{,}849{,}962{,}572+11{,}301{,}475{,}999\lambda)}{71{,}341{,}055{,}621}, r_{14} = -\frac{4096(-1{,}500{,}792{,}372{,}416+228{,}734{,}011\lambda)}{15{,}508{,}925{,}135}, \\ r_{15} = q_{15}, r_{16} = \frac{43{,}641{,}510{,}974{,}266{,}076-6{,}354{,}680{,}006{,}961\lambda}{713{,}410{,}556{,}210}, r_{17} = -\frac{2(-1{,}060{,}205{,}894{,}022{,}116+202{,}907{,}726{,}307\lambda)}{71{,}341{,}055{,}621}, \\ r_{18} = \frac{2(-2{,}870{,}055{,}173{,}156{,}756+475{,}573{,}395{,}275\lambda)}{71{,}341{,}055{,}621}, r_{21} = \frac{q_{21}}{2}, r_{22} = -1, r_{23} = -q_1, r_{24} = -q_2, \\ r_{25} = -q_3, r_{26} = -1 - q_4, r_{27} = -2 - q_1 - q_5, \end{cases} \tag{62}$$

where $\lambda = r_{28}$. In addition, the corresponding J_f is given by:

$$J_f(s,u,v) = K_f(s,u) \cdot \frac{1+\sum_{i=1}^{3} q_i s^i + u\sum_{i=4}^{8} q_i s^{i-4} + u^2 \sum_{i=9}^{14} q_i s^{i-9} + u^3 \sum_{i=15}^{21} q_i s^{i-15}}{\mathcal{J}(s,u)+v\cdot[\sum_{i=22}^{25} r_i s^{i-22} + u(r_{26}+r_{27}s+\lambda s^2)]}, \quad (63)$$

where $\mathcal{J}(s,u)$ is given by (42).

With the coefficients given by (62), we also find $H(z)$, $\mathcal{G}_9(t)$, $\mathcal{W}_7(t)$ as follows:

$$\begin{cases} H(z) = \frac{1}{8}\frac{(1+3t)\cdot\mathcal{G}_9(t)}{(1+t)(1+6t+t^2)\cdot\mathcal{W}_7(t)}, \text{ with } t = z^2, \\ \mathcal{G}_9(t) = (1+10t+5t^2)(1+92t+134t^2+28t^3+t^4)\cdot\gamma(t;\lambda), \\ \mathcal{W}_7(t) = (1+28t+70t^2+28t^3+t^4)\cdot\gamma(t;\lambda). \end{cases} \quad (64)$$

where $\gamma(t;\lambda) = 918{,}470{,}889{,}768 - 136{,}352{,}293\lambda + t(8{,}801{,}039{,}652{,}064 - 1{,}443{,}018{,}049\lambda) + t^2(9{,}126{,}540{,}551{,}048 - 2{,}824{,}686{,}623\lambda) + 55t^3(-1{,}172{,}805{,}939{,}824 + 80{,}073{,}763\lambda)$.

In Proposition 1 of Section 4, it is shown that $\mathcal{G}_9(t)$ and $\mathcal{W}_7(t)$ have such factorizations as well as a common factor $\gamma(t;\lambda)$. Consequently, after cancelling out the common factor, $H(z)$ reduces to:

$$H(z) = \frac{1}{8}\frac{(1+3t)(1+10t+5t^2)(1+92t+134t^2+28t^3+t^4)}{(1+t)(1+6t+t^2)(1+28t+70t^2+28t^3+t^4)}, \text{ with } t = z^2, \quad (65)$$

Holding true regardless of λ. Hence, it is regretfully infeasible to directly use $\gamma(t;\lambda)$ for obtaining the desired purely imaginary extraneous fixed points with a possible λ. Nevertheless, it is interesting to observe that the roots of the right-hand side of (65) are all negative, namely, $\{-\frac{1}{3}, -1.89443, -0.105573, -22.1335, -5.04468, -0.810727, and -0.0110469\}$, which leads us to all the desired purely imaginary extraneous fixed points. It is also surprising to note that this $H(z)$ is identical with that of (4), being derived from different forms of weight function J_f.

Nine values of $\lambda \in \{0, -\frac{3{,}055{,}820{,}263{,}252}{76{,}497{,}245}, -\frac{56{,}884{,}034{,}112{,}404}{44{,}614{,}515{,}451}, -\frac{45{,}802{,}209{,}949{,}332}{44{,}308{,}526{,}471}, -\frac{17{,}778{,}426{,}888{,}128}{67{,}929{,}066{,}997}, -\frac{21{,}034{,}820{,}227{,}211}{132{,}665{,}343{,}294}, \frac{1{,}353{,}974{,}063{,}793{,}787}{212{,}746{,}858{,}830}, \frac{226{,}231{,}159{,}891{,}830}{34{,}083{,}208{,}621}, \frac{918{,}470{,}889{,}768}{136{,}352{,}293}\}$ are selected for subcases 3G1, 3G2, 3G3, 3G4, 3G5, 3G6, 3G7, 3G8, 3G9 in order. These subcases further simplify J_f with integer coefficients q_i, r_i of J_f with $r_{28} = 0, (q_1 = r_1 = r_{23} = 0), (q_2 = r_2 = r_{24} = 0), (q_3 = r_3 = r_{25} = 0), (q_4 = r_4 = 0, r_{26} = -1), (q_5 = r_5 = 0), (q_9 = r_9 = 0), (q_{15} = r_{15} = 0), (q_{21} = r_{21} = 0)$ in order.

In the next section, we investigate how to select appropriate free parameters giving purely imaginary extraneous fixed points.

3. Extraneous Fixed Points and Their Dynamics

The dynamics behind the extraneous fixed points [20] of iterative map (5) have been investigated by many authors with the aid of the relevant basins of attraction. Such dynamics were studied by Stewart [24], Amat et al., e.g., References [25,26], Andreu et al. [27], Argyros-Magreñan [28], Chun et al. [29], Chicharro et al. [30], Chun-Neta [31], Cordero et al. [32], Geum et al. [22,33–35], Rhee at al. [23], Magreñan [36,37], Neta et al. [38–40], and Scott et al. [41].

To find a root α of $f(x)$ under consideration, we usually locate a fixed point ξ of the iterative map:

$$x_{n+1} = R_f(x_n), n = 0, 1, \cdots, \quad (66)$$

where R_f is the iteration function associated with f. Usually, R_f is expressed with weight function H_f in the form: $R_f(x_n) = x_n - \frac{f(x_n)}{f'(x_n)}H_f(x_n)$. Thus the zeros of H_f are other fixed points $\xi \neq \alpha$ called extraneous fixed points of R_f. The presence of extraneous fixed points may induce attractive, indifferent, or repulsive, and other periodic or chaotic orbits influencing the underlying dynamics of R_f. The dynamics behind the extraneous fixed points motivates the current analysis. To make our analysis more feasible, we rewrite Iterative Map (66) in a more specific form:

$$x_{n+1} = R_f(x_n) = x_n - \frac{f(x_n)}{f'(x_n)} H_f(x_n), \tag{67}$$

where $H_f(x_n) = 1 + sQ_f(s) + suK_f(s,u) + suvJ_f(s,u,v)$ plays a role of a weight function in the classical Newton's method. It is clear that α is a fixed point of R_f. Points $\xi \neq \alpha$ for which $H_f(\xi) = 0$ are extraneous fixed points of R_f.

The influence of extraneous fixed points on the convergence behavior of the iterative dynamical system was extensively demonstrated for simple zeros via König functions and Schröder functions [20] with applications to a family of functions $\{f_k(x) = x^k - 1, k \geq 2\}$.

For ease of dynamics behind the extraneous fixed points of Iterative Maps (67), we select a simple member $f(z) = (z^2 - 1)$. By a similar approach made by Chun et al. [42,43] and Neta et al. [38,40,44], we are able to construct $H_f(x_n) = s \cdot Q_f(s) + s \cdot u \cdot K_f(s,u) + s \cdot u \cdot v \cdot J_f(s,u,v)$ in (67). Applying $f(z) = (z^2 - 1)$ to H_f, we construct a rational function $H(z)$ with $t = z^2$ in the form:

$$H(z) = \frac{\mathcal{N}(t)}{\mathcal{D}(t)}, \tag{68}$$

where both $\mathcal{D}(t)$ and $\mathcal{N}(t)$ are coprime polynomial functions of t. The underlying dynamics of Iterative Map (67) can be favorably investigated on the Riemann sphere [45] with possible fixed points "0(zero)" and "∞". As can be seen in Section 5, the relevant dynamics will be illustrated in a 6×6 square region centered at the origin.

Indeed, the roots t of $\mathcal{N}(t)$ provide the extraneous fixed points ξ of R_f in Map (67) by the relation:

$$\xi = \begin{cases} t^{\frac{1}{2}}, & \text{if } t \neq 0, \\ 0(\text{double root}), & \text{if } t = 0. \end{cases} \tag{69}$$

Extraneous Fixed Points and their Stability

Among a number of case studies with $f(z) = z^2 - 1$ in the preceding section, we list in Table 1 the resulting extraneous fixed points for selected ones. The dynamics of the methods highlighted in yellow in this table is investigated in more detail in Section 5. In Proposition 1, regarding Case 3G, we first show that the relevant governing equation $H(z)$ in (64) has a common factor $\gamma(t;\lambda)$ in $\mathcal{G}_9(t)$ and $\mathcal{W}_7(t)$.

Proposition 1. *By the same technique of lowering the degrees as used in Case 3G, we let $\mathcal{G}(t)$ and $\mathcal{W}(t)$ in (64) reduce to, respectively,* 11- *and* 7-*degree polynomials:*

$$\begin{aligned}
&\mathcal{G}_{11}(t) = -30{,}232{,}602{,}7148{,}844 - 2{,}645{,}460{,}523{,}447\lambda - 3{,}549{,}604{,}642{,}962\omega + \\
&t(-19{,}045{,}217{,}693{,}092{,}380 - 184{,}372{,}611{,}780{,}771\lambda - 246{,}980{,}050{,}245{,}050\omega) + \\
&t^2(-229{,}583{,}218{,}935{,}759{,}412 - 2{,}478{,}905{,}521{,}740{,}161\lambda - 3{,}314{,}558{,}301{,}486{,}462\omega) + \\
&t^3(-1{,}057{,}943{,}643{,}253{,}576{,}164 - 12{,}706{,}921{,}698{,}909{,}725\lambda - 16{,}952{,}179{,}986{,}855{,}494\omega) + \\
&t^4(-572{,}802{,}383{,}853{,}145016 - 12{,}399{,}265{,}091{,}556{,}262\lambda - 16, 312, 777, 192, 063, 668\omega) + \\
&t^5(4{,}980{,}100{,}685{,}582{,}204{,}520 + 54{,}259{,}611{,}564{,}699{,}906\lambda + 73{,}129{,}514{,}966{,}886{,}908\omega) + \\
&t^6(2{,}227{,}938{,}789{,}630{,}354{,}968 + 59{,}080{,}193{,}903{,}387{,}422\lambda + 79{,}126{,}227{,}393{,}874{,}628\omega) + \\
&t^7(-8{,}090{,}329{,}150{,}129{,}283{,}784 - 37{,}514{,}311{,}315{,}868{,}154\lambda - 50{,}730{,}685{,}869{,}384{,}460\omega) + \\
&t^8(-5{,}993{,}458{,}057{,}271{,}207{,}260 - 41{,}435{,}500{,}104{,}681{,}395\lambda - 55{,}790{,}114{,}824{,}122{,}010\omega) + \\
&t^9(-89{,}1021{,}737{,}210{,}685{,}324 - 6{,}595{,}611{,}949{,}775{,}471\lambda - 8, 874, 933, 288, 600, 610\omega) - \\
&43t^{10}(70{,}287{,}000705{,}004 + 530{,}278{,}886{,}951\lambda + 713{,}410{,}556{,}210\omega) + \\
&t^{11}(70{,}287{,}000{,}705{,}004 + 530{,}278{,}886{,}951\lambda + 713{,}410{,}556{,}210\omega),
\end{aligned}$$

$$\begin{aligned}
\bar{W}_7(t) &= 14,502,439,152,740 + 132,198,852,281\lambda + 177,262,130,422\omega + \\
& t(306,041,922,910,212 + 3,165,697,676,185\lambda + 4,236,146,358,902\omega) + \\
& t^2(1,965,140,873,452,948 + 22,777,852,783,125\lambda + 30,415,220,584,670\omega) + \\
& t^3(2,689,571,152,941,748 + 41,132,482,603,381\lambda + 54,558,749,294,302\omega) + \\
& t^4(-9,677,123,180,363,412 - 95,019,090,176,101\lambda - 128,332,636,641,790\omega) + \\
& t^5(-10,883,768,984,025,268 - 186,269,794,428,805\lambda - 249,609,933,175,038\omega) + \\
& t^6(22,205,775,217,829,788 + 123,422,809,056,087\lambda + 166,569,579,351,722\omega) + \\
& t^7(12,226,341,638,552,316 + 9,065,7843,633,847\lambda + 121,985,612,096,810\omega),
\end{aligned}$$

with $\omega = r_{27}$ and $\lambda = r_{28}$. Then the following will hold:

(a) *$\mathcal{G}_{11}(t)$ and $\bar{W}_7(t)$ have a common factor $\gamma(t;\lambda)$, if relation $\omega = \frac{-702,87,000,705,004-530,278,886,951\lambda}{713,410,556,210}$ holds.*
(b) *With the use of the above ω, the two polynomials $\mathcal{G}_{11}(t)$ and $\bar{W}_7(t)$ respectively reduce to $\mathcal{G}_9(t)$ and $\mathcal{W}_7(t)$.*

Proof. (a) Let $\mathcal{G}_{11}(t) = 0$ and $\bar{W}_7(t) = 0$ for some extraneous fixed points t with some values of ω. Solving for ω from $\bar{W}_7(t) = 0$ in terms of t and substituting into $\mathcal{G}_{11}(t) = 0$, we find the relation after simplification:

$$\begin{aligned}
&918{,}470{,}889{,}768 - 136{,}352{,}293\lambda + t(8{,}801{,}039{,}652{,}064 - 1{,}443{,}018{,}049\lambda) + \\
&\quad t^2(912{,}654{,}0551{,}048 - 2{,}824{,}686{,}623\lambda) + 55t^3(-1{,}172{,}805{,}939{,}824 + 80{,}073{,}763\lambda) = 0,
\end{aligned}$$

whose left-hand side represents $\gamma(t;\lambda)$. Since $\gamma(t;\lambda)$ divides $\mathcal{G}_{11}(t)$ and $\bar{W}_7(t)$ simultaneously, we must have $\omega = \frac{-70,287,000,705,004-530,278,886,951\lambda}{713,410,556,210}$. This value of ω indeed annihilates the coefficients of tenth- and eleventh-degree terms of $\mathcal{G}_{11}(t)$ reducing to $\mathcal{G}_9(t)$ and makes $\bar{W}_7(t)$ become $\mathcal{W}_7(t)$. Note also that the resulting $\mathcal{G}_9(t)$ and $\mathcal{W}_7(t)$ have more factors $(1 + 10t + 5t^2)(1 + 92t + 134t^2 + 28t^3 + t^4)$ and $(1 + 28t + 70t^2 + 28t^3 + t^4)$, respectively. □

Proposition 2. *Let $f(z) = z^2 - 1$. Then the extraneous fixed points ξ for Cases 1–3 discussed in Section 3 are all found to be indifferent.*

Proof. All subcases of Cases 1 and 2 have the same procedure for their stability proofs. Hence, it suffices to show the relevant proof for typical Subcases 1A and 2A.

(a) **Case 1A:** $H_f(z) = \frac{\mathcal{N}_1}{144115188075855872\ t^{42}}$,
where $\mathcal{N}_1 = 1 - 79t + 3138t^2 - 83{,}096t^3 + 1{,}643{,}075t^4 - 25{,}782{,}937t^5 + 333{,}401{,}042t^6 - 3{,}644{,}000{,}008t^7 + 34{,}277{,}812{,}964t^8 - 281{,}244{,}319{,}820t^9 + 2{,}033{,}394{,}013{,}704t^{10} - 13{,}058{,}032{,}832{,}736t^{11} + 74{,}952{,}295{,}448{,}348t^{12} - 386{,}486{,}800{,}674{,}676t^{13} + 1{,}797{,}643{,}814{,}328{,}584t^{14} - 7{,}567{,}394{,}816{,}098{,}464t^{15} + 28{,}911{,}173{,}797{,}459{,}454t^{16} - 100{,}477{,}288{,}658{,}961{,}282t^{17} + 318{,}278{,}083{,}152{,}645{,}340t^{18} - 920{,}489{,}401{,}136{,}647{,}760t^{19} + 2{,}434{,}168{,}954{,}635{,}747{,}562t^{20} - 5{,}893{,}562{,}472{,}356{,}381{,}374t^{21} + 13{,}080{,}156{,}529{,}292{,}380{,}700t^{22} - 26{,}638{,}016{,}093{,}569{,}726{,}832t^{23} + 4{,}982{,}024{,}5302{,}963{,}260{,}564t^{24} - 85{,}618{,}609{,}075{,}224{,}925{,}164t^{25} + 135{,}228{,}559{,}944{,}098{,}725{,}576t^{26} - 196{,}231{,}613{,}423{,}323{,}919{,}456t^{27} + 261{,}372{,}115{,}317{,}868{,}080{,}012t^{28} - 319{,}007{,}941{,}784{,}285{,}577{,}732t^{29} + 355{,}856{,}755{,}223{,}008{,}219{,}368t^{30} - 36{,}150{,}275{,}3628{,}693{,}456{,}160t^{31} + 332{,}819{,}155{,}551{,}846{,}207{,}241t^{32} - 275{,}945{,}259{,}967{,}246{,}039{,}591t^{33} + 204{,}389{,}167{,}991{,}420{,}430{,}066t^{34} - 133{,}893{,}883{,}120{,}492{,}105{,}688t^{35} + 76{,}642{,}870{,}050{,}787{,}847{,}979t^{36} - 37{,}798{,}416{,}021{,}193{,}245{,}233t^{37} + 15{,}817{,}272{,}569{,}020{,}366{,}882t^{38} - 5{,}542{,}723{,}320{,}845{,}857{,}224t^{39} + 163{,}232{,}333{,}182{,}7226{,}240t^{40} - 446{,}689{,}637{,}549{,}264{,}080t^{41} + 25{,}066{,}2959{,}380{,}507{,}424t^{42}$, with $t = z^2$.

By direct computation of $R'_f(z)$ with $f(z) = z^2 - 1$, we write it as with $t = z^2$:

$$R'_f(z) = 1 - \frac{(1+t)\mathcal{N}_1}{288{,}230{,}376{,}151{,}711{,}744t^{43}},$$

from which we find $R'_f(\xi) = 1$ due to the fact that $\mathcal{N}_1 = 0$ at an extraneous fixed point ξ.

(b) **Case 2A:** $H_f(z) = \frac{\mathcal{N}_2}{128t^3(1+t)(-1+8t+22t^2+32t^3+3t^4)^2\cdot\mathcal{D}_2}$,
where $\mathcal{N}_2 = 9 - 331t + 4574t^2 - 28{,}818t^3 + 79{,}831t^4 - 132{,}541t^5 + 642{,}364t^6 + 2{,}086{,}716t^7 - 28{,}562{,}743t^8 - 124{,}659{,}835t^9 + 327{,}234{,}066t^{10} + 3{,}643{,}707{,}810t^{11} + 10{,}401{,}729{,}279t^{12} + 5{,}434{,}132{,}091t^{13} - 46{,}499{,}708{,}408t^{14} - 124{,}126{,}445{,}432t^{15} + 27{,}709{,}201{,}187t^{16} + 960{,}201{,}127{,}911t^{17} + 3{,}094{,}219{,}885{,}346t^{18} + 5{,}954{,}358{,}575{,}826t^{19} + 8{,}009{,}701{,}202{,}029t^{20} + 7{,}837{,}767{,}988{,}593t^{21} + 5{,}561{,}044{,}852{,}220t^{22} + 2{,}771{,}393{,}311{,}100t^{23} + 911{,}814{,}978{,}699t^{24} + 183{,}378{,}733{,}407t^{25} + 22{,}088{,}672{,}174t^{26} + 1{,}598{,}681{,}822t^{27} + 63{,}656{,}509t^{28} + 1{,}143{,}377t^{29}$, and $\mathcal{D}_2 = 3 - 6t - 644t^2 + 2001t^3 + 17{,}642t^4 + 37{,}027t^5 - 45{,}424t^6 - 308{,}455t^7 - 287{,}756t^8 + 1{,}273{,}543t^9 + 5{,}022{,}636t^{10} + 9{,}037{,}291t^{11} + 9{,}769{,}070t^{12} + 6{,}466{,}449t^{13} + 2{,}240{,}392t^{14} + 309{,}803t^{15} + 20{,}337t^{16} + 523t^{17}$.

By direct computation of $R'_f(z)$ with $f(z) = z^2 - 1$, we write it as:

$$R'_f(z) = 1 - \frac{\mathcal{N}_2}{256t^4(-1+8t+22t^2+32t^3+3t^4)^2\cdot\mathcal{D}_2},$$

where $\mathcal{D}_2 = 3 - 6t - 644t^2 + 2001t^3 + 17{,}642t^4 + 37{,}027t^5 - 45{,}424t^6 - 308{,}455t^7 - 287{,}756t^8 + 1{,}273{,}543t^9 + 5{,}022{,}636t^{10} + 9{,}037{,}291t^{11} + 9{,}769{,}070t^{12} + 6{,}466{,}449t^{13} + 2{,}240{,}392t^{14} + 309{,}803t^{15} + 20{,}337t^{16} + 523t^{17}$. Consequently, we find $R'_f(\xi) = 1$ due to the fact that $\mathcal{N}_2 = 0$ at an extraneous fixed point ξ.

(c) **Case 3:** $H_f(z) = \frac{\mathcal{A}\cdot G(t)}{(1+t)(1+6t+t^2)\cdot W(t)}$, with $t = z^2$,

In this case, $H_f(z)$ is given by (54). By direct computation of $R'_f(z)$ with $f(z) = z^2 - 1$, we write it as:

$$R'_f(z) = \frac{(z^2-1)\cdot\Psi_n(z)}{8z^2(z^4+6z^2+1)\cdot\Psi_d(z)}, \tag{70}$$

where $\Psi_n(z)$ and $\Psi_d(z)$ are, respectively, 36- and 32-degree polynomials (being too lengthy to be listed here) with their coefficients containing at most 44 free parameters. With the help of Mathematica, we find the concise relation, with $t = z^2$, below:

$$(z^2-1)\cdot\Psi_n(z) - 8z^2(z^4+6z^2+1)\cdot\Psi_d(z) = -G(t),$$

where $G(t)$ is stated in Case 3, investigated in Section 3. Consequently, $G(t) = 0$ holds at an extraneous fixed point ξ. Hence, the right side of the above equation vanishes, which yields $R'_f(\xi) = 1$ for any extraneous fixed point ξ.

□

Remark 2. *Although not described here in detail due to limited space, by means of a similar proof as shown in Proposition 2, extraneous fixed points ξ for methods KT16, MBM (with $G(u,s) = \frac{2\beta+u(2\beta+2(\beta^2-2\beta-4)s-5)-(4\beta+1)S^2+2(\beta^2-4\beta+1)s-5}{2\beta+2(\beta^2-6\beta+6)s-5}$) and SAK (Case 3E) were also all found to be indifferent.*

If $f(z) = p(z)$ is a generic polynomial other than $z^2 - 1$, then required dynamical analysis may not be feasible due to the increased algebraic complexity. Despite that, we explore such dynamics by means of Iterative Map (67) applied to $f(z) = p(z)$, which is denoted by R_p as follows:

$$z_{n+1} = R_p(z_n) = z_n - \frac{p(z_n)}{p'(z_n)}H_p(z_n). \tag{71}$$

Basins of attraction for various polynomials are illustrated in Section 5 to observe the complicated dynamics behind the fixed points or the extraneous fixed points. In order to better describe the numerical and dynamical aspects of Iterative Maps (67), the letter W was conveniently prefixed to each case number to designate the relevant map in Table 1.

Table 1. Indifferent extraneous fixed points ξ for selected cases.

Case	ξ	No. of ξ
1A	$(\pm 0.716774 \pm 1.463148i, \pm 0.704031 \pm 0.887970i, \pm 0.509303 \pm 0.680849i, \pm 0.552293 \pm 0.680029i, \pm 0.644300 \pm 0.566138i, \pm 0.397046 \pm 0.197178i, \pm 0.397224 \pm 0.197072i, \pm 0.400017 \pm 0.199994i, \pm 0.413388 \pm 0.196956i, \pm 0.505408 \pm 0.345309i, \pm 0.426144 \pm 0.191947i, \pm 0.428139 \pm 0.194873i, \pm 0.445334 \pm 0.199428i, \pm 0.457883 \pm 0.195982i, \pm 0.535617 \pm 0.336251i, \pm 0.617710 \pm 0.411357i, \pm 0.512710 \pm 0.079271i, \pm 0.578694 \pm 0.261228i, \pm 0.538024 \pm 0.076528i, \pm 0.573985 \pm 0.147071i, \pm 0.576737 \pm 0.039167i)$	84
1B	$(\pm 0.646260 \pm 1.558534i, \pm 0.573863 \pm 1.105870i, \pm 0.471892 \pm 0.956889i, \pm 0.463707 \pm 0.936790i, \pm 0.623620 \pm 0.933466i, \pm 0.728444 \pm 0.733722i, \pm 0.596963 \pm 0.602781i, \pm 0.610195 \pm 0.612559i, \pm 0.367176 \pm 0.200525i, \pm 0.367171 \pm 0.200504i, \pm 0.367206 \pm 0.200525i, \pm 0.367203 \pm 0.200503i, \pm 0.403542 \pm 0.210550i, \pm 0.404007 \pm 0.210413i, \pm 0.398037 \pm 0.197796i, \pm 0.398110 \pm 0.197758i, \pm 0.400017 \pm 0.199994i, \pm 0.404260 \pm 0.205204i, \pm 0.404714 \pm 0.206052i, \pm 0.415410 \pm 0.198555i, \pm 0.422913 \pm 0.197803i, \pm 0.424091 \pm 0.196011i, \pm 0.424365 \pm 0.192891i, \pm 0.425095 \pm 0.193221i, \pm 0.572137 \pm 0.360397i, \pm 0.580912 \pm 0.361119i, \pm 0.709477 \pm 0.522949i, \pm 0.532515 \pm 0.178828i, \pm 0.538313 \pm 0.180727i, \pm 0.677061 \pm 0.398466i, \pm 0.553714 \pm 0.055820i, \pm 0.558693 \pm 0.057370i, \pm 0.638428 \pm 0.291676i, \pm 0.610031 \pm 0.211182i, \pm 0.595402 \pm 0.144287i, \pm 0.598418 \pm 0.089490i, \pm 0.604384 \pm 0.028030i)$	148
1C	$(\pm 0.713491 \pm 1.415954i, \pm 0.707173 \pm 0.851056i, \pm 0.519235 \pm 0.642663i, \pm 0.364297i, \pm 0.000022 \pm 0.363832i, \pm 0.363375i, \pm 0.352066i, \pm 0.005456 \pm 0.331225i, \pm 0.005445 \pm 0.331187i, \pm 0.315964i, \pm 0.310239i, \pm 1.315080 \times 10^{-6} \pm 0.310157i, \pm 0.310076i, \pm 0.562171 \pm 0.636866i, \pm 0.652814 \pm 0.539061i, \pm 0.429965 \pm 0.182919i, \pm 0.430103 \pm 0.182740i, \pm 0.433599 \pm 0.185209i, \pm 0.443643 \pm 0.185271i, \pm 0.443643 \pm 0.185271i, \pm 0.453605 \pm 0.182740i, \pm 0.455896 \pm 0.185816i, \pm 0.526768 \pm 0.321862i, \pm 0.471346 \pm 0.191479i, \pm 0.484057 \pm 0.190171i, \pm 0.559869 \pm 0.307515i, \pm 0.632768 \pm 0.389804i, \pm 0.539733 \pm 0.074353i, \pm 0.595236 \pm 0.244838i, \pm 0.568623 \pm 0.069543i, \pm 0.595878 \pm 0.138896i, \pm 0.596352 \pm 0.035885i)$	116
1D	$(\pm 0.693506 \pm 1.481750i, \pm 0.646018 \pm 0.909001i, \pm 0.432738 \pm 0.750275i, \pm 0.466393 \pm 0.746826i, \pm 0.490498 \pm 0.620922i, \pm 0.401531 \pm 0.531569i, \pm 0.396684 \pm 0.527092i, \pm 0.449030 \pm 0.560230i, \pm 0.354408 \pm 0.437112i, \pm 0.354619 \pm 0.436426i, \pm 0.350402 \pm 0.431032i, \pm 0.348367 \pm 0.427902i, \pm 0.343316 \pm 0.421996i, \pm 0.344575 \pm 0.422127i, \pm 0.347625 \pm 0.424013i, \pm 0.348758 \pm 0.424037i, \pm 0.447477 \pm 0.506034i, \pm 0.476627 \pm 0.483846i, \pm 0.490204 \pm 0.414270i, \pm 0.487326 \pm 0.409475i, \pm 0.596892 \pm 0.434853i, \pm 0.486449 \pm 0.143965i, \pm 0.486601 \pm 0.143625i, \pm 0.490877 \pm 0.143958i, \pm 0.492128 \pm 0.147922i, \pm 0.493560 \pm 0.142372i, \pm 0.495266 \pm 0.146616i, \pm 0.504122 \pm 0.149725i, \pm 0.506887 \pm 0.149278i, \pm 0.562405 \pm 0.247162i, \pm 0.581085 \pm 0.236682i, \pm 0.635515 \pm 0.314572i, \pm 0.582651 \pm 0.064339i, \pm 0.632191 \pm 0.202663i, \pm 0.612019 \pm 0.051508i, \pm 0.635905 \pm 0.119464i, \pm 0.628650 \pm 0.033021i)$	148
2A	$(\pm 1.215819 \pm 4.035890i, \pm 1.102438 \pm 2.967150i, \pm 2.241089i, \pm 1.220542i, \pm 0.261095 \pm 0.917722i, \pm 0.697374i, \pm 0.331560 \pm 0.730142i, \pm 0.319535 \pm 0.723699i, \pm 0.229878 \pm 0.683312i, \pm 0.451009 \pm 0.727253i, \pm 0.383724 \pm 0.589629i, \pm 0.298813 \pm 0.344439i, \pm 0.309931 \pm 0.000351i, \pm 0.315612, \pm 0.346876 \pm 0.141468i, \pm 0.373582, \pm 0.589697 \pm 0.120450i)$	58
2B	$(\pm 0.703647 \pm 3.885385i, \pm 0.989685 \pm 2.635847i, \pm 2.164505i, \pm 1.286567i, \pm 0.260580 \pm 0.911410i, \pm 0.309220 \pm 0.741605i, \pm 0.291998 \pm 0.732634i, \pm 0.185014 \pm 0.663522i, \pm 0.629854i, \pm 0.413668 \pm 0.719695i, \pm 0.384645 \pm 0.575356i, \pm 0.287111 \pm 0.344554i, \pm 0.307800 \pm 0.085927i, 0.392478 \pm 0.257430i, \pm 0.393477 \pm 0.257656i, \pm 0.397932 \pm 0.259322i, \pm 0.396745 \pm 0.249085i, \pm 0.586328 \pm 0.114556i)$	66
2C	$(\pm 1.139431 \pm 4.494531i, \pm 1.213401 \pm 3.046207i, \pm 2.334024i, \pm 0.982423i, \pm 0.253445 \pm 0.977548i, \pm 0.770039i, \pm 0.249856 \pm 0.694092i, \pm 0.344109 \pm 0.724114i, \pm 0.334043 \pm 0.718452i, \pm 0.450458 \pm 0.731506i, \pm 0.378413 \pm 0.587508i, \pm 0.312875 \pm 0.311712i, \pm 0.117795, \pm 0.331434, \pm 0.359204 \pm 0.094005i, \pm 0.548263 \pm 0.118782i, \pm 0.617066, \pm 0.617191, \pm 0.617195, \pm 0.618643)$	62
2D	$(\pm 4.174483i, \pm 3.351009i, \pm 0.831592 \pm 2.322153i, \pm 0.831592 \pm 2.322153i, \pm 2.051498i, \pm 1.196162i, \pm 0.222824 \pm 0.907174i, \pm 0.739639i, \pm 0.321367 \pm 0.670016i, \pm 0.319479 \pm 0.661126i, \pm 0.285673 \pm 0.638307i, \pm 0.403431 \pm 0.668736i, \pm 0.409548 \pm 0.568880i, \pm 0.301333 \pm 0.374698i, \pm 0.136063, \pm 0.323016 \pm 0.165062i, \pm 0.333373, \pm 0.595240 \pm 0.110274i)$	54
2E	$(\pm 4.028172i, \pm 3.333460i, \pm 0.791332 \pm 2.329323i, \pm 2.078603i, \pm 1.179385i, \pm 0.238209 \pm 0.943463i, \pm 0.769852i, \pm 0.323937 \pm 0.711924i, \pm 0.315584 \pm 0.686046i, \pm 0.440797 \pm 0.718355i, \pm 0.260437 \pm 0.588361i, \pm 0.444869 \pm 0.581503i, 0.222659 \pm 0.404903i, \pm 0.394114 \pm 0.414184i, \pm 0.306062 \pm 0.105541i, \pm 0.407419 \pm 0.225151i, \pm 0.419289 \pm 0.070455i, \pm 0.604410 \pm 0.114408i)$	62
2F	$(\pm 1.557947 \pm 4.115911i, \pm 1.109587 \pm 3.132135i, \pm 2.265833i, \pm 1.241460i, \pm 0.267741 \pm 0.910860i, \pm 0.211278 \pm 0.708600i, \pm 0.309188 \pm 0.733453i, \pm 0.323000 \pm 0.736854i, \pm 0.661079i, \pm 0.446348 \pm 0.723011i, \pm 0.360912 \pm 0.583603i, \pm 0.135267 \pm 0.429714i, \pm 0.208763 \pm 0.210558i, \pm 0.313956 \pm 0.303659i, \pm 0.269639, \pm 0.360586 \pm 0.139483i, \pm 0.369640, \pm 0.585139 \pm 0.120025i)$	62
2G	$(\pm 1.071407 \pm 4.005965i, \pm 1.081529 \pm 2.877503i, \pm 2.224120i, \pm 1.232495i, \pm 0.260348 \pm 0.916945i, \pm 0.689223i, \pm 0.325053 \pm 0.731468i, \pm 0.311919 \pm 0.723932i, \pm 0.220708 \pm 0.675887i, \pm 0.441978 \pm 0.723369i, \pm 0.384371 \pm 0.587584i, \pm 0.048717 \pm 0.211128i, \pm 0.297157 \pm 0.352937i, \pm 0.194713, \pm 0.335696 \pm 0.156300i, \pm 0.346829, \pm 0.589764 \pm 0.119214i,)$	58
3A	$(\pm 5.563983i, \pm 2.072687i, \pm 0.467473 \pm 1.943965i, \pm 1.246626i, \pm 0.890987i, \pm 0.400706 \pm 0.686399I, \pm 0.508469i, \pm 0.387955i, \pm 0.176043i, \pm 0.144413)$	24
3B	$(\pm 5.236295i, \pm 2.126880i, \pm 0.467075 \pm 1.895012i, \pm 1.263571i, \pm 0.889748i, \pm 0.372130 \pm 0.702582i, \pm 0.107022 \pm 0.541652i, \pm 0.294306i, \pm 0.212418i, \pm 0.108571i)$	26
3C	$(\pm 5.256567i, \pm 2.085555i, \pm 0.437080 \pm 1.929301i, \pm 0.162679 \pm 1.027673i, \pm 0.732921i, \pm 0.332147 \pm 0.671778i, \pm 0.462118i, \pm 0.398698i, \pm 0.187754i, \pm 0.178917, \pm 0.434668 \pm 0.202087i)$	30
3D	$(\pm 5.268735i, \pm 2.060259i, \pm 0.425108 \pm 1.951005i, \pm 0.253847 \pm 1.016418i, \pm 0.771471i, \pm 0.319566 \pm 0.657762i, \pm 0.441401i, \pm 0.412724i, \pm 0.187723i, \pm 0.179750, \pm 0.511371 \pm 0.158351i)$	30
3E	$\pm i/\sqrt{3}, \pm 4.70463i, \pm 2.24604i, \pm 1.37638i, \pm 0.900404i, \pm 0.32492i, \pm 0.105104i$	14
3F1	$\pm i/\sqrt{3}, \pm 2.07652i, \pm 0.797473i, \pm 0.228243i, \pm 1.29347i, \pm 0.488058i, \pm 0.00751051i$	14
3F2	$\pm i/\sqrt{3}, \pm 2.07652i, \pm 0.797473i, \pm 0.228243i, \pm 1.22867i, \pm 0.444743i, \pm 0.0488449i$	14
3F3	$\pm i/\sqrt{3}, \pm 2.07652i, \pm 0.797473i, \pm 0.228243i, \pm 1.09904i, \pm 0.215173i, \pm 0.19567i$	14
3G	$\pm i/\sqrt{3}, \pm 4.70463i, \pm 2.24604i, \pm 1.37638i, \pm 0.900404i, \pm 0.32492i, \pm 0.105104i$	14
KT16	$(\pm 0.476916 \pm 2.650557i, \pm 0.051459 \pm 2.050952i, \pm 0.400896 \pm 1.369080i, \pm 0.280274 \pm 1.208123i, \pm 0.284743 \pm 1.142729i, \pm 0.014198 \pm 1.079244i, \pm 0.014345 \pm 1.058107i, \pm 0.247407 \pm 1.068248i, \pm 1.010977i, \pm 0.393030 \pm 0.996591i, \pm 0.325607 \pm 0.569521i, \pm 0.166390 \pm 0.432774i, \pm 0.204517 \pm 0.433095i, \pm 0.155288 \pm 0.407042i, \pm 0.206799 \pm 0.426139i, \pm 0.285928 \pm 0.468747i, \pm 0.207797 \pm 0.414946i, \pm 0.200778 \pm 0.407067i, \pm 0.295255 \pm 0.456380i, \pm 0.196392 \pm 0.392413i, \pm 0.196007 \pm 0.384202i, \pm 0.177968 \pm 0.371249i, \pm 0.185475 \pm 0.371981i, \pm 0.309647 \pm 0.437999i, \pm 0.366211 \pm 0.424552i, \pm 0.250036 \pm 0.263593i, \pm 0.240486 \pm 0.249258i, \pm .263038 \pm 0.268698i, \pm 0.323134 \pm 0.267378i, \pm 0.264732 \pm 0.147746i)$	118
MBM	$(\pm 4.2114i, \pm 1.87158i, \pm 0.494294 \pm 1.19157i, \pm 0.800597i, \pm 0.316819 \pm 0.781436i, \pm 0.364426i, \pm 0.0786966 \pm 0.191185i, \pm 0.187217)$	22
SAK	$\pm i/\sqrt{3}, \pm 4.70463i, \pm 2.24604i, \pm 1.37638i, \pm 0.900404i, \pm 0.32492i, \pm 0.105104i$	14

The highlighted methods are investigated for their dynamics in Section 5.

4. Results and Discussion on Numerical and Dynamical Aspects

For various test functions, we begin by numerical aspects of (5), as well as three existing methods, KT16, MBM (with $G(u,s) = \frac{2\beta+u(2\beta+2(\beta^2-2\beta-4)s-5)-(4\beta+1)S^2+2(\beta^2-4\beta+1)s-5}{2\beta+2(\beta^2-6\beta+6)s-5}$, $\beta = 0$), and SAK; then we explore the dynamics underlying extraneous fixed points based on Iterative Maps (71) from the illustrated relevant attractor basins. Numerical experiments have been implemented for all listed methods in Table 1. Computational experiments on dynamical aspects have carried out with only 17 members of (5) and three methods KT16, MBM, and SAK. For each map, numerical experiments have strongly confirmed the desired convergence properties.

Throughout the computational experiments with the aid of Mathematica, $\$MinPrecision = 400$ has been assigned to maintain 400 digits of minimum number of precision. If α is not exact, then it should be given by an approximate value with more precision digits than $\$MinPrecision$.

The value of α is approximately given with 450 precision digits unless exact. Limited paper space allows us to list x_n and α with up to 15 significant digits. We set error bound $\epsilon = \frac{1}{2} \times 10^{-360}$ to meet $|x_n - \alpha| < \epsilon$.

Methods W3G1, W2D, and W1C successfully located desired zeros of test functions $F_1 - F_3$:

$$\begin{cases} \textbf{W3G1}: F_1(x) = \sin(\frac{\pi}{x^7+1}) + x^{15}e^{-5x} - 2x^3\log(1+\frac{1}{x}) + 4, \ \alpha \approx 1.7284526755304, \\ \textbf{W2D}: F_2(x) = 2x - \pi - \cos x \log(x^2+1), \ \alpha = \frac{\pi}{2}, \\ \textbf{W1C}: F_3(x) = \cos\left[(x-3)^2+3\right] - \log\left[(x-3)^2+4\right] - 1, \ \alpha = 3 + i\sqrt{3}, \\ \text{where } \log z (z \in \mathbb{C}) \text{ is a principal analytic branch with } -\pi < Im(\log z) \leq \pi. \end{cases} \tag{72}$$

Ensured in Table 2 is sixteenth-order convergence. The computational asymptotic error constant $|e_n|/|e_{n-1}|^{16}$ is in agreement with $\eta = \lim_{n\to\infty} |e_n|/|e_{n-1}|^{16}$ up to 10 significant digits. The computational convergence order $p_n = \log|e_n/\eta|/\log|e_{n-1}|$ well approaches 16.

Additional test functions in Table 3 confirm the convergence of scheme (5). The n*th* iterate errors $|x_n - \alpha|$ are listed in Table 4 for comparison among methods W1A–W3G9 and three methods KT16, MBM, SAK. Bold-face numbers indicate the least errors for the selected test functions. In the current experiments, MBM has slightly better convergence for f_4, f_6, W3E or SAK for f_2, f_3, W1D for f_5 as well as W1C for f_1 and f_7. According to the definition of the asymptotic error constant $\eta(c_i, Q_f, K_f, J_f) = \lim_{n\to\infty} |R_f(x_n) - \alpha| / \ |x_n - \alpha|^{16}$, the convergence is dependent on iterative map $R_f(x_n)$, $f(x)$, x_0, α and the weight functions Q_f, K_f and J_f. Consequently, it is hard to believe that one method always achieves better convergence than the others for any test functions.

Table 2. Convergence of methods W3G1, W2D, W1C for test functions $F_1(x) - F_3(x)$ shown in (72).

MT	F	n	x_n	$\|F(x_n)\|$	$\|x_n - \alpha\|$	$\|e_n/e_{n-1}^{16}\|$	η	p_n
W3G1	F_1	0	1.8	0.270467	0.0715473			
		1	1.72845267553040	1.431×10^{-18}	3.695×10^{-19}	0.7836895849	1.895228371	16.3348
		2	1.72845267553040	8.871×10^{-295}	2.290×10^{-295}	1.895228371		16.00000
W2D	F_2	0	1.3	0.806294	0.270796			
		1	1.57079632679455	1.120×10^{-12}	3.454×10^{-13}	0.0004131418075	$5.084558258 \times 10^{-6}$	12.6338
		2	1.57079632679490	6.782×10^{-205}	2.091×10^{-205}	$5.084558258 \times 10^{-6}$		16.00000
W1C	F_3	0	$\begin{pmatrix} 2.95 \\ 1.76 \end{pmatrix}^*$	0.198116	0.0572814			
		1	$\begin{pmatrix} 3.00000000000000 \\ 1.73205080756888 \end{pmatrix}$	4.820×10^{-15}	1.391×10^{-15}	103591.4791	3210.28764	14.7852
		2	$\begin{pmatrix} 3.00000000000000 \\ 1.73205080756888 \end{pmatrix}$	2.201×10^{-234}	6.356×10^{-235}	3210.287640		16.00000

MT = method, $\begin{pmatrix} 2.95 \\ 1.76 \end{pmatrix}^* = 2.95 + 1.76i$, $i = \sqrt{-1}$, $\eta = \lim_{n\to\infty} \frac{|e_n|}{|e_{n-1}|^{16}}$, $p_n = \frac{\log|e_n/\eta|}{\log|e_{n-1}|}$.

Table 3. Additional test functions $f_i(x)$ with zeros α and initial values x_0.

i	$f_i(x)$	α	x_0
1	$e^{-x^3}\sin(x) - (x^2+3)\log(x-\pi+1)$	π	3.5
2	$\cos(x^2+1) - \log(2x^2+e+2-\pi)+1$	$\sqrt{-1+\frac{\pi}{2}}$	0.9
3	$1-\sqrt{\cos(x^2-4x+6)} + x\log(\frac{3}{2}+\frac{1}{(x-2)^2})$	$2+i\sqrt{2}$	$1.95+1.28i$
4	$x^5-2+\sqrt{x+4}\cdot\log(e-\frac{x}{1-x^2})$	0	0.08
5	$x^2e^x + x\cos(\frac{1}{x^3})+1$	-1.56506028675084	-1.3
6	$x^3+\sin(\frac{\pi}{x^2})\log(x^2-2)-3\sqrt{3}$	$\sqrt{3}$	2.0
7	$x^2\sin(\frac{\pi}{2x+1})-3x+2$	0.917990036268013	1.25

Here, $\log z$ $(z\in\mathbb{C})$ represents a principal analytic branch with $-\pi \le Im(\log z) < \pi$.

The proposed Family of Methods (5) has efficiency index EI [11], which is $16^{1/5} \approx 1.741101$ and larger than that of Newton's method. In general, the local convergence of iterative methods (71) is guaranteed with good initial values x_0 that are close to α. Selection of good initial values is a difficult task, depending on precision digits, error bound, and the given function $f(x)$.

The influence of initial values x_0 on the global convergence is effectively described by means of a basin of attraction that is the set of initial values leading to long-time behavior approaching the attractors under the iterative action of R_f. Basins of attraction contain information about the region of convergence. A method occupying a larger region of convergence is likely to be a more robust method. A quantitative analysis undoubtedly measures the size of region of convergence.

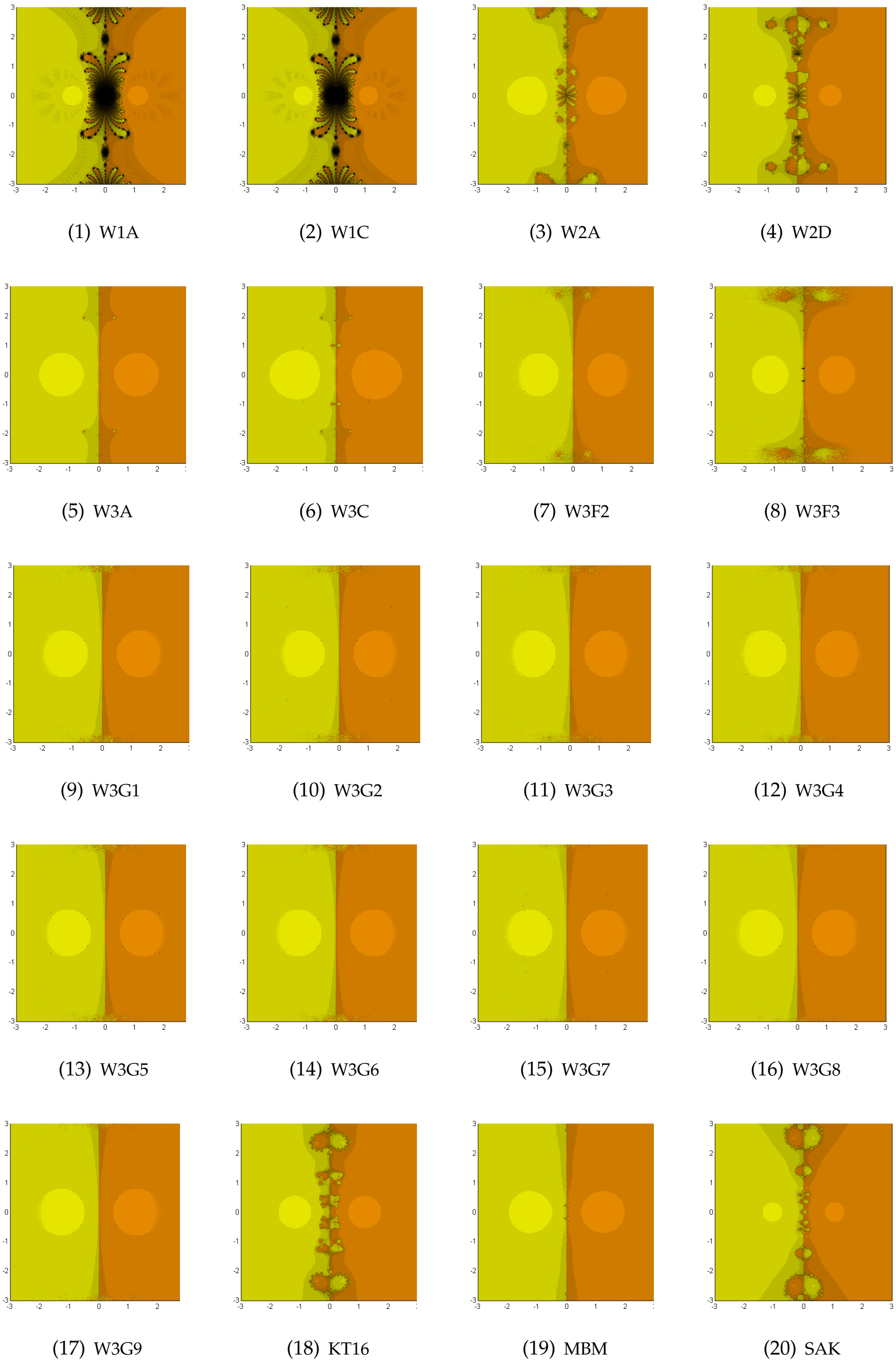
The basins of attraction, as well as the relevant statistical data, are constructed in a similar manner shown in the work of Geum-Kim-Neta [22].

Owing to the limited space, in Table 1, we explore the dynamics of 17 maps, W1A, W1C, W2A, W2D, W3A, W3C, W3F2, W3F3, W3G1, W3G2, W3G3, W3G4, W3G5, W3G6, W3G7, W3G8, W3G9, and three existing methods KT16, MBM, SAK, with applications to $p_k(z)$, $(1 \le k \le 6)$ and one nonpolynomial equation through the following seven examples. In each example, we have shown dynamical planes for the convergence behavior of iterative maps $x_{n+1} = R_f(x_n)$ (67) with $f(z) = p_k(z)$ by illustrating the relevant basins of attraction through Figures 1–6.

Example 1. *We have taken $p_1(z)$ as a quadratic polynomial with all real roots:*

$$p_1(z) = z^2 - 1. \tag{73}$$

The roots are obviously ± 1. Note that the extraneous fixed points are computed based on this example. Clearly the best methods have basins separated by the imaginary axis. Basins of attraction for W1A–W3G9, KT16, MBM with $\beta = 0$ and SAK are given in Figure 1. Methods W3A, W3G1–W3G8 show basins separated by the imaginary axis. Consulting Tables 5–7, we find that the methods W3G7–W3G9 use the least number of iterations per point on average (ANIP), followed closely by W3C, W3G1, W3G3–W3G6. The fastest method is W3A. The methods KT16, MBM, and SAK have the least number of black points. Methods W1A and W1C have the highest number of black points. Notice that these methods use polynomial weight functions.



(1) W1A (2) W1C (3) W2A (4) W2D

(5) W3A (6) W3C (7) W3F2 (8) W3F3

(9) W3G1 (10) W3G2 (11) W3G3 (12) W3G4

(13) W3G5 (14) W3G6 (15) W3G7 (16) W3G8

(17) W3G9 (18) KT16 (19) MBM (20) SAK

Figure 1. The top row for W1A (left), W1C (center left), W2A (center right) and W2D (right). The second row for W3A (left), W3C (center left), W3F2 (center right) and W3F3 (right). The third row for W3G1 (left), W3G2 (center left), W3G3 (center right), and W3G4 (right). The fourth row for W3G5 (left), W3G6 (center left), W3G7 (center right), and W3G8 (right). The bottom row for W3G9 (left), KT16 (center left), MBM (center right) and SAK (right), for the roots of the polynomial equation $(z^2 - 1)$

Table 4. Comparison of $|x_n - \alpha|$ for selected methods applied to various test functions.

		$f(x);\ x_0$						
Method	$\|x_n - \alpha\|$	f_1; 3.5	f_2; 0.9	f_3; $1.95 + 1.28i$	f_4; 0.08	f_5; -1.3	f_6; 2.0	f_7; 1.25
W1A	$\|x_1 - \alpha\|$ $\|x_2 - \alpha\|$	1.27e-17* 2.32e-278	1.16e-14 1.74e-222	1.26e-12 1.40e-187	2.12e-19 2.04e-299	3.06e-14 6.71e-221	2.49e-13 3.55e-203	3.89e-20 2.38e-322
W1B	$\|x_1 - \alpha\|$ $\|x_2 - \alpha\|$	2.64e-17 5.35e-273	1.75e-14 1.68e-219	1.42e-12 1.34e-186	5.21e-19 4.14e-293	3.18e-14 2.17e-220	3.62e-13 2.41e-200	1.01e-19 5.02e-316
W1C	$\|x_1 - \alpha\|$ $\|x_2 - \alpha\|$	1.13e-17 **3.73e-279**	1.47e-14 1.08e-220	3.25e-13 5.78e-197	6.36e-18 1.79e-276	5.31e-14 4.06e-217	2.425e-13 2.10e-203	6.72e-21 **2.03e-336**
W1D	$\|x_1 - \alpha\|$ $\|x_2 - \alpha\|$	1.05e-17 1.24e-279	1.58e-14 4.94e-220	6.86e-14 9.10e-208	4.28e-19 5.24e-296	1.42e-14 **3.17e-226**	1.72e-13 9.51e-206	2.21e-19 1.51e-309
W2A	$\|x_1 - \alpha\|$ $\|x_2 - \alpha\|$	2.24e-16 4.22e-261	3.82e-17 2.86e-265	1.48e-11 4.11e-173	1.01e-17 4.96e-273	1.78e-10 1.27e-161	5.71e-13 2.58e-199	3.33e-17 1.42e-271
W2B	$\|x_1 - \alpha\|$ $\|x_2 - \alpha\|$	2.70e-16 8.30e-260	1.739e-17 3.33e-271	1.04e-11 9.27e-176	4.740e-18 2.15e-278	1.26e-10 4.22e-164	7.65e-13 2.51e-197	2.74e-17 4.84e-273
W2C	$\|x_1 - \alpha\|$ $\|x_2 - \alpha\|$	1.88e-16 2.66e-262	8.79e-17 6.78e-259	1.79e-11 1.65e-171	1.82e-17 1.09e-268	1.94e-10 4.42e-161	4.57e-13 8.62e-201	3.57e-17 4.62e-271
W2D	$\|x_1 - \alpha\|$ $\|x_2 - \alpha\|$	2.66e-16 6.67e-260	1.53e-17 7.71e-272	8.09e-12 1.61e-177	5.94e-18 7.60e-277	9.75e-11 3.22e-166	7.73e-13 3.33e-197	2.36e-17 4.05e-274
W2E	$\|x_1 - \alpha\|$ $\|x_2 - \alpha\|$	2.58e-16 4.10e-260	1.26e-17 1.28e-273	1.33e-11 4.35e-174	2.81e-17 1.07e-265	1.49e-10 3.37e-163	7.09e-13 8.28e-198	2.72e-17 4.97e-273
W2F	$\|x_1 - \alpha\|$ $\|x_2 - \alpha\|$	2.26e-16 4.96e-261	3.98e-17 4.51e-265	1.37e-11 1.26e-173	5.19e-18 9.26e-278	1.68e-10 5.57e-162	5.83e-13 3.46e-199	3.30e-17 1.20e-271
W2G	$\|x_1 - \alpha\|$ $\|x_2 - \alpha\|$	2.33e-16 7.83e-261	3.32e-17 2.51e-266	1.37e-11 1.04e-173	8.62e-18 3.71e-274	1.64e-10 3.52e-162	6.08e-13 6.92e-199	3.19e-17 6.87e-272
W3A	$\|x_1 - \alpha\|$ $\|x_2 - \alpha\|$	2.17e-16 2.68e-261	2.12e-18 2.61e-287	1.13e-13 1.73e-207	3.62e-18 2.69e-280	4.97e-12 3.36e-186	6.73e-13 6.01e-198	2.86e-18 5.10e-290
W3B	$\|x_1 - \alpha\|$ $\|x_2 - \alpha\|$	2.25e-16 5.05e-261	2.28e-18 1.10e-286	1.11e-13 1.25e-207	2.50e-18 6.12e-283	6.75e-12 4.66e-184	7.18e-13 1.71e-197	3.60e-18 2.77e-288
W3C	$\|x_1 - \alpha\|$ $\|x_2 - \alpha\|$	2.32e-16 8.27e-261	2.19e-18 4.83e-287	1.34e-13 2.53e-206	2.45e-18 4.40e-283	9.99e-12 2.59e-181	7.52e-13 3.59e-197	4.72e-18 3.15e-286
W3D	$\|x_1 - \alpha\|$ $\|x_2 - \alpha\|$	2.39e-16 1.30e-260	2.11e-18 2.06e-287	1.54e-13 2.32e-205	2.41e-18 3.25e-283	1.26e-11 1.12e-179	7.82e-13 6.77e-197	5.91e-18 1.54e-284
W3E	$\|x_1 - \alpha\|$ $\|x_2 - \alpha\|$	7.41e-17 4.02e-270	1.13e-18 **1.05e-291**	4.49e-14 **2.58e-214**	1.90e-18 6.03e-285	1.77e-12 6.04e-194	3.37e-13 2.06e-203	4.80e-18 3.05e-286
W3F1	$\|x_1 - \alpha\|$ $\|x_2 - \alpha\|$	7.07e-14 3.46e-217	1.39e-15 5.18e-238	1.28e-11 2.57e-170	3.41e-15 2.01e-228	2.01e-10 1.23e-156	7.64e-11 3.56e-161	2.15e-15 5.70e-240
W3F2	$\|x_1 - \alpha\|$ $\|x_2 - \alpha\|$	4.43e-14 1.47e-220	9.32e-16 6.99e-241	9.06e-12 8.64e-173	2.25e-15 2.16e-231	1.02e-10 1.92e-161	5.04e-11 3.48e-164	1.18-15 2.74e-244
W3F3	$\|x_1 - \alpha\|$ $\|x_2 - \alpha\|$	8.02e-15 1.94e-233	3.45e-16 5.93e-248	3.67e-12 2.36e-179	6.84e-16 5.94e-240	6.30e-11 4.88e-166	1.12e-11 2.03e-176	7.51e-17 3.93e-264
W3G1	$\|x_1 - \alpha\|$ $\|x_2 - \alpha\|$	8.11e-13 3.52e-200	2.78e-14 4.29e-217	6.06e-11 2.41e-159	5.85e-14 1.64e-208	2.54e-10 4.75e-155	4.74e-10 1.82e-148	7.06e-15 9.27e-233
W3G2	$\|x_1 - \alpha\|$ $\|x_2 - \alpha\|$	3.18e-13 8.83e-206	4.15e-15 1.95e-229	4.17e-11 4.46e-161	1.54e-14 6.62e-217	1.42e-10 3.96e-158	8.29e-10 1.15e-143	4.21e-15 4.83e-235
W3G3	$\|x_1 - \alpha\|$ $\|x_2 - \alpha\|$	2.88e-12 2.73e-191	1.32e-14 3.53e-222	5.92e-11 1.98e-159	8.83e-13 1.42e-189	2.35e-10 1.72e-155	6.92e-10 9.65e-146	2.34e-14 6.62e-225
W3G4	$\|x_1 - \alpha\|$ $\|x_2 - \alpha\|$	2.04e-12 1.06e-193	1.44e-14 1.42e-221	5.95e-11 2.12e-159	2.68e-13 7.32e-198	2.38e-10 2.10e-155	6.44e-10 2.95e-146	1.73e-14 7.47e-227
W3G5	$\|x_1 - \alpha\|$ $\|x_2 - \alpha\|$	9.79e-13 7.45e-199	2.20e-14 1.09e-218	6.04e-11 2.42e-159	7.52e-14 9.37e-207	2.50e-10 3.89e-155	5.12e-10 6.68e-148	8.57e-15 1.79e-231
W3G6	$\|x_1 - \alpha\|$ $\|x_2 - \alpha\|$	9.07e-13 2.15e-199	2.39e-14 4.05e-218	6.05e-11 2.43e-159	6.78e-14 1.75e-207	2.51e-10 4.21e-155	4.97e-10 4.01e-148	7.92e-15 5.44e-232
W3G7	$\|x_1 - \alpha\|$ $\|x_2 - \alpha\|$	1.27e-14 4.98e-230	1.00e-17 3.62e-274	2.97e-12 5.80e-182	1.05e-16 2.63e-254	3.57e-10 2.21e-153	1.38e-11 7.26e-174	9.64e-16 5.88e-246
W3G8	$\|x_1 - \alpha\|$ $\|x_2 - \alpha\|$	2.07e-14 1.77e-226	1.82e-16 2.62e-253	1.74e-12 3.10e-185	5.40e-16 2.59e-242	3.62e-10 3.46e-153	2.49e-11 1.19e-169	1.04e-15 2.27e-245
W3G9	$\|x_1 - \alpha\|$ $\|x_2 - \alpha\|$	2.34e-14 1.45e-225	2.40e-16 2.83e-251	1.80e-12 6.66e-185	6.90e-16 1.64e-240	3.64e-10 4.02e-153	2.88e-11 1.32e-168	1.07e-15 3.55e-245
KT16	$\|x_1 - \alpha\|$ $\|x_2 - \alpha\|$	5.62e-17 7.30e-273	4.41e-17 4.73e-264	1.87e-12 3.87e-187	1.85e-19 1.11e-300	1.47e-13 1.31e-211	2.44e-13 1.42e-206	6.41e-19 4.32e-301
MBM	$\|x_1 - \alpha\|$ $\|x_2 - \alpha\|$	5.81e-17 1.64e-270	2.81e-17 5.23e-268	2.30e-12 7.95e-186	8.74e-20 **6.25e-306**	5.26e-12 7.47e-186	1.12e-13 **3.34e-210**	9.07e-18 1.47e-281
SAK	$\|x_1 - \alpha\|$ $\|x_2 - \alpha\|$	7.41e-17 4.02e-270	1.13e-18 **1.05e-291**	4.49e-14 **2.58e-214**	1.90e-18 6.03e-285	1.77e-12 6.04e-194	3.37e-13 2.06e-203	4.80e-18 3.05e-286

* 1.27e-17 $\equiv 1.27 \times 10^{-17}$.

Example 2. *We have taken $p_2(z)$ as a cubic polynomial:*

$$p_2(z) = z^3 + 4z^2 - 10. \tag{74}$$

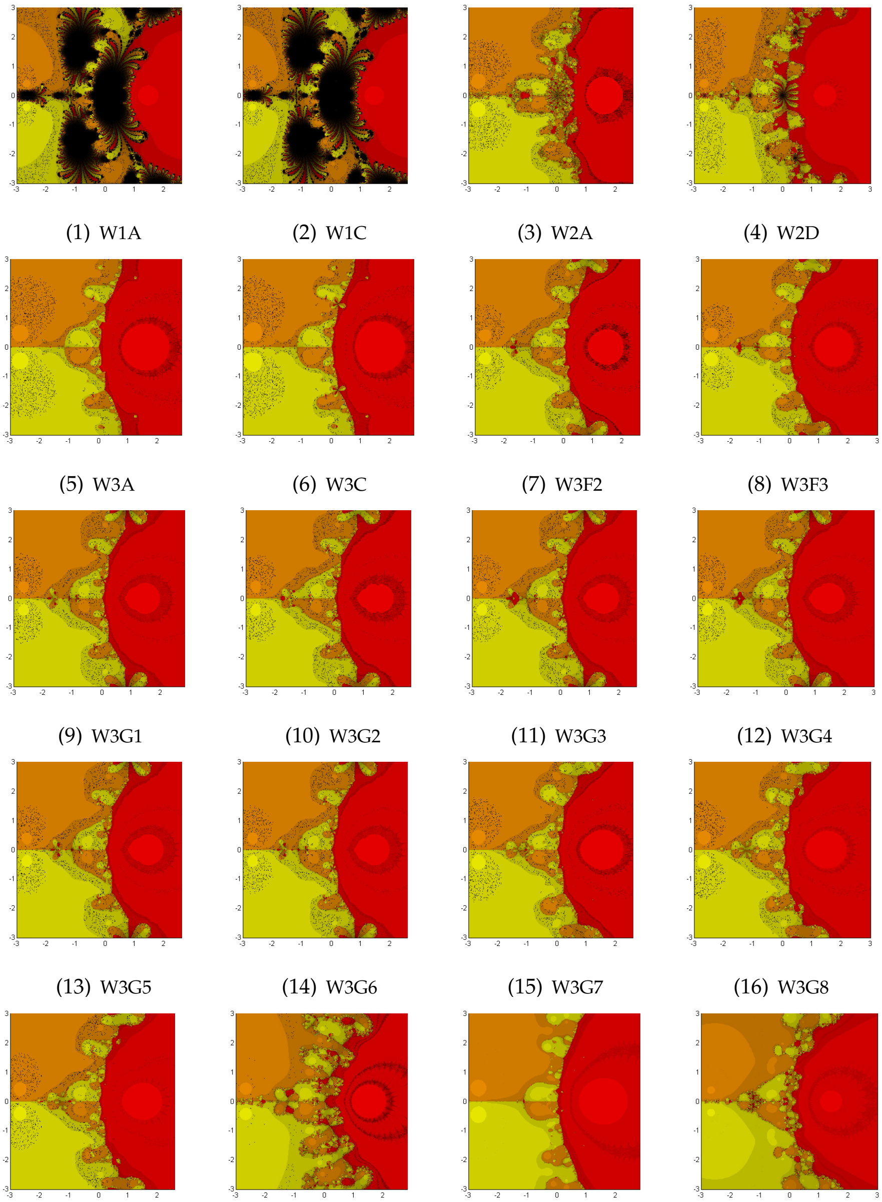
Basins of attraction are given in Figure 2. Notice that the basins for W1A and W1C have many black points and therefore will not be considered in the rest of the examples. Besides that, in view of a close inspection that the basins for W3G3–W3G6 have similarities to the other remaining members of the listed W3G-family, we will omit them in the rest of the examples. Consulting Tables 5–7, we find that the method with the fewest ANIP is MBM with 2.22 iteration. All the others require between 2.71 and 12.12. In terms of CPU timein seconds, the fastest is W3A (472.715 s) and the slowest is W1C (2399.093 s). The methods W1C and W1A have the most black points (59,904 and 58,910, respectively) and SAK has the least number (6 points). We will not consider W1A and W1C any further.

Table 5. Average number of iterations per point for each example (1–7).

	Example							
Map	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**Average**
W1A	4.00	11.90	-	-	-	-	-	-
W1C	4.08	12.12	-	-	-	-	-	-
W2A	2.24	3.27	3.53	25.22	-	-	-	-
W2D	2.46	3.56	2.82	5.11	4.92	-	-	-
W3A	2.04	2.86	2.30	2.70	7.91	6.59	2.17	3.80
W3C	2.02	2.82	2.24	2.58	2.64	5.65	2.19	2.88
W3F2	2.07	2.89	2.47	2.80	2.91	6.47	2.38	3.14
W3F3	2.21	2.72	2.45	2.77	2.86	6.18	2.34	3.08
W3G1	2.02	2.95	2.51	2.89	3.02	6.69	2.54	3.23
W3G2	2.03	2.98	2.50	2.88	2.99	6.67	2.48	3.22
W3G3	2.02	2.94	-	-	-	-	-	-
W3G4	2.02	2.94	-	-	-	-	-	-
W3G5	2.02	2.94	-	-	-	-	-	-
W3G6	2.02	2.93	-	-	-	-	-	-
W3G7	2.01	2.82	2.37	2.78	2.82	6.06	2.27	3.02
W3G8	2.01	2.84	2.40	2.83	2.88	6.18	2.37	3.07
W3G9	2.01	2.82	2.41	2.83	2.89	6.26	2.38	3.09
KT16	2.31	3.02	2.76	3.83	3.99	3.92	2.45	3.18
MBM	2.04	2.22	2.28	2.64	2.76	2.41	2.02	2.34
SAK	2.35	2.76	3.02	3.78	5.40	4.50	3.83	3.66

Table 6. CPU time (in seconds) required for each example(1–7) using a Dell Multiplex-990.

	Example							
Map	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**Average**
W1A	631.680	2331.092	-	-	-	-	-	-
W1C	635.548	2399.093	-	-	-	-	-	-
W2A	380.768	604.769	695.952	5117.503	-	-	-	-
W2D	446.646	664.393	582.336	1123.160	1143.846	-	-	-
W3A	316.370	472.715	406.102	497.284	1579.775	2199.567	461.358	847.596
W3C	354.466	529.495	451.421	542.509	615.892	1869.921	507.815	695.931
W3F2	561.791	850.642	771.487	843.607	948.190	2371.028	786.229	1018.996
W3F3	628.418	795.964	798.741	849.410	933.650	2378.079	785.855	1024.302
W3G1	541.432	820.160	726.855	836.680	978.782	2557.308	806.525	1038.249
W3G2	545.754	828.677	739.960	863.824	934.306	2553.424	799.552	1037.928
W3G3	575.986	575.986	-	-	-	-	-	-
W3G4	555.738	851.375	-	-	-	-	-	-
W3G5	577.750	867.132	-	-	-	-	-	-
W3G6	573.116	836.446	-	-	-	-	-	-
W3G7	562.321	809.676	724.812	828.927	896.834	2385.099	768.789	996.637
W3G8	557.828	803.140	720.288	837.039	909.143	2552.878	772.424	1021.820
W3G9	551.416	784.529	715.530	824.590	936.240	2462.929	770.333	1006.510
KT16	386.414	679.572	529.482	783.359	888.784	2427.189	564.770	894.224
MBM	831.189	1039.919	993.804	1157.589	1205.014	2045.282	923.011	1170.830
SAK	467.785	712.831	687.824	833.653	1376.600	2947.202	849.831	1125.104



(1) W1A (2) W1C (3) W2A (4) W2D

(5) W3A (6) W3C (7) W3F2 (8) W3F3

(9) W3G1 (10) W3G2 (11) W3G3 (12) W3G4

(13) W3G5 (14) W3G6 (15) W3G7 (16) W3G8

(17) W3G9 (18) KT16 (19) MBM (20) SAK

Figure 2. The top row for W1A (left), W1C (center left), W2A (center right) and W2D (right). The second row for W3A (left), W3C (center left), W3F2 (center right), and W3F3 (right). The third row for W3G1 (left), W3G2 (center left), W3G7 (center right), and W3G8 (right). The bottom row for W3G9 (left), KT16 (center left), MBM (center right) and SAK (right), for the roots of the polynomial equation $(z^3 + 4z^2 - 10)$.

Example 3. *We have taken $p_3(z)$ as another cubic polynomial:*

$$p_3(z) = z^3 - z. \tag{75}$$

All roots were easily found to be real. The basins for this example are plotted in Figure 3. The basins for W2A, W2D, KT16, and SAK are too chaotic. Based on Table 5 we see that W3C has the lowest ANIP followed closely by MBM. The fastest method is again W3A (406.102 s) and the slowest is W3G8 (2041.663 seconds). The methods KT16, MBM, and SAK have no black points, and the rest have between 58 and 204 black points.

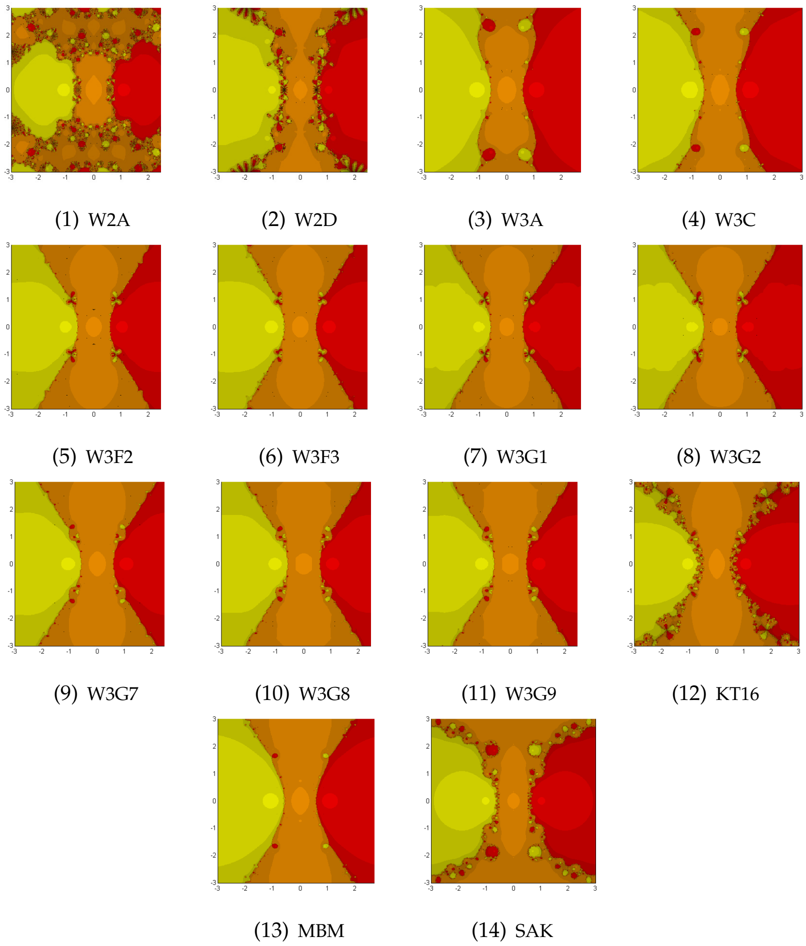


Figure 3. The top row for W2A (left), W2D (center left), W3A (center right) and W3C (right). The second row for W3F2 (left), W3F3 (center left), W3G1 (center right), and W3G2 (right). The third row for W3G7 (left), W3G8 (center left), W3G9 (center right), and KT16 (right). The bottom row for MBM (left), SAK (center left), for the roots of the polynomial equation $(z^3 - z)$.

Table 7. Number of points requiring 40 iterations for each example (1–7).

	Example							
Map	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**Average**
W1A	5425	58,910	-	-	-	-	-	-
W1C	6051	59,904	-	-	-	-	-	-
W2A	645	5852	118	218,849	-	-	-	-
W2D	703	7494	204	10,785	13,828	-	-	-
W3A	627	5948	138	1333	49,167	36,443	1765	13,632
W3C	623	5485	150	1373	46	29,107	2283	5581
W3F2	683	4476	172	1349	94	35,728	3148	6521
W3F3	791	4196	70	1257	44	33,438	3036	6119
W3G1	685	5121	88	1325	66	36,524	4313	6875
W3G2	701	5170	86	1329	65	36,697	3859	6844
W3G3	679	4973	-	-	-	-	-	-
W3G4	681	5007	-	-	-	-	-	-
W3G5	679	5008	-	-	-	-	-	-
W3G6	687	4893	-	-	-	-	-	-
W3G7	703	4548	96	1365	54	32,565	2470	5972
W3G8	665	4753	114	1341	67	33,236	3203	6197
W3G9	677	4778	122	1301	52	33,935	3290	6308
KT16	601	640	0	1241	45	1457	3725	1101
MBM	601	89	0	1889	1704	5	644	705
SAK	601	6	0	1201	18	1	13720	2221

Example 4. *We have taken $p_4(z)$ as a quartic polynomial:*

$$p_4(z) = z^4 - 1. \tag{76}$$

The basins are given in Figure 4. We now see that W2A is the worst, followed by W2D. The best are those with smaller lobes along the diagonals. In terms of ANIP, W3C is the best (2.58), followed by MBM (2.64), and the worst is W2A (25.22). The fastest is again W3A (497.284 s), followed by W3C (542.509 s), and the slowest is W2A (5117.503 s). Most of the methods have between 1201 and 1889 black points with the worst being W2A with 218,849 points and W2D with 10,785 black points. We remove W2A from further consideration.

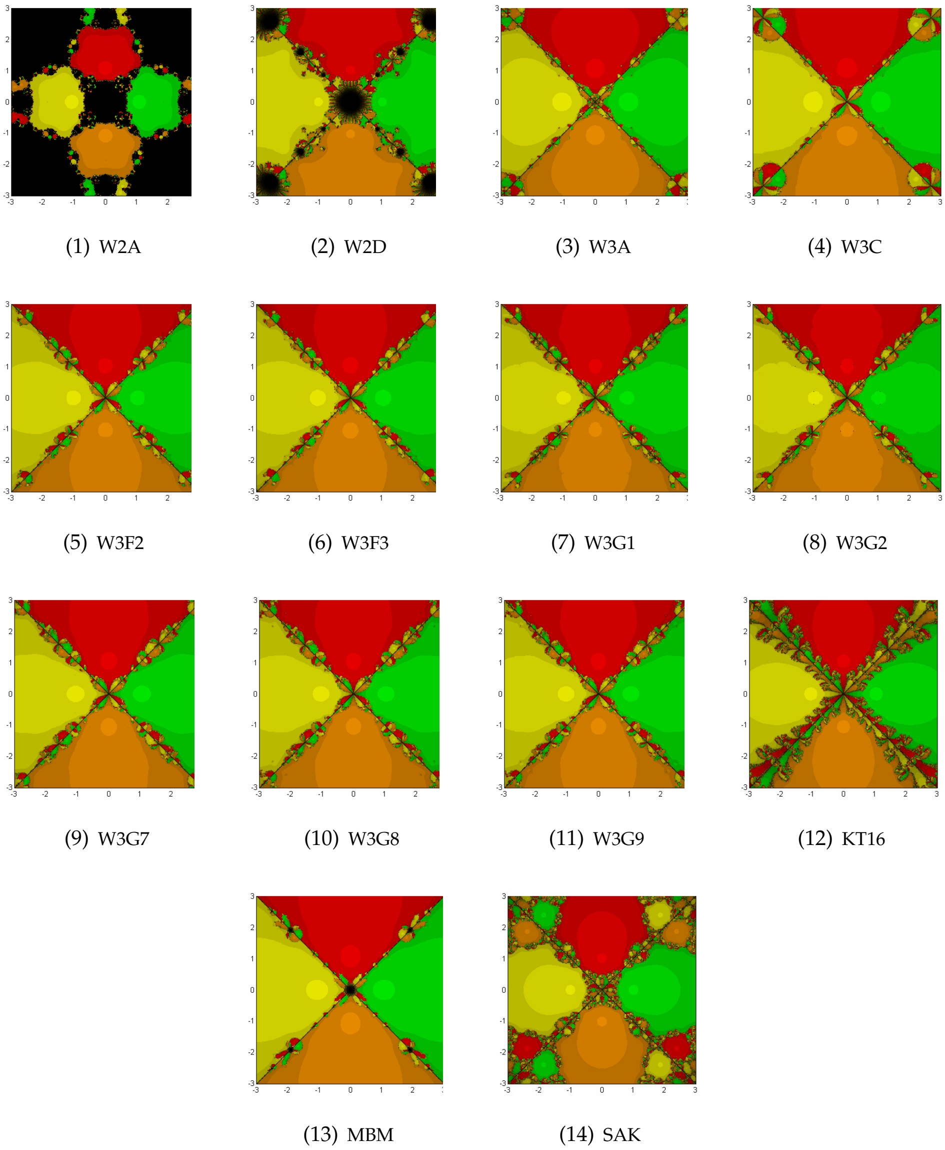


Figure 4. The top row for W2A (left), W2D (center left), W3A (center right), and W3C (right). The second row for W3F2 (left), W3F3 (center left), W3G1 (center right), and W3G2 (right). The third row for W3G7 (left), W3G8 (center left), W3G9 (center right), and KT16 (right). The bottom row for MBM (left), SAK (center left), for the roots of the polynomial equation $(z^4 - 1)$.

Example 5. *We have taken $p_5(z)$ as a quintic polynomial:*

$$p_5(z) = z^5 - 1. \tag{77}$$

The basins for the best methods left are plotted in Figure 5. The worst are W3A, W2D, and SAK. In terms of ANIP, the best is W3C (2.64), followed closely by MBM (2.76), and the worst are W3A (7.91) and SAK (5.40). The fastest is W3C using 615.892 s, followed by KT16 using 888.784 s, and the slowest was W3G1 (2409.685 s). SAK has 18 black points, but the basins are chaotic. The highest number of black points is for W3A (49,176), preceded by W2D with 13,828 black points. We remove W2D from further consideration.

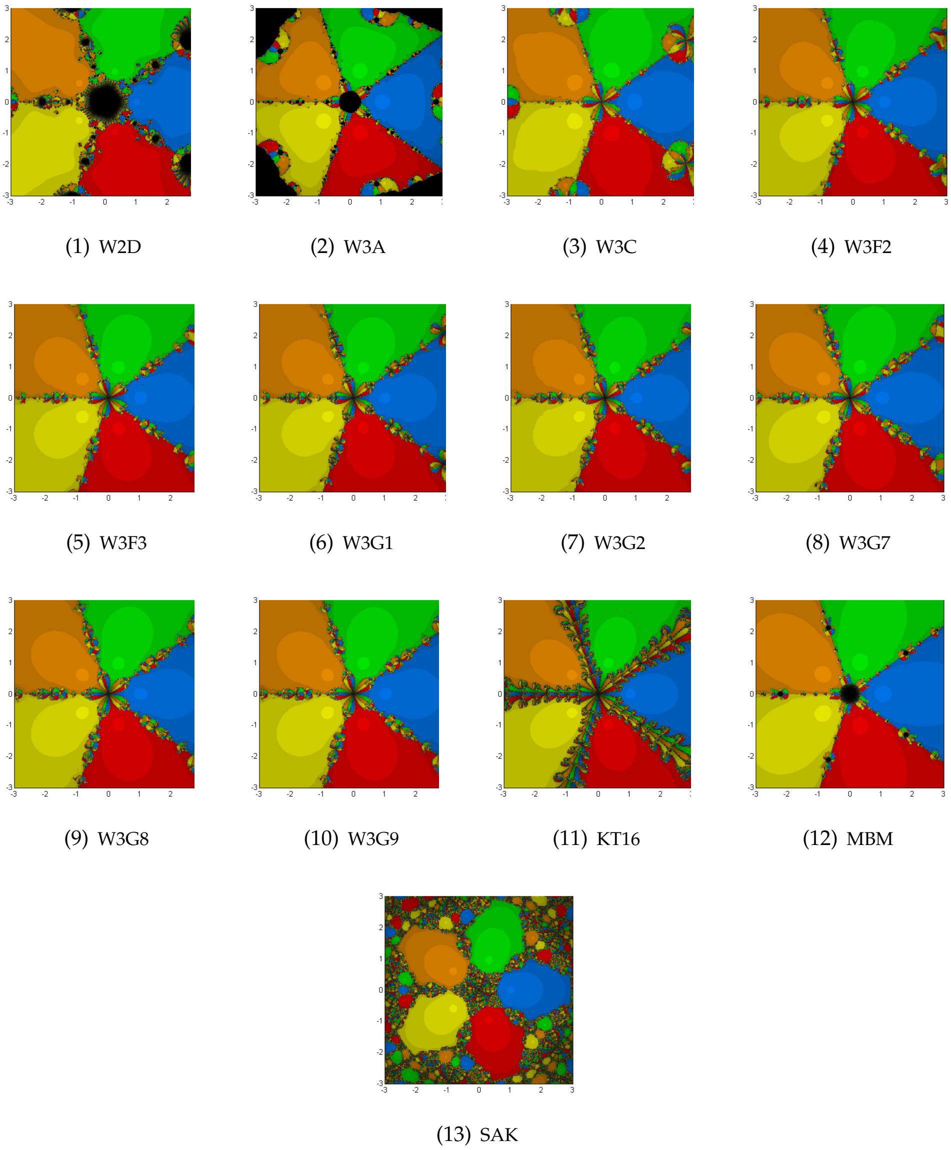


Figure 5. The top row for W2D (left), W3A (center left), W3C (center right) and W3F2 (right). The second row for W3F3 (left), W3G1 (center left), W3G2 (center right) and W3G7 (right). The third row for **W3G8** (left), W3G9 (center left), KT16 (center right) and MBM (right). The bottom row for SAK (center), for the roots of the polynomial equation $(z^5 - 1)$.

Example 6. *We have taken $p_6(z)$ as a sextic polynomial with complex coefficients:*

$$p_6(z) = z^6 - \frac{1}{2}z^5 + \frac{11(i+1)}{4}z^4 - \frac{3i+19}{4}z^3 + \frac{5i+11}{4}z^2 - \frac{i+11}{4}z + \frac{3}{2} - 3i. \tag{78}$$

The basins for the best methods left are plotted in Figure 6. It is clear that the SAK is very chaotic. Based on Table 5, we find that MBM has the lowest ANIP (2.41) followed by KT16 (3.92). The fastest method is W3C (1869.921 s), followed by MBM (2045.282 s), and W3A (2199.567 s). The slowest is SAK taking 2947.202 s. There are two methods with five black points or fewer, namely, SAK and MBM. The highest number is for W3G2 with 36,697 black points. In fact, all our new methods have over 29,000 black points.

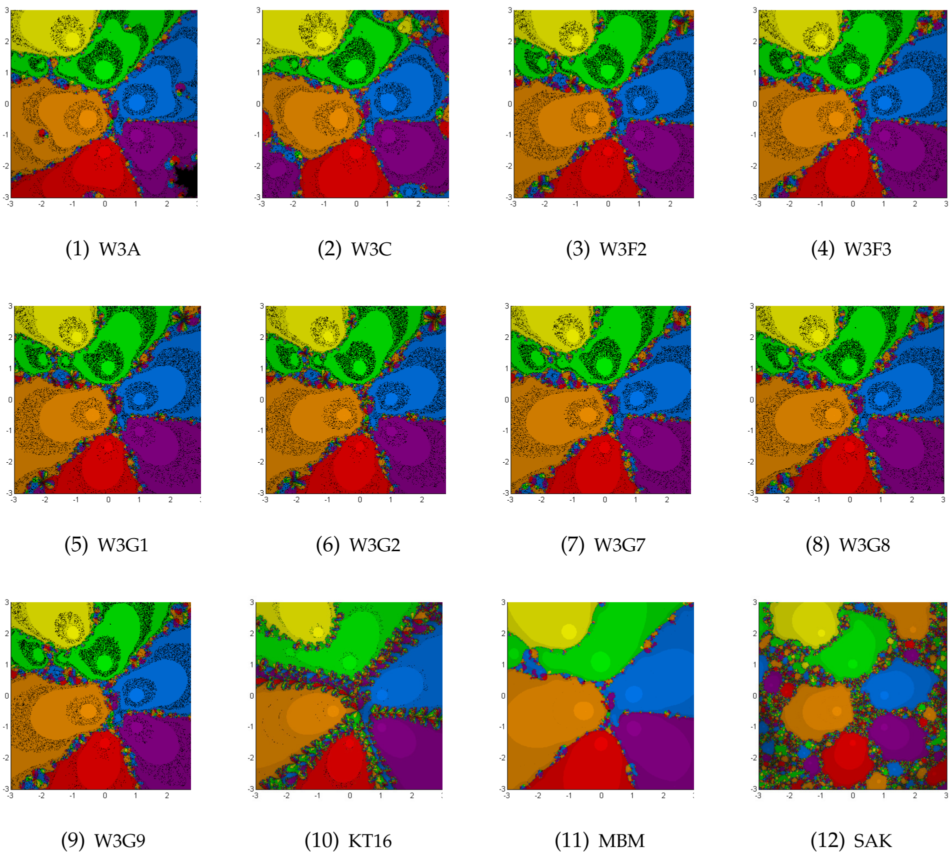


Figure 6. The top row for W3A (left), W3C (center left), W3F2 (center right), and W3F3 (right). The second row for W3G1 (left), W3G2 (center left), W3G7 (center right), and W3G8 (right). The bottom row for W3G9 (left), KT16 (center left), and MBM (center right) and SAK (right), for the roots of the polynomial equation $z^6 - \frac{1}{2}z^5 + \frac{11(i+1)}{4}z^4 - \frac{3i+19}{4}z^3 + \frac{5i+11}{4}z^2 - \frac{i+11}{4}z + \frac{3}{2} - 3i$.

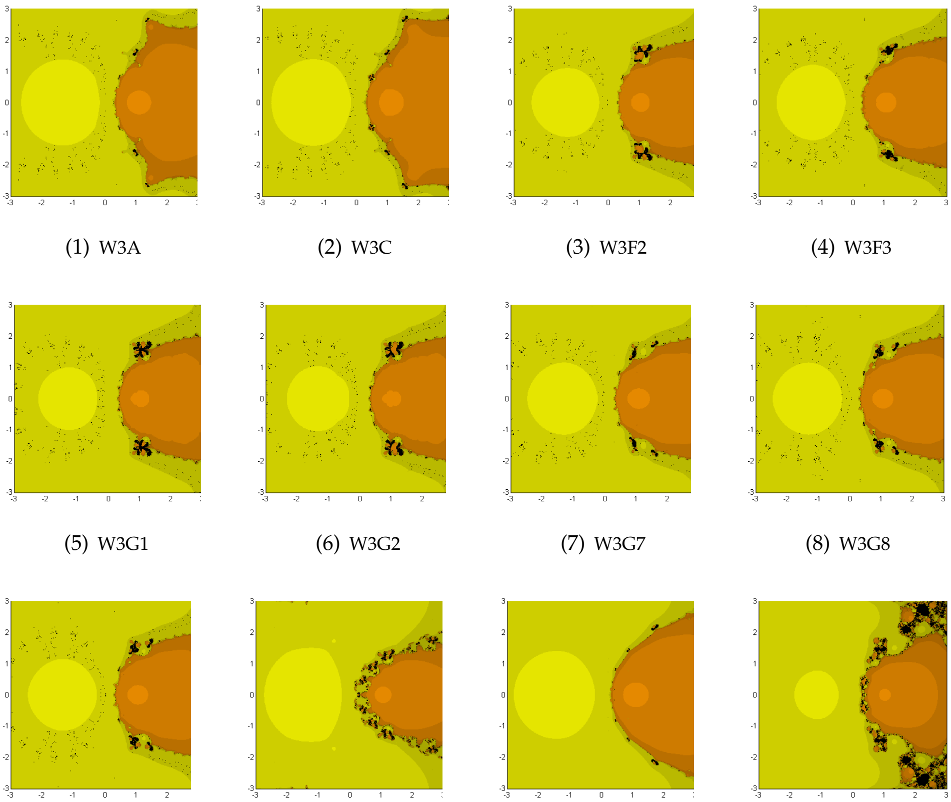
Example 7. *We have taken $p_7(z)$ as a nonpolynomial equation:*

$$p_7(z) = \left(e^{z+1} - 1\right)(z-1). \tag{79}$$

The basins for this example are plotted in Figure 7. The roots are at ± 1 and it is expected that the boundary will be close to the imaginary axis as in Example 1. All methods show a larger basin for the root at -1. The methods with the largest basin for $+1$ are W3A, W3C, and MBM. In terms of ANIP, MBM was best (2.02), followed closely by W3A (2.17) and W3C (2.19). The worst is SAK with 3.83. The fastest method is W3A (461.358 s) and the slowest is MBM (923.011 s). MBM has the least number of black points (644) and SAK has the highest (13720) such number.

We now average all these results across the seven examples to try and pick the best method. MBM had the lowest ANIP (2.34), followed by W3C with 2.88. The fastest method was W3C (695.931 s), followed by W3A (847.596 s). MBM has the lowest number of black points on average (705), followed by KT16 (1101 black points).

Based on these seven examples we see that MBM and W3C have three examples with the lowest ANIP, W3G7, W3G8, and W3G9 each with one example. W3A is the fastest in five examples and W3C in two examples. Thus, we recommend W3C and W3G7, since W3C is in the top four and W3G7 in the top five in all three categories. MBM was at the top in only two categories, KT16 is in top three in two categories, and W3F3 and W3G9 were in the top six in two categories.



(1) W3A (2) W3C (3) W3F2 (4) W3F3

(5) W3G1 (6) W3G2 (7) W3G7 (8) W3G8

(9) W3G9 (10) KT16 (11) MBM (12) SAK

Figure 7. The top row for W3A (left), W3C (center left), W3F2 (center right), and W3F3 (right). The second row for W3G1 (left), W3G2 (center left), W3G7 (center right), and W3G8 (right). The bottom row for W3G9 (left), KT16 (center left), and MBM (center right) and SAK (right), for the roots of non-polynomial equation $\left(e^{z+1}-1\right)(z-1)$.

5. Conclusions

Both numerical and dynamical aspects of Iterative Maps (5) support the main theorem well through a number of test equations and examples. The W3F and W3G methods were observed to occupy relatively slower CPU time due to the intensive use of rational coefficients for weight function J_f. If less number of the rational coefficients were employed, it would take less CPU time to build the relevant basins of attraction.

The proposed Family of Methods (5) favorably cover most of optimal sixteenth-order simple-root finders with certain weight functions developed (or to be developed), since they employ fairly generic weight functions. The dynamics behind the purely imaginary extraneous fixed points will choose best members of the family with improved convergence behavior. However, due to the order of convergence is rather high, the required algebra encounters difficulty resolving its increased complexity. The current work is limited to univariate nonlinear equations; its extension to multivariate ones becomes another task. In future work, as a follow-up study, we will not only extend Case 3 with other combinations of simple coefficients q_i, r_i, but also investigate different types of weight functions possessing less number of rational coefficients to obtain purely imaginary extraneous fixed points, and strengthen the desired computational as well as dynamical behavior.

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