Computers and Mathematics with Applications 59 (2010) 254–273

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

# Large time asymptotic and numerical solution of a nonlinear diffusion model with memory

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ARTICLE INFO

*Article history:*
Received 18 May 2009
Accepted 28 July 2009

*Keywords:*
System of nonlinear integro-differential equations
Large time behavior
Finite difference scheme

ABSTRACT

Large time behavior of solutions and finite difference approximation of a nonlinear system of integro-differential equations associated with the penetration of a magnetic field into a substance is studied. Two initial-boundary value problems are investigated: the first with homogeneous conditions on whole boundary and the second with nonhomogeneous boundary data on one side of lateral boundary. The rates of convergence are also given. Mathematical results presented show that there is a difference between stabilization rates of solutions with homogeneous and nonhomogeneous boundary conditions. The convergence of the corresponding finite difference scheme is also proved. The decay of the numerical solution is compared with the analytical results.

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## 1. Introduction

Integro-differential equations and systems of such equations arise in the study of various problems in physics, chemistry, technology, economics etc. Such systems arise, for instance, for mathematical modelling of the process of penetrating of magnetic field in the substance. If the coefficient of thermal heat capacity and electroconductivity of the substance is highly dependent on temperature, then Maxwell's system, that describe the process of penetration of a magnetic field into a substance [1], can be rewritten in the following form [2]:

$$\frac{\partial H}{\partial t} = -rot\left[a\left(\int_0^t |rotH|^2\,d\tau\right) rotH\right], \tag{1.1}$$

where $H = (H_1, H_2, H_3)$ is a vector of the magnetic field and the function $a = a(S)$ is defined for $S \in [0, \infty)$.

If the magnetic field has the form $H = (0, U, V)$ and $U = U(x, t)$, $V = V(x, t)$, then we have

$$rot\,(a(S)rotH) = \left(0, -\frac{\partial}{\partial x}\left(a(S)\frac{\partial U}{\partial x}\right), -\frac{\partial}{\partial x}\left(a(S)\frac{\partial V}{\partial x}\right)\right).$$

Therefore, we obtain the following system of nonlinear integro-differential equations:

$$\begin{aligned}
\frac{\partial U}{\partial t} &= \frac{\partial}{\partial x}\left[a\left(\int_0^t\left[\left(\frac{\partial U}{\partial x}\right)^2+\left(\frac{\partial V}{\partial x}\right)^2\right]d\tau\right)\frac{\partial U}{\partial x}\right],\\
\frac{\partial V}{\partial t} &= \frac{\partial}{\partial x}\left[a\left(\int_0^t\left[\left(\frac{\partial U}{\partial x}\right)^2+\left(\frac{\partial V}{\partial x}\right)^2\right]d\tau\right)\frac{\partial V}{\partial x}\right].
\end{aligned} \tag{1.2}$$

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0898-1221/$ – see front matter. Published by Elsevier Ltd
doi:10.1016/j.camwa.2009.07.052

Note that the system (1.2) is complex, but special cases were investigated, see [2–7]. The existence of global solutions for initial-boundary value problems of such models have been proven in [2,3,7] by using the Galerkin and compactness methods [8,9]. For solvability and uniqueness properties for initial-boundary value problems (1.2), see e.g. [4–6]. The asymptotic behavior of the solutions of (1.2) have been the subject of intensive research in recent years, (see e.g. [7,10]).

Laptev [5] proposed some generalization of equations of type (1.1). Assume that the temperature of the considered body is constant throughout the material, i.e., depending on time, but independent of the space coordinates. If the magnetic field again has the form $H = (0, U, V)$ and $U = U(x, t)$, $V = V(x, t)$, then the same process of penetration of the magnetic field into the material is modeled by the following system of integro-differential equations [5]:

$$\begin{aligned} \frac{\partial U}{\partial t} &= a\left(\int_0^t \int_0^1 \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] \mathrm{d}x\mathrm{d}\tau\right) \frac{\partial^2 U}{\partial x^2}, \\ \frac{\partial V}{\partial t} &= a\left(\int_0^t \int_0^1 \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] \mathrm{d}x\mathrm{d}\tau\right) \frac{\partial^2 V}{\partial x^2}. \end{aligned} \tag{1.3}$$

The purpose of this work is to study the asymptotic behavior of solutions of the initial-boundary value problem for the system (1.3) and the convergence of the finite difference approximation for the case $a(S) = 1+S$. The solvability, uniqueness and asymptotics to the solutions of (1.3) type scalar models are studied in [7,11].

Note that in [12,13] difference schemes for (1.2) type models were investigated. Difference schemes for one nonlinear parabolic integro-differential scalar model similar to (1.2) were studied in [14]. Difference schemes for the scalar equation of (1.3) type with $a(S) = 1 + S$ were studied in [15].

The rest of the paper is organized as follows. In Section 2 large time behavior of solutions of the initial-boundary value problem with zero lateral boundary data for the system (1.3) with $a(S) = 1 + S$ is discussed. Section 3 is devoted to the study of the problem with non-zero boundary data in part of lateral boundary. In Section 4 the finite difference scheme for (1.3) is investigated. We close with a section on numerical implementations and present the numerical results comparing the decay rate to the theoretical results.

## 2. The problem with zero boundary conditions

Consider the following initial-boundary value problem:

$$\frac{\partial U}{\partial t} = (1+S)\frac{\partial^2 U}{\partial x^2}, \qquad \frac{\partial V}{\partial t} = (1+S)\frac{\partial^2 V}{\partial x^2}, \quad (x,t) \in Q = (0,1) \times (0,\infty), \tag{2.1}$$

$$U(0,t) = U(1,t) = V(0,t) = V(1,t) = 0, \quad t \geq 0, \tag{2.2}$$

$$U(x,0) = U_0(x), \qquad V(x,0) = V_0(x), \quad x \in [0,1], \tag{2.3}$$

where

$$S(t) = \int_0^t \int_0^1 \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] \mathrm{d}x\mathrm{d}\tau$$

and $U_0 = U_0(x)$, $V_0 = V_0(x)$ are given functions.

The existence and uniqueness of the solution of such problems in suitable classes are proved in [7]. Now we are going to estimate the solution of the problem (2.1)–(2.3).

Recall the $L_2$-inner product and norm:

$$(u, v) = \int_0^1 u(x)v(x)\mathrm{d}x, \qquad \|u\| = (u, u)^{1/2}.$$

We use the well-known Sobolev spaces $H^k(0, 1)$ and $H_0^k(0, 1)$.

**Theorem 2.1.** *If $U_0, V_0 \in H_0^1(0, 1)$, then for the solution of problem (2.1)–(2.3) the following estimate is true*

$$\|U\| + \left\|\frac{\partial U}{\partial x}\right\| + \|V\| + \left\|\frac{\partial V}{\partial x}\right\| \leq C \exp\left(-\frac{t}{2}\right).$$

Remark: Note that here and in the following second and third sections, $C$, $C_i$ and $c$ denote positive constants independent of $t$.

**Proof.** Let us multiply the first equation of the system (2.1) by $U$ and integrate on the interval $(0, 1)$. Using the boundary conditions (2.2) and integration by parts, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|U\|^2 + \int_0^1 (1+S)\left(\frac{\partial U}{\partial x}\right)^2 \mathrm{d}x = 0.$$

From this, using Poincare's inequality and the nonnegativity of $S(t)$ we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|U\|^2 + \left\|\frac{\partial U}{\partial x}\right\|^2 \le 0, \qquad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|U\|^2 + \|U\|^2 \le 0. \tag{2.4}$$

Analogously,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|V\|^2 + \left\|\frac{\partial V}{\partial x}\right\|^2 \le 0, \qquad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|V\|^2 + \|V\|^2 \le 0. \tag{2.5}$$

Let us multiply the first equation of the system (2.1) by $\partial^2 U/\partial x^2$. Using again integration by parts we have

$$\left.\frac{\partial U}{\partial t}\frac{\partial U}{\partial x}\right|_0^1 - \int_0^1 \frac{\partial^2 U}{\partial t\partial x}\frac{\partial U}{\partial x}\mathrm{d}x = \int_0^1 (1+S)\left(\frac{\partial^2 U}{\partial x^2}\right)^2 \mathrm{d}x.$$

Taking into account (2.2), from the last equality we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{\partial U}{\partial x}\right\|^2 + (1+S)\left\|\frac{\partial^2 U}{\partial x^2}\right\|^2 = 0, \tag{2.6}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{\partial U}{\partial x}\right\|^2 \le 0. \tag{2.7}$$

Analogously,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{\partial V}{\partial x}\right\|^2 \le 0. \tag{2.8}$$

Using inequalities (2.4), (2.5), (2.7) and (2.8) we receive

$$\begin{aligned}
&\exp(t)\frac{\mathrm{d}}{\mathrm{d}t}\left(\|U\|^2 + \|V\|^2\right) + \exp(t)\left(\|U\|^2 + \|V\|^2\right) \\
&\quad + \exp(t)\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\frac{\partial U}{\partial x}\right\|^2 + \left\|\frac{\partial V}{\partial x}\right\|^2\right) + \exp(t)\left(\left\|\frac{\partial U}{\partial x}\right\|^2 + \left\|\frac{\partial V}{\partial x}\right\|^2\right) \le 0.
\end{aligned}$$

From this we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\exp(t)\left(\|U\|^2 + \|V\|^2 + \left\|\frac{\partial U}{\partial x}\right\|^2 + \left\|\frac{\partial V}{\partial x}\right\|^2\right)\right] \le 0.$$

This inequality immediately proves Theorem 2.1. □

Note that Theorem 2.1 gives exponential stabilization of the solution of the problem (2.1)–(2.3) in the norm of the space $H^1(0, 1)$. Let us show that the stabilization is also achieved in the norm of the space $C^1(0, 1)$. In particular, let us show that the following statement holds.

**Theorem 2.2.** *If $U_0, V_0 \in H^4(0, 1) \cap H_0^1(0, 1)$, then for the solution of problem (2.1)–(2.3) the following relations hold:*

$$\left|\frac{\partial U(x,t)}{\partial x}\right| \le C\exp\left(-\frac{t}{2}\right), \qquad \left|\frac{\partial V(x,t)}{\partial x}\right| \le C\exp\left(-\frac{t}{2}\right),$$
$$\left|\frac{\partial U(x,t)}{\partial t}\right| \le C\exp\left(-\frac{t}{2}\right), \qquad \left|\frac{\partial V(x,t)}{\partial t}\right| \le C\exp\left(-\frac{t}{2}\right).$$

To this end we need the following Lemma.

**Lemma 2.1.** *For the solution of problem (2.1)–(2.3) the following estimate holds*

$$\left\|\frac{\partial U(x,t)}{\partial t}\right\| + \left\|\frac{\partial V(x,t)}{\partial t}\right\| \le C\exp\left(-\frac{t}{2}\right).$$

**Proof.** Let us differentiate the first equation of the system (2.1) with respect to $t$

$$\frac{\partial^2 U}{\partial t^2} = (1+S)\frac{\partial^3 U}{\partial x^2 \partial t} + \left(\int_0^1 \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] dx\right)\frac{\partial^2 U}{\partial x^2} \tag{2.9}$$

and multiply by $\partial U/\partial t$. Using integration by parts and boundary conditions (2.2), we deduce

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \left(\frac{\partial U}{\partial t}\right)^2 dx + (1+S)\int_0^1 \left(\frac{\partial^2 U}{\partial x\partial t}\right)^2 dx + \left(\int_0^1 \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] dx\right)\int_0^1 \frac{\partial U}{\partial x}\frac{\partial^2 U}{\partial x\partial t} dx = 0,$$

or

$$\frac{d}{dt}\int_0^1 \left(\frac{\partial U}{\partial t}\right)^2 dx + 2(1+S)\int_0^1 \left(\frac{\partial^2 U}{\partial x\partial t}\right)^2 dx = -2\left(\int_0^1 \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] dx\right)\int_0^1 \frac{\partial U}{\partial x}\frac{\partial^2 U}{\partial x\partial t} dx.$$

Now the right-hand side can be estimated as follows:

$$\begin{aligned}
&-2\left(\int_0^1 \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] dx\right)\int_0^1 \frac{\partial U}{\partial x}\frac{\partial^2 U}{\partial x\partial t} dx \\
&\quad\le \left|\int_0^1 2\left\{(1+S)^{-1/2}\left(\int_0^1 \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] dx\right)\frac{\partial U}{\partial x}\right\}\left\{(1+S)^{1/2}\frac{\partial^2 U}{\partial x\partial t}\right\} dx\right| \\
&\quad\le \int_0^1 (1+S)^{-1}\left\{\left(\int_0^1 \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] dx\right)\frac{\partial U}{\partial x}\right\}^2 dx + \int_0^1 (1+S)\left(\frac{\partial^2 U}{\partial x\partial t}\right)^2 dx.
\end{aligned}$$

Therefore,

$$\frac{d}{dt}\int_0^1 \left(\frac{\partial U}{\partial t}\right)^2 dx + (1+S)\int_0^1 \left(\frac{\partial^2 U}{\partial x\partial t}\right)^2 dx \le (1+S)^{-1}\left\{\int_0^1 \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] dx\right\}^2 \int_0^1 \left(\frac{\partial U}{\partial x}\right)^2 dx. \tag{2.10}$$

Using Poincare's inequality, Theorem 2.1, the nonnegativity of $S(t)$ and relation (2.10) we arrive at

$$\frac{d}{dt}\left(\exp(t)\left\|\frac{\partial U}{\partial t}\right\|^2\right) \le C\exp(-2t),$$

or

$$\left\|\frac{\partial U}{\partial t}\right\| \le C\exp\left(-\frac{t}{2}\right).$$

A similar argument show that

$$\left\|\frac{\partial V}{\partial t}\right\| \le C\exp\left(-\frac{t}{2}\right).$$

This proves Lemma 2.1. □

Now we turn to the proof of Theorem 2.1.

**Proof.** Let us estimate $\partial^2 U/\partial x^2$ in the norm of the space $L_1(0,1)$. From the first equation of the system (2.1) we have

$$\frac{\partial^2 U}{\partial x^2} = (1+S)^{-1}\frac{\partial U}{\partial t}. \tag{2.11}$$

Integrating on $(0,1)$ and using Schwarz's inequality we get

$$\int_0^1 \left|\frac{\partial^2 U}{\partial x^2}\right| dx = \int_0^1 \left|(1+S)^{-1}\frac{\partial U}{\partial t}\right| dx \le \left[\int_0^1 (1+S)^{-2} dx\right]^{1/2}\left[\int_0^1 \left(\frac{\partial U}{\partial t}\right)^2 dx\right]^{1/2}.$$

Applying Lemma 2.1 and taking into account the nonnegativity of $S(t)$ we derive

$$\int_0^1 \left|\frac{\partial^2 U}{\partial x^2}\right| dx \le C \exp\left(-\frac{t}{2}\right).$$

From this, taking into account the relation

$$\frac{\partial U(x,t)}{\partial x} = \int_0^1 \frac{\partial U(y,t)}{\partial y} dy + \int_0^1 \int_y^x \frac{\partial^2 U(\xi,t)}{\partial \xi^2} d\xi\, dy$$

and the boundary conditions (2.2), it follows that

$$\left|\frac{\partial U(x,t)}{\partial x}\right| = \left|\int_0^1 \int_y^x \frac{\partial^2 U(\xi,t)}{\partial \xi^2} d\xi\, dy\right| \le \int_0^1 \left|\frac{\partial^2 U(y,t)}{\partial y^2}\right| dy \le C \exp\left(-\frac{t}{2}\right).$$

Analogously,

$$\left|\frac{\partial V(x,t)}{\partial x}\right| \le C \exp\left(-\frac{t}{2}\right).$$

At the next step, let us estimate $\partial U/\partial t$ in the norm of the space $C^1(0,1)$. Let us multiply the first equation of the system (2.1) by $\partial^3 U/\partial x^2 \partial t$. Using integration by parts we get

$$\frac{\partial U}{\partial t} \frac{\partial^2 U}{\partial x \partial t}\bigg|_0^1 - \left\|\frac{\partial^2 U}{\partial x \partial t}\right\|^2 = (1+S) \int_0^1 \frac{\partial^2 U}{\partial x^2} \frac{\partial^3 U}{\partial x^2 \partial t} dx. \tag{2.12}$$

Taking into account the equality

$$\int_0^1 \frac{\partial^3 U}{\partial x^2 \partial t} \frac{\partial^2 U}{\partial x^2} dx = \frac{1}{2} \frac{d}{dt} \left\|\frac{\partial^2 U}{\partial x^2}\right\|^2$$

and the boundary conditions (2.2) we arrive at

$$\frac{1+S}{2} \frac{d}{dt} \left\|\frac{\partial^2 U}{\partial x^2}\right\|^2 + \left\|\frac{\partial^2 U}{\partial x \partial t}\right\|^2 = 0,$$

or

$$\frac{d}{dt} \left\|\frac{\partial^2 U}{\partial x^2}\right\|^2 \le 0. \tag{2.13}$$

Note that from (2.12) we have

$$\left\|\frac{\partial^2 U}{\partial x \partial t}\right\|^2 \le \frac{1+S}{2} \left\|\frac{\partial^2 U}{\partial x^2}\right\|^2 + \frac{1+S}{2} \left\|\frac{\partial^3 U}{\partial x^2 \partial t}\right\|^2. \tag{2.14}$$

Let us multiply the Eq. (2.9) by $\partial^3 U/\partial x^2 \partial t$. Integration by parts gives

$$\frac{\partial^2 U}{\partial t^2} \frac{\partial^2 U}{\partial x \partial t}\bigg|_0^1 - \int_0^1 \frac{\partial^3 U}{\partial x \partial t^2} \frac{\partial^2 U}{\partial x \partial t} dx = (1+S) \left\|\frac{\partial^3 U}{\partial x^2 \partial t}\right\|^2 + \left(\int_0^1 \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] dx\right) \int_0^1 \frac{\partial^2 U}{\partial x^2} \frac{\partial^3 U}{\partial x^2 \partial t} dx.$$

The last equality, by taking into account boundary conditions (2.2), can be rewritten as follows

$$\frac{d}{dt} \left\|\frac{\partial^2 U}{\partial x \partial t}\right\|^2 + 2(1+S) \left\|\frac{\partial^3 U}{\partial x^2 \partial t}\right\|^2 = -2 \left(\int_0^1 \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] dx\right) \int_0^1 \frac{\partial^2 U}{\partial x^2} \frac{\partial^3 U}{\partial x^2 \partial t} dx.$$

We estimate the right-hand side in a similar fashion as we have done to obtain (2.10). It is easy to see that

$$\frac{d}{dt} \left\|\frac{\partial^2 U}{\partial x \partial t}\right\|^2 + (1+S) \left\|\frac{\partial^3 U}{\partial x^2 \partial t}\right\|^2 \le (1+S)^{-1} \left\{\int_0^1 \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] dx\right\}^2 \int_0^1 \left(\frac{\partial^2 U}{\partial x^2}\right)^2 dx.$$

Using Theorem 2.1, relation (2.11) and Lemma 2.1 we arrive at

$$\frac{d}{dt} \left\|\frac{\partial^2 U}{\partial x \partial t}\right\|^2 + (1+S) \left\|\frac{\partial^3 U}{\partial x^2 \partial t}\right\|^2 \le C \exp(-3t). \tag{2.15}$$

Combining (2.4), (2.6) and (2.13)–(2.15) we get

$$\begin{aligned}
&\|U\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\|U\|^2 + \left\|\frac{\partial U}{\partial x}\right\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{\partial U}{\partial x}\right\|^2 + 2(1+S)\left\|\frac{\partial^2 U}{\partial x^2}\right\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{\partial^2 U}{\partial x^2}\right\|^2 \\
&\quad + \left\|\frac{\partial^2 U}{\partial x \partial t}\right\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{\partial^2 U}{\partial x \partial t}\right\|^2 + (1+S)\left\|\frac{\partial^3 U}{\partial x^2 \partial t}\right\|^2 \\
&\quad \le \frac{1+S}{2}\left\|\frac{\partial^2 U}{\partial x^2}\right\|^2 + \frac{1+S}{2}\left\|\frac{\partial^3 U}{\partial x^2 \partial t}\right\|^2 + C\exp(-3t).
\end{aligned}$$

From this, keeping in mind the nonnegativity of $S(t)$, we deduce

$$\|U\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\|U\|^2 + \left\|\frac{\partial U}{\partial x}\right\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{\partial U}{\partial x}\right\|^2 + \left\|\frac{\partial^2 U}{\partial x^2}\right\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{\partial^2 U}{\partial x^2}\right\|^2 + \left\|\frac{\partial^2 U}{\partial x \partial t}\right\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{\partial^2 U}{\partial x \partial t}\right\|^2 \le C\exp(-3t).$$

After multiplying by the function $\exp(t)$ we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\exp(t)\left(\|U\|^2 + \left\|\frac{\partial U}{\partial x}\right\|^2 + \left\|\frac{\partial^2 U}{\partial x^2}\right\|^2 + \left\|\frac{\partial^2 U}{\partial x \partial t}\right\|^2\right)\right] \le C\exp(-2t).$$

Integration from 0 to $t$ gives

$$\|U\|^2 + \left\|\frac{\partial U}{\partial x}\right\|^2 + \left\|\frac{\partial^2 U}{\partial x^2}\right\|^2 + \left\|\frac{\partial^2 U}{\partial x \partial t}\right\|^2 \le C\exp(-t).$$

From this, taking into account Lemma 2.1, it follows that

$$\begin{aligned}
\left|\frac{\partial U(x,t)}{\partial t}\right| &= \left|\int_0^1 \frac{\partial U(y,t)}{\partial t}\mathrm{d}y + \int_0^1\int_y^x \frac{\partial^2 U(\xi,t)}{\partial \xi \partial t}\mathrm{d}\xi\,\mathrm{d}y\right| \\
&\le \left[\int_0^1 \left(\frac{\partial U(x,t)}{\partial t}\right)^2 \mathrm{d}x\right]^{1/2} + \int_0^1 \left|\frac{\partial^2 U(y,t)}{\partial y \partial t}\right| \mathrm{d}y \le C\exp\left(-\frac{t}{2}\right).
\end{aligned}$$

Analogously,

$$\left|\frac{\partial V(x,t)}{\partial t}\right| \le C\exp\left(-\frac{t}{2}\right).$$

This completes the proof of Theorem 2.2. □

## 3. The problem with non-zero data on one side of lateral boundary

Consider again the system:

$$\frac{\partial U}{\partial t} = (1+S)\frac{\partial^2 U}{\partial x^2}, \qquad \frac{\partial V}{\partial t} = (1+S)\frac{\partial^2 V}{\partial x^2}, \tag{3.1}$$

where as before

$$S(t) = \int_0^t \int_0^1 \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] \mathrm{d}x\mathrm{d}\tau. \tag{3.2}$$

In the domain $Q$ for the system (3.1) and (3.2) let us consider the following initial-boundary value problem:

$$U(0,t) = V(0,t) = 0, \qquad U(1,t) = \psi_1, \qquad (1,t) = \psi_2, \quad t \ge 0, \tag{3.3}$$

$$U(x,0) = U_0(x), \qquad V(x,0) = V_0(x), \quad x \in [0,1], \tag{3.4}$$

where $\psi_1 = Const \ge 0$, $\psi_2 = Const \ge 0$, $\psi_1^2 + \psi_2^2 \ne 0$; $U_0 = U_0(x)$ and $V_0 = V_0(x)$ are given functions.

The main result of this section can be formulated as follow.

**Theorem 3.1.** *If $U_0(0) = V_0(0) = 0$, $U_0(1) = \psi_1$, $V_0(1) = \psi_2$, $\psi_1^2 + \psi_2^2 \neq 0$, $U_0, V_0 \in H^3(0,1)$, then for the solution of problem (3.1)–(3.4) the following estimates are true:*

$$\left|\frac{\partial U(x,t)}{\partial x} - \psi_1\right| \le C(1+t)^{-2}, \qquad \left|\frac{\partial V(x,t)}{\partial x} - \psi_2\right| \le C(1+t)^{-2}, \quad t \ge 0,$$
$$\left|\frac{\partial U(x,t)}{\partial t}\right| \le C(1+t)^{-1}, \qquad \left|\frac{\partial V(x,t)}{\partial t}\right| \le C(1+t)^{-1}, \quad t \ge 0.$$

Before we proceed to the proof of Theorem 3.1, we state and prove some auxiliary lemmas.

**Lemma 3.1.** *Following estimates are true:*

$$\varphi^{\frac{1}{3}}(t) \le 1 + S(t) \le C\varphi^{\frac{1}{3}}(t), \quad t \ge 0,$$

*where*

$$\varphi(t) = 1 + \int_0^t \int_0^1 (1+S)^2 \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] \mathrm{d}x\mathrm{d}\tau. \tag{3.5}$$

**Proof.** From (3.2) it follows that

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \int_0^1 \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] \mathrm{d}x, \qquad S(0) = 0. \tag{3.6}$$

Let us multiply the first equality of (3.6) by $(1+S)^2$ and introduce following notations:

$$\sigma_1 = (1+S)\frac{\partial U}{\partial x}, \qquad \sigma_2 = (1+S)\frac{\partial V}{\partial x}.$$

We have

$$\frac{1}{3}\frac{\mathrm{d}(1+S)^3}{\mathrm{d}t} = \int_0^1 \left(\sigma_1^2 + \sigma_2^2\right) \mathrm{d}x. \tag{3.7}$$

Integrating Eq. (3.7) on $(0,t)$ we get

$$\frac{(1+S)^3}{3} = \int_0^t \int_0^1 \left(\sigma_1^2 + \sigma_2^2\right) \mathrm{d}x\mathrm{d}\tau + \frac{1}{3},$$

or, taking into account (3.5)

$$\varphi^{\frac{1}{3}}(t) \le 1 + S(t) \le [3\varphi(t)]^{\frac{1}{3}}.$$

So, Lemma 3.1 is proved. □

**Lemma 3.2.** *The following estimates are true:*

$$c\varphi^{\frac{2}{3}}(t) \le \int_0^1 \left(\sigma_1^2 + \sigma_2^2\right) \mathrm{d}x \le C\varphi^{\frac{2}{3}}(t), \quad t \ge 0.$$

**Proof.** Taking into account Lemma 3.1 and the boundary conditions we get

$$\int_0^1 \left(\sigma_1^2 + \sigma_2^2\right) \mathrm{d}x = \int_0^1 (1+S)^2 \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] \mathrm{d}x \ge \varphi^{\frac{2}{3}}(t) \left(\int_0^1 \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] \mathrm{d}x\right)$$
$$\ge \varphi^{\frac{2}{3}}(t) \left\{\left[\int_0^1 \frac{\partial U}{\partial x}\mathrm{d}x\right]^2 + \left[\int_0^1 \frac{\partial V}{\partial x}\mathrm{d}x\right]^2\right\} = \left(\psi_1^2 + \psi_2^2\right) \varphi^{\frac{2}{3}}(t),$$

or

$$\int_0^1 \left(\sigma_1^2 + \sigma_2^2\right) \mathrm{d}x \ge c\varphi^{\frac{2}{3}}(t). \tag{3.8}$$

Let us multiply the first equation of (3.1) by $(1+S)^{-1}\partial U/\partial t$ and integrate on the domain $(0,1)\times(0,t)$. Using conditions (3.3), (3.4) and integration by parts we have

$$\int_0^t \int_0^1 (1+S)^{-1} \left(\frac{\partial U}{\partial \tau}\right)^2 \mathrm{d}x\mathrm{d}\tau + \frac{1}{2}\int_0^1 \left(\frac{\partial U}{\partial x}\right)^2 \mathrm{d}x - \frac{1}{2}\int_0^1 \left(\frac{\mathrm{d}U_0}{\mathrm{d}x}\right)^2 \mathrm{d}x = 0.$$

From this we get

$$\int_0^1 \left(\frac{\partial U}{\partial x}\right)^2 \mathrm{d}x \le C. \tag{3.9}$$

Analogously,

$$\int_0^1 \left(\frac{\partial V}{\partial x}\right)^2 \mathrm{d}x \le C. \tag{3.10}$$

From (3.9), (3.10) and Lemma 3.1 we conclude

$$\int_0^1 \left(\sigma_1^2 + \sigma_2^2\right) \mathrm{d}x = (1+S)^2 \int_0^1 \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] \mathrm{d}x \le C\varphi^{\frac{2}{3}}(t).$$

So, the last inequality with estimate (3.8), proves Lemma 3.2. □

From Lemma 3.2 and relation (3.5) we receive following estimates:

$$c\varphi^{\frac{2}{3}}(t) \le \frac{\mathrm{d}\varphi(t)}{\mathrm{d}t} \le C\varphi^{\frac{2}{3}}(t), \quad t \ge 0.$$

Integrating this inequalities one can easily get

$$\left(1 + \frac{c}{3}t\right)^3 \le \varphi(t) \le \left(1 + \frac{C}{3}t\right)^3,$$

or

$$c\,(1+t)^3 \le \varphi(t) \le C\,(1+t)^3.$$

From this, taking into account Lemma 3.1 we get the following estimate:

$$c\,(1+t) \le 1 + S(x,t) \le C\,(1+t), \quad t \ge 0. \tag{3.11}$$

**Lemma 3.3.** *The derivatives $\partial U/\partial t$ and $\partial V/\partial t$ satisfy the inequality*

$$\left\|\frac{\partial U}{\partial t}\right\| + \left\|\frac{\partial V}{\partial t}\right\| \le C(1+t)^{-1}, \quad t \ge 0.$$

**Proof.** Note that inequality (2.10) is valid for the problem (3.1)–(3.4) as well. So, from (2.10), using Poincare's inequality and relations (3.9)–(3.11) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \left(\frac{\partial U}{\partial t}\right)^2 \mathrm{d}x + c(1+t)\int_0^1 \left(\frac{\partial U}{\partial t}\right)^2 \mathrm{d}x \le C(1+t)^{-1}.$$

Using Gronwall's inequality we arrive at

$$\begin{aligned}\int_0^1 \left(\frac{\partial U}{\partial t}\right)^2 \mathrm{d}x &\le \exp\left(-c\int_0^t (1+\tau)\,\mathrm{d}\tau\right)\left\{\int_0^1 \left(\frac{\partial U}{\partial t}\right)^2 \mathrm{d}x\bigg|_{t=0} + C\int_0^t \exp\left(c\int_0^\tau (1+\xi)\,\mathrm{d}\xi\right)(1+\tau)^{-1}\,\mathrm{d}\tau\right\} \\ &= C_1 \exp\left(-\frac{c(1+t)^2}{2}\right)\left[C_2 + C_3\int_0^t \exp\left(\frac{c(1+\tau)^2}{2}\right)(1+\tau)^{-1}\mathrm{d}\tau\right].\end{aligned} \tag{3.12}$$

Applying L'Hospital's rule we obtain

$$\begin{aligned}\lim_{t\to\infty}\frac{\int_0^t \exp\left(\frac{c(1+\tau)^2}{2}\right)(1+\tau)^{-1}\mathrm{d}\tau}{\exp\left(\frac{c(1+t)^2}{2}\right)(1+t)^{-2}} &= \lim_{t\to\infty}\frac{\exp\left(\frac{c(1+t)^2}{2}\right)(1+t)^{-1}}{\exp\left(\frac{c(1+t)^2}{2}\right)(1+t)^{-1}\left[c - 2(1+t)^{-2}\right]} \\ &= \lim_{t\to\infty}\frac{1}{c - 2(1+t)^{-2}} = C.\end{aligned} \tag{3.13}$$

Therefore, from (3.12) and (3.13) we get

$$\int_0^1 \left(\frac{\partial U}{\partial t}\right)^2 \mathrm{d}x \le C(1+t)^{-2}.$$

Analogously,

$$\int_0^1 \left(\frac{\partial V}{\partial t}\right)^2 \mathrm{d}x \le C(1+t)^{-2}.$$

So, Lemma 3.3 is proved. □

Now we are ready to prove Theorem 3.1.

**Proof.** According to the method applied in Section 2, taking into account Lemma 3.3 and the estimate (3.11), we derive

$$\begin{aligned}\left|\frac{\partial U(x,t)}{\partial x} - \psi_1\right| &= \left|\int_0^1 \int_y^x \frac{\partial^2 U(\xi,t)}{\partial \xi^2}\mathrm{d}\xi\,\mathrm{d}y\right| \le \int_0^1 \left|\frac{\partial^2 U(x,t)}{\partial x^2}\right| \mathrm{d}x \\ &\le \int_0^1 \left|(1+S)^{-1}\frac{\partial U}{\partial t}\right| \mathrm{d}x \le \left[\int_0^1 (1+S)^{-2}\mathrm{d}x\right]^{1/2} \left[\int_0^1 \left|\frac{\partial U}{\partial t}\right|^2 \mathrm{d}x\right]^{1/2} \le C(1+t)^{-2}.\end{aligned}$$

Hence, we have

$$\left|\frac{\partial U(x,t)}{\partial x} - \psi_1\right| \le C(1+t)^{-2}. \tag{3.14}$$

Analogously,

$$\left|\frac{\partial V(x,t)}{\partial x} - \psi_2\right| \le C(1+t)^{-2}. \tag{3.15}$$

Now let us estimate $\partial U/\partial t$ and $\partial V/\partial t$. For this let us multiply (2.10) by $(1+t)^2$. Keeping in mind estimates (3.9)–(3.11), we arrive at

$$\int_0^t (1+\tau)^2 \frac{d}{\mathrm{d}\tau}\left(\int_0^1 \left(\frac{\partial U}{\partial \tau}\right)^2 \mathrm{d}x\right)\mathrm{d}\tau + c\int_0^t (1+\tau)^3 \left(\int_0^1 \left(\frac{\partial^2 U}{\partial \tau \partial x}\right)^2 \mathrm{d}x\right)\mathrm{d}\tau \le C\int_0^t (1+\tau)\mathrm{d}\tau.$$

Integrating last inequality on $(0,t)$, using integration by parts, estimate (3.11) and Lemma 3.3 we get

$$\begin{aligned} c\int_0^t (1+\tau)^3 \left(\int_0^1 \left(\frac{\partial^2 U}{\partial \tau \partial x}\right)^2 \mathrm{d}x\right)\mathrm{d}\tau &\le -(1+t)^2 \int_0^1 \left(\frac{\partial U}{\partial t}\right)^2 \mathrm{d}x + \int_0^1 \left(\frac{\partial U}{\partial t}\right)^2 \mathrm{d}x\bigg|_{t=0} \\ &\quad + 2\int_0^t (1+\tau)\left(\int_0^1 \left(\frac{\partial U}{\partial \tau}\right)^2 \mathrm{d}x\right)\mathrm{d}\tau + \frac{1}{2}\left[(1+t)^2 - 1\right] \\ &\le C_1 + C_2\int_0^t (1+\tau)^{-1}\mathrm{d}\tau - \frac{1}{2} + \frac{1}{2}(1+t)^2 \le C(1+t)^2,\end{aligned}$$

or

$$\int_0^t (1+\tau)^3 \left(\int_0^1 \left(\frac{\partial^2 U}{\partial \tau \partial x}\right)^2 \mathrm{d}x\right)\mathrm{d}\tau \le C(1+t)^2. \tag{3.16}$$

In an analogous way we can obtain

$$\int_0^t (1+\tau)^3 \left(\int_0^1 \left(\frac{\partial^2 V}{\partial \tau \partial x}\right)^2 \mathrm{d}x\right)\mathrm{d}\tau \le C(1+t)^2. \tag{3.17}$$

Let us multiply (2.9) by $(1+t)^3\partial^2 U/\partial t^2$

$$\begin{aligned}\int_0^1 (1+t)^3 \left(\frac{\partial^2 U}{\partial t^2}\right)^2 \mathrm{d}x &= \int_0^1 (1+t)^3 (1+S)\frac{\partial^3 U}{\partial x^2 \partial t}\frac{\partial^2 U}{\partial t^2}\mathrm{d}x \\ &\quad + \int_0^1 (1+t)^3 \left(\int_0^1 \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right]\mathrm{d}x\right)\frac{\partial^2 U}{\partial x^2}\frac{\partial^2 U}{\partial t^2}\mathrm{d}x.\end{aligned}$$

Integration by parts and using the boundary conditions (3.3), gives

$$\int_0^1 (1+t)^3 \left(\frac{\partial^2 U}{\partial t^2}\right)^2 \mathrm{d}x + \int_0^1 (1+t)^3 (1+S) \frac{\partial^2 U}{\partial x \partial t} \frac{\partial^3 U}{\partial t^2 \partial x} \mathrm{d}x$$
$$+ \int_0^1 (1+t)^3 \left(\int_0^1 \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] \mathrm{d}x\right) \frac{\partial U}{\partial x} \frac{\partial^3 U}{\partial t^2 \partial x} \mathrm{d}x = 0.$$

After integrating over $(0, t)$ we arrive at

$$\int_0^t \int_0^1 (1+\tau)^3 \left(\frac{\partial^2 U}{\partial \tau^2}\right)^2 \mathrm{d}x\mathrm{d}\tau + \frac{1}{2}\int_0^t \int_0^1 (1+\tau)^3 (1+S) \frac{\partial}{\partial \tau} \left(\frac{\partial^2 U}{\partial \tau \partial x}\right)^2 \mathrm{d}x\mathrm{d}\tau$$
$$+ \int_0^t (1+\tau)^3 \left(\int_0^1 \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] \mathrm{d}x\right) \left(\int_0^1 \frac{\partial U}{\partial x} \frac{\partial}{\partial \tau} \left(\frac{\partial^2 U}{\partial \tau \partial x}\right) \mathrm{d}x\right) \mathrm{d}\tau = 0.$$

Integration by parts again and taking into account (3.6) we get

$$\frac{(1+t)^3(1+S)}{2} \int_0^1 \left(\frac{\partial^2 U}{\partial t \partial x}\right)^2 \mathrm{d}x - \frac{1}{2}\int_0^1 \left(\frac{\partial^2 U}{\partial t \partial x}\right)^2 \mathrm{d}x \bigg|_{t=0}$$
$$\le \frac{3}{2}\int_0^t \int_0^1 (1+\tau)^2 (1+S) \left(\frac{\partial^2 U}{\partial \tau \partial x}\right)^2 \mathrm{d}x\mathrm{d}\tau$$
$$+ \frac{1}{2}\int_0^t (1+\tau)^3 \left(\int_0^1 \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] \mathrm{d}x\right) \left(\int_0^1 \left(\frac{\partial^2 U}{\partial \tau \partial x}\right)^2 \mathrm{d}x\right) \mathrm{d}\tau$$
$$- (1+t)^3 \left(\int_0^1 \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] \mathrm{d}x\right) \left(\int_0^1 \frac{\partial U}{\partial x} \frac{\partial^2 U}{\partial t \partial x} \mathrm{d}x\right)$$
$$+ \left(\int_0^1 \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] \mathrm{d}x\right) \left(\int_0^1 \frac{\partial U}{\partial x} \frac{\partial^2 U}{\partial t \partial x} \mathrm{d}x\right) \bigg|_{t=0}$$
$$+ 3\int_0^t (1+\tau)^2 \left(\int_0^1 \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] \mathrm{d}x\right) \left(\int_0^1 \frac{\partial U}{\partial x} \frac{\partial^2 U}{\partial \tau \partial x} \mathrm{d}x\right) \mathrm{d}\tau$$
$$+ \int_0^t (1+\tau)^3 \frac{d}{\mathrm{d}\tau} \left(\int_0^1 \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] \mathrm{d}x\right) \left(\int_0^1 \frac{\partial U}{\partial x} \frac{\partial^2 U}{\partial \tau \partial x} \mathrm{d}x\right) \mathrm{d}\tau$$
$$+ \int_0^t (1+\tau)^3 \left(\int_0^1 \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] \mathrm{d}x\right) \left(\int_0^1 \left(\frac{\partial^2 U}{\partial \tau \partial x}\right)^2 \mathrm{d}x\right) \mathrm{d}\tau.$$

By using Schwarz's inequality in the last relation, keeping in mind estimates (3.9)–(3.11), we deduce

$$\frac{c}{2}(1+t)^4 \int_0^1 \left(\frac{\partial^2 U}{\partial t \partial x}\right)^2 \mathrm{d}x \le C_1 + C_2 \int_0^t \int_0^1 (1+\tau)^3 \left(\frac{\partial^2 U}{\partial \tau \partial x}\right)^2 \mathrm{d}x\mathrm{d}\tau$$
$$+ C_3 \int_0^t (1+\tau)^3 \left(\int_0^1 \left(\frac{\partial^2 U}{\partial \tau \partial x}\right)^2 \mathrm{d}x\right) \mathrm{d}\tau + \frac{c}{4}(1+t)^4 \int_0^1 \left(\frac{\partial^2 U}{\partial t \partial x}\right)^2 \mathrm{d}x$$
$$+ C_4 (1+t)^2 \int_0^1 \left(\frac{\partial U}{\partial x}\right)^2 \mathrm{d}x + \left(\left\|\frac{\partial U}{\partial x}\right\|^2 + \left\|\frac{\partial V}{\partial x}\right\|^2\right) \left\|\frac{\partial U}{\partial x}\right\| \left\|\frac{\partial^2 U}{\partial x \partial t}\right\| \bigg|_{t=0}$$
$$+ \int_0^t (1+\tau)^3 \left(\int_0^1 \left(\frac{\partial^2 U}{\partial \tau \partial x}\right)^2 \mathrm{d}x\right) \mathrm{d}\tau + C_5 \int_0^t (1+\tau)\mathrm{d}\tau$$
$$+ C_6 \int_0^t (1+\tau)^3 \left\{\left[\int_0^1 \left(\frac{\partial U}{\partial x}\right)^2 \mathrm{d}x\right]^{1/2} \left[\int_0^1 \left(\frac{\partial^2 U}{\partial x \partial \tau}\right)^2 \mathrm{d}x\right]^{1/2}\right.$$

$$+\left[\int_0^1\left(\frac{\partial V}{\partial x}\right)^2\mathrm{d}x\right]^{1/2}\left[\int_0^1\left(\frac{\partial^2 V}{\partial x\partial\tau}\right)^2\mathrm{d}x\right]^{1/2}\Bigg\}\left[\int_0^1\left(\frac{\partial U}{\partial x}\right)^2\mathrm{d}x\right]^{1/2}\left[\int_0^1\left(\frac{\partial^2 U}{\partial x\partial\tau}\right)^2\mathrm{d}x\right]^{1/2}\mathrm{d}\tau$$
$$+C_7\int_0^t(1+\tau)^3\left(\int_0^1\left(\frac{\partial^2 U}{\partial\tau\partial x}\right)^2\mathrm{d}x\right)\mathrm{d}\tau.$$

From this, taking into account estimates (3.9)–(3.11), (3.16) and (3.17), we get

$$\frac{c}{4}(1+t)^4\int_0^1\left(\frac{\partial^2 U}{\partial t\partial x}\right)^2\mathrm{d}x\le C_8+C_9(1+t)^2+C_{10}\int_0^t(1+\tau)\mathrm{d}\tau+C_{11}\int_0^t(1+\tau)^3\left(\int_0^1\left(\frac{\partial^2 U}{\partial\tau\partial x}\right)^2\mathrm{d}x\right)\mathrm{d}\tau$$
$$+C_{11}\int_0^t(1+\tau)^3\left[\int_0^1\left(\frac{\partial^2 V}{\partial\tau\partial x}\right)^2\mathrm{d}x\int_0^1\left(\frac{\partial^2 U}{\partial\tau\partial x}\right)^2\mathrm{d}x\right]^{1/2}\mathrm{d}\tau$$
$$\le C_{12}(1+t)^2+C_{13}\int_0^t(1+\tau)^3\left(\int_0^1\left(\frac{\partial^2 V}{\partial\tau\partial x}\right)^2\mathrm{d}x\right)\mathrm{d}\tau+C_{13}\int_0^t(1+\tau)^3\left(\int_0^1\left(\frac{\partial^2 U}{\partial\tau\partial x}\right)^2\mathrm{d}x\right)\mathrm{d}\tau\le C_{14}(1+t)^2,$$

or at last

$$\int_0^1\left(\frac{\partial^2 U}{\partial t\partial x}\right)^2\mathrm{d}x\le C(1+t)^{-2}.$$

From this, according to the scheme of the second section, we obtain

$$\left|\frac{\partial U(x,t)}{\partial t}\right|\le C(1+t)^{-1}.$$

Analogously,

$$\left|\frac{\partial V(x,t)}{\partial t}\right|\le C(1+t)^{-1}.$$

So, the proof of the main Theorem 3.1 of this section is over. □

Remarks:

1. Note that in this section we used a scheme similar to the scheme of [16] in which the adiabatic shearing of incompressible fluids with temperature-dependent viscosity is studied.
2. The existence of globally defined solutions of the problems (2.1)–(2.3) and (3.1)–(3.3) can be obtained by a routine procedure. One first establishes the existence of local solutions on a maximal time interval and then uses the derived a priori estimates to show that the solutions cannot escape in finite time (see, for example, [7–9]).
3. Mathematical results, that are given in the second and third sections, show difference between stabilization rates of solutions with homogeneous and nonhomogeneous boundary conditions.

## 4. Finite difference scheme

In the rectangle $Q_T=(0,1)\times(0,T)$, where $T$ is a positive constant, we discuss finite difference approximation of the nonlinear integro-differential problem:

$$\begin{aligned}&\frac{\partial U}{\partial t}-\left\{1+\int_0^t\int_0^1\left[\left(\frac{\partial U}{\partial x}\right)^2+\left(\frac{\partial V}{\partial x}\right)^2\right]\mathrm{d}x\mathrm{d}\tau\right\}\frac{\partial^2 U}{\partial x^2}=f_1(x,t),\\&\frac{\partial V}{\partial t}-\left\{1+\int_0^t\int_0^1\left[\left(\frac{\partial U}{\partial x}\right)^2+\left(\frac{\partial V}{\partial x}\right)^2\right]\mathrm{d}x\mathrm{d}\tau\right\}\frac{\partial^2 V}{\partial x^2}=f_2(x,t),\end{aligned}\tag{4.1}$$

$$U(0,t)=U(1,t)=V(0,t)=V(1,t)=0,\tag{4.2}$$

$$U(x,0)=U_0(x),\qquad V(x,0)=V_0(x).\tag{4.3}$$

Here $f_1=f_1(x,t)$, $f_2=f_2(x,t)$, $U_0=U_0(x)$ and $V_0=V_0(x)$ are given sufficiently smooth functions of their arguments.

We introduce a net in the rectangle $Q_T$ whose mesh points are denoted by $(x_i,t_j)=(ih,j\tau)$, where $i=0,1,\dots,M$ and $j=0,1,\dots,N$ with $h=1/M$, $\tau=T/N$. The initial line is denoted by $j=0$. The discrete approximation at $(x_i,t_j)$ is denoted by $u_i^j$, $v_i^j$ and the exact solution to the problem (4.1)–(4.3) at those points by $U_i^j$, $V_i^j$. We will use the following notations for the differences and norms:

$$\Delta_x r_i^j = \frac{r_{i+1}^j - r_i^j}{h}, \qquad \nabla_x r_i^j = \frac{r_i^j - r_{i-1}^j}{h},$$
$$\Delta_t r_i^j = \frac{r_i^{j+1} - r_i^j}{\tau}, \qquad \nabla_t r_i^j = \Delta_t r_i^{j-1} = \frac{r_i^j - r_i^{j-1}}{\tau},$$
$$\|r\|_h = \left(\sum_{i=1}^{M-1} r_i^2 h\right)^{1/2}, \qquad \| r]|_h = \left(\sum_{i=1}^{M} r_i^2 h\right)^{1/2}.$$

Thus we have

$$\begin{aligned} &\nabla_t u_i^j - \left\{1 + \tau h \sum_{l=1}^{M} \sum_{k=1}^{j+1} \left[(\nabla_x u_l^k)^2 + (\nabla_x v_l^k)^2\right]\right\} \Delta_x \nabla_x u_i^{j+1} = f_{1,i}^j, \\ &\nabla_t v_i^j - \left\{1 + \tau h \sum_{l=1}^{M} \sum_{k=1}^{j+1} \left[(\nabla_x u_l^k)^2 + (\nabla_x v_l^k)^2\right]\right\} \Delta_x \nabla_x v_i^{j+1} = f_{2,i}^j, \quad i = 1, 2, \ldots, M-1; j = 0, 1, \ldots, N-1, \end{aligned} \tag{4.4}$$

$$u_0^j = u_M^j = v_0^j = v_M^j = 0, \quad j = 0, 1, \ldots, N, \tag{4.5}$$

$$u_i^0 = U_{0,i}, \qquad v_i^0 = V_{0,i}, \quad i = 0, 1, \ldots, M. \tag{4.6}$$

Multiplying the first equality of (4.4) by $\tau h u_i^{j+1}(t)$, summing for each $i$ from 1 to $M-1$ and using the discrete analogue of the integration by parts we get

$$\|u^{j+1}\|_h^2 - h \sum_{i=1}^{M-1} u_i^{j+1} u_i^j + \tau h \sum_{i=1}^{M} \left(1 + \tau h \sum_{l=1}^{M} \sum_{k=1}^{j+1} \left[(\nabla_x u_l^k)^2 + (\nabla_x v_l^k)^2\right]\right) \left(\nabla_x u_i^{j+1}\right)^2 = \tau h \sum_{i=1}^{M-1} f_{1,i}^j u_i^{j+1}. \tag{4.7}$$

Taking into account the following relations

$$h \sum_{i=1}^{M-1} u_i^{j+1} u_i^j \le \frac{1}{2}\|u^{j+1}\|_h^2 + \frac{1}{2}\|u^j\|_h^2, \qquad h \sum_{i=1}^{M-1} f_i^j u_i^{j+1} \le \frac{1}{2}\|f^j\|_h^2 + \frac{1}{2}\|u^{j+1}\|_h^2$$

and discrete analogue of Poincare's inequality

$$\|u^{j+1}\|_h \le \| \nabla_x u^{j+1}]|_h \tag{4.8}$$

from (4.7) we get

$$\frac{1}{2}\|u^{j+1}\|_h^2 - \frac{1}{2}\|u^j\|_h^2 + \tau \ \| \nabla_x u^{j+1}]\,|_h^2 \le \tau \|f_1^j\|_h^2 + \frac{\tau}{2} \ \| \nabla_x u^{j+1}]\,|_h^2 .$$

From this inequality it is not difficult to get the following estimation

$$\|u^n\|_h^2 + \sum_{j=1}^{n} \| \nabla_x u^j]\,|_h^2 \, \tau < C, \quad n = 1, 2, \ldots, N. \tag{4.9}$$

Analogously, we can show that

$$\|v^n\|_h^2 + \sum_{j=1}^{n} \| \nabla_x v^j]\,|_h^2 \, \tau < C, \quad n = 1, 2, \ldots, N. \tag{4.10}$$

In (4.9) and (4.10) the constant $C$ depends on $T$ and on $f_1$ and $f_2$ respectively.

The a priori estimates (4.9) and (4.10) guarantee the stability and existence, see [9], of solution of the scheme (4.4)–(4.6).

The main result of this section is:

**Theorem 4.1.** *If problem (4.1)–(4.3) has a sufficiently smooth solution $U = U(x,t)$, $V = V(x,t)$, then the solution $u^j = (u_1^j, u_2^j, \ldots, u_{M-1}^j)$, $v^j = (v_1^j, v_2^j, \ldots, v_{M-1}^j)$, $j = 1, 2, \ldots, N$ of the difference scheme (4.4)–(4.6) tends to $U^j = (U_1^j, U_2^j, \ldots, U_{M-1}^j)$, $V^j = (V_1^j, V_2^j, \ldots, V_{M-1}^j)$, $j = 1, 2, \ldots, N$ as $\tau \to 0$, $h \to 0$ and the following estimates are true*

$$\|u^j - U^j\|_h \le C(\tau + h), \qquad \|v^j - V^j\|_h \le C(\tau + h), \quad j = 1, 2, \ldots, N. \tag{4.11}$$

**Proof.** For $U = U(x,t)$ and $V = V(x,t)$ we have:

$$\begin{aligned}
&\nabla_t U_i^j - \left\{1 + \tau h \sum_{l=1}^{M}\sum_{k=1}^{j+1}\left[(\nabla_x U_l^k)^2 + (\nabla_x V_l^k)^2\right]\right\} \Delta_x \nabla_x U_i^{j+1} = f_{1,i}^j - \psi_{1,i}^j,\\
&\nabla_t V_i^j - \left\{1 + \tau h \sum_{l=1}^{M}\sum_{k=1}^{j+1}\left[(\nabla_x U_l^k)^2 + (\nabla_x V_l^k)^2\right]\right\} \Delta_x \nabla_x V_i^{j+1} = f_{2,i}^j - \psi_{2,i}^j,
\end{aligned} \tag{4.12}$$

$$U_0^j = U_M^j = V_0^j = V_M^j = 0, \tag{4.13}$$

$$U_i^0 = U_{0,i}, \qquad V_i^0 = V_{0,i}, \tag{4.14}$$

where

$$\psi_{k,i}^j = O(\tau + h), \quad k = 1, 2.$$

Solving (4.4)–(4.6) instead of the problem (4.1)–(4.3) we have the errors $y_i^j = u_i^j - U_i^j$ and $z_i^j = v_i^j - V_i^j$. From (4.4)–(4.6) and (4.12)–(4.14) we get

$$\begin{aligned}
&\nabla_t y_i^j - \Delta_x\left\{\left(1 + \tau h \sum_{l=1}^{M}\sum_{k=1}^{j+1}\left[(\nabla_x u_l^k)^2 + (\nabla_x v_l^k)^2\right]\right)\nabla_x u_i^{j+1}\right.\\
&\qquad \left. - \left(1 + \tau h \sum_{l=1}^{M}\sum_{k=1}^{j+1}\left[(\nabla_x U_l^k)^2 + (\nabla_x V_l^k)^2\right]\right)\nabla_x U_i^{j+1}\right\} = \psi_{1,i}^j,\\
&\nabla_t z_i^j - \Delta_x\left\{\left(1 + \tau h \sum_{l=1}^{M}\sum_{k=1}^{j+1}\left[(\nabla_x u_l^k)^2 + (\nabla_x v_l^k)^2\right]\right)\nabla_x v_i^{j+1}\right.\\
&\qquad \left. - \left(1 + \tau h \sum_{l=1}^{M}\sum_{k=1}^{j+1}\left[(\nabla_x U_l^k)^2 + (\nabla_x V_l^k)^2\right]\right)\nabla_x V_i^{j+1}\right\} = \psi_{2,i}^j,
\end{aligned} \tag{4.15}$$

$$y_0^j = y_M^j = z_0^j = z_M^j = 0, \tag{4.16}$$

$$y_i^0 = z_i^0 = 0. \tag{4.17}$$

Multiplying Eq. (4.15) by $\tau h y_i^{j+1}$ and $\tau h z_i^{j+1}$, respectively, summing for each $i$ from 1 to $M-1$, using (4.16) and the discrete analogue of formula of integration by parts we get

$$\begin{aligned}
&\|y^{j+1}\|_h^2 - h\sum_{i=1}^{M-1} y_i^{j+1} y_i^j + \tau h \sum_{i=1}^{M}\left\{\left(1 + \tau h \sum_{l=1}^{M}\sum_{k=1}^{j+1}\left[(\nabla_x u_l^k)^2 + (\nabla_x v_l^k)^2\right]\right)\nabla_x u_i^{j+1}\right.\\
&\qquad \left. - \left(1 + \tau h \sum_{l=1}^{M}\sum_{k=1}^{j+1}\left[(\nabla_x U_l^k)^2 + (\nabla_x V_l^k)^2\right]\right)\nabla_x U_i^{j+1}\right\}\nabla_x y_i^{j+1} = \tau h \sum_{i=1}^{M-1}\psi_{1,i}^j y_i^{j+1},\\
&\|z^{j+1}\|_h^2 - h\sum_{i=1}^{M-1} z_i^{j+1} z_i^j + \tau h \sum_{i=1}^{M}\left\{\left(1 + \tau h \sum_{l=1}^{M}\sum_{k=1}^{j+1}\left[(\nabla_x u_l^k)^2 + (\nabla_x v_l^k)^2\right]\right)\nabla_x v_i^{j+1}\right.\\
&\qquad \left. - \left(1 + \tau h \sum_{l=1}^{M}\sum_{k=1}^{j+1}\left[(\nabla_x U_l^k)^2 + (\nabla_x V_l^k)^2\right]\right)\nabla_x V_i^{j+1}\right\}\nabla_x z_i^{j+1} = \tau h \sum_{i=1}^{M-1}\psi_{2,i}^j z_i^{j+1}.
\end{aligned} \tag{4.18}$$

Note that

$$h\sum_{i=1}^{M-1} r_i^{j+1} r_i^j = \frac{1}{2}\|r^{j+1}\|_h^2 + \frac{1}{2}\|r^j\|_h^2 - \frac{1}{2}\|r^{j+1} - r^j\|_h^2, \tag{4.19}$$

and

$$\begin{aligned}
&\left(\left[(\nabla_x u_i^k)^2 + (\nabla_x v_i^k)^2\right]\nabla_x u_i^{j+1} - \left[(\nabla_x U_i^k)^2 + (\nabla_x V_i^k)^2\right]\nabla_x U_i^{j+1}\right)(\nabla_x u_i^{j+1} - \nabla_x U_i^{j+1})\\
&\quad = \left[(\nabla_x u_i^k)^2 + (\nabla_x v_i^k)^2\right](\nabla_x u_i^{j+1})^2 + \left[(\nabla_x U_i^k)^2 + (\nabla_x V_i^k)^2\right](\nabla_x U_i^{j+1})^2\\
&\qquad - \nabla_x u_i^{j+1}\nabla_x U_i^{j+1}\left[(\nabla_x U_i^k)^2 + (\nabla_x V_i^k)^2 + (\nabla_x u_i^k)^2 + (\nabla_x v_i^k)^2\right]\\
&\quad = \frac{1}{2}\left(\nabla_x u_i^{j+1} - \nabla_x U_i^{j+1}\right)^2\left[(\nabla_x u_i^k)^2 + (\nabla_x v_i^k)^2 + (\nabla_x U_i^k)^2 + (\nabla_x V_i^k)^2\right]
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}(\nabla_x u_i^{j+1})^2\left[(\nabla_x U_i^k)^2+(\nabla_x V_i^k)^2\right]-\frac{1}{2}(\nabla_x U_i^{j+1})^2\left[(\nabla_x u_i^k)^2+(\nabla_x v_i^k)^2\right]\\
&+\frac{1}{2}(\nabla_x u_i^{j+1})^2\left[(\nabla_x u_i^k)^2+(\nabla_x v_i^k)^2\right]+\frac{1}{2}(\nabla_x U_i^{j+1})^2\left[(\nabla_x U_i^k)^2+(\nabla_x V_i^k)^2\right]\\
\geq{}&\frac{1}{2}\left[(\nabla_x u_i^{j+1})^2-(\nabla_x U_i^{j+1})^2\right]\left[(\nabla_x u_i^k)^2+(\nabla_x v_i^k)^2-(\nabla_x U_i^k)^2-(\nabla_x V_i^k)^2\right].
\end{aligned}
\tag{4.20}
$$

Analogously,

$$
\begin{aligned}
&\left(\left[(\nabla_x u_i^k)^2+(\nabla_x v_i^k)^2\right]\nabla_x v_i^{j+1}-\left[(\nabla_x U_i^k)^2+(\nabla_x V_i^k)^2\right]\nabla_x V_i^{j+1}\right)(\nabla_x v_i^{j+1}-\nabla_x V_i^{j+1})\\
\geq{}&\frac{1}{2}\left[(\nabla_x v_i^{j+1})^2-(\nabla_x V_i^{j+1})^2\right]\left[(\nabla_x u_i^k)^2+(\nabla_x v_i^k)^2-(\nabla_x U_i^k)^2-(\nabla_x V_i^k)^2\right].
\end{aligned}
\tag{4.21}
$$

Taking into account relations (4.19)–(4.21), from (4.18) for all $\varepsilon > 0$ we have

$$
\begin{aligned}
&\|y^{j+1}\|_h^2+\frac{1}{2}\|y^{j+1}-y^j\|_h^2-\frac{1}{2}\|y^{j+1}\|_h^2-\frac{1}{2}\|y^j\|_h^2+\tau\parallel\nabla_x y^{j+1}]|_h^2\\
&+\|z^{j+1}\|_h^2+\frac{1}{2}\|z^{j+1}-z^j\|_h^2-\frac{1}{2}\|z^{j+1}\|_h^2-\frac{1}{2}\|z^j\|_h^2+\tau\parallel\nabla_x z^{j+1}]|_h^2\\
&+\frac{\tau^2h^2}{2}\sum_{i=1}^{M}\sum_{l=1}^{M}\sum_{k=1}^{j+1}\left[(\nabla_x u_l^k)^2+(\nabla_x v_l^k)^2-(\nabla_x U_i^k)^2-(\nabla_x V_i^k)^2\right]\Big[(\nabla_x u_i^{j+1})^2+(\nabla_x v_i^{j+1})^2\\
&-(\nabla_x U_i^{j+1})^2-(\nabla_x V_i^{j+1})^2\Big]\leq\tau\varepsilon(\|\psi_1^j\|_h^2+\|\psi_2^j\|_h^2)+\frac{\tau}{4\varepsilon}(\|y^{j+1}\|_h^2+\|z^{j+1}\|_h^2),\quad j=0,1,\ldots,N-1.
\end{aligned}
\tag{4.22}
$$

Let us introduce the notation

$$
\xi^j=\tau h\sum_{k=1}^{j}\sum_{l=1}^{M}\left[(\nabla_x u_l^k)^2+(\nabla_x v_l^k)^2-(\nabla_x U_l^k)^2-(\nabla_x V_l^k)^2\right],\quad \xi^0=0,
$$

then

$$
\Delta_t\xi^j=h\sum_{l=1}^{M}\left[(\nabla_x u_l^{j+1})^2+(\nabla_x v_l^{j+1})^2-(\nabla_x U_l^{j+1})^2-(\nabla_x V_l^{j+1})^2\right].
$$

So, from (4.22) we get

$$
\begin{aligned}
&\|y^{j+1}\|_h^2-\|y^j\|_h^2+\tau^2\|\nabla_t y^{j+1}\|_h^2+\tau\parallel\nabla_x y^{j+1}]|_h^2+\|z^{j+1}\|_h^2-\|z^j\|_h^2+\tau^2\|\nabla_t z^{j+1}\|_h^2+\tau\parallel\nabla_x z^{j+1}]|_h^2+\tau^2\left(\Delta_t\xi^j\right)^2\\
&+\tau\xi^j\Delta_t\xi^j\leq\frac{\tau}{\varepsilon}(\|\psi_1^j\|_h^2+\|\psi_2^j\|_h^2)+4\varepsilon\tau(\|y^{j+1}\|_h^2+\|z^{j+1}\|_h^2).
\end{aligned}
\tag{4.23}
$$

Using (4.17), discrete analogue of Poincare's inequality

$$
\|r^{j+1}\|_h^2\leq\parallel\nabla_x r_i^{j+1}]|_h^2
$$

and the relation

$$
\tau\xi^j\Delta_t\xi^j=\frac{1}{2}\left(\xi^{j+1}\right)^2-\frac{1}{2}\left(\xi^j\right)^2-\frac{\tau^2}{2}\left(\Delta_t\xi^j\right)^2,
$$

from (4.23) when $\varepsilon = 1$, we have

$$
\begin{aligned}
&\|y^n\|_h^2+\tau^2\sum_{j=0}^{n-1}\|\Delta_t y_i^j\|_h^2+\frac{\tau}{2}\sum_{j=0}^{n-1}\parallel\nabla_x y_i^{j+1}]|_h^2+\|z^n\|_h^2+\tau^2\sum_{j=0}^{n-1}\|\Delta_t z_i^j\|_h^2\\
&+\frac{\tau}{2}\sum_{j=0}^{n-1}\parallel\nabla_x z_i^{j+1}]|_h^2+\frac{\tau^2}{2}\sum_{j=0}^{n-1}\left(\Delta_t\xi^j\right)^2+\frac{1}{2}\left(\xi^n\right)^2\leq\tau\sum_{j=0}^{n-1}\left(\|\psi_1^j\|_h^2+\|\psi_2^j\|_h^2\right),\quad n=1,2,\ldots,N.
\end{aligned}
\tag{4.24}
$$

From (4.24) we get (4.11) and thus Theorem 4.1 has been proven. □

Remark: Note, that according to the scheme of proving convergence theorem, the uniqueness of the solution of the scheme (4.4)–(4.6) can be proven. In particular, assuming the existence of two solutions $(u, v)$ and $(\bar{u}, \bar{v})$ of the scheme (4.4)–(4.6), then for the differences $\bar{y} = u - \bar{u}$ and $\bar{z} = v - \bar{v}$ we get $\|\bar{y}^n\|_h^2 + \|\bar{z}^n\|_h^2 \leq 0$, $n = 1, 2, \ldots, N$. So, $\bar{y} = \bar{z} \equiv 0$.

## 5. Numerical implementation

The finite difference scheme (4.4)–(4.6) can be rewritten as follows:

$$\begin{aligned}
&\frac{u_i^{j+1}-u_i^j}{\tau} - A^{j+1}\frac{u_{i+1}^{j+1}-2u_i^{j+1}+u_{i-1}^{j+1}}{h^2} = f_{1,i}^j,\\
&\frac{v_i^{j+1}-v_i^j}{\tau} - A^{j+1}\frac{v_{i+1}^{j+1}-2v_i^{j+1}+v_{i-1}^{j+1}}{h^2} = f_{2,i}^j, \quad i=1,2,\dots,M-1, j=0,1,\dots,N-1,
\end{aligned} \tag{5.1}$$

where

$$A^j = 1 + \tau h \sum_{\ell=1}^{M}\sum_{k=1}^{j}\left[\left(\frac{u_\ell^k - u_{\ell-1}^k}{h}\right)^2 + \left(\frac{v_\ell^k - v_{\ell-1}^k}{h}\right)^2\right]. \tag{5.2}$$

In order to rewrite this in matrix form, we define the vectors

$$\mathbf{u}^j = \begin{bmatrix} u_1^j \\ u_2^j \\ \vdots \\ u_{M-1}^j \end{bmatrix}$$

and similarly $\mathbf{v}^j$, $\mathbf{f}_1^j$, and $\mathbf{f}_2^j$. We also define the symmetric tridiagonal $(M-1)\times(M-1)$ matrix $\mathbf{T}$ as follows

$$\mathbf{T}_{rs}^{j+1} = \begin{cases} -\dfrac{1}{h^2}A^{j+1}, & s = r-1,\\ \dfrac{2}{h^2}A^{j+1}, & s = r,\\ -\dfrac{1}{h^2}A^{j+1}, & s = r+1,\\ 0, & \text{otherwise}. \end{cases}$$

Thus the system (5.1) becomes

$$\frac{1}{\tau}\begin{bmatrix}\mathbf{u}^{j+1}\\ \mathbf{v}^{j+1}\end{bmatrix} - \frac{1}{\tau}\begin{bmatrix}\mathbf{u}^{j}\\ \mathbf{v}^{j}\end{bmatrix} + \begin{bmatrix}\mathbf{T}^{j+1} & 0\\ 0 & \mathbf{T}^{j+1}\end{bmatrix}\begin{bmatrix}\mathbf{u}^{j+1}\\ \mathbf{v}^{j+1}\end{bmatrix} - \begin{bmatrix}\mathbf{f}_1^{j}\\ \mathbf{f}_2^{j}\end{bmatrix} = 0. \tag{5.3}$$

We will use Newton's method to solve the nonlinear system (5.3). Let

$$\mathbf{P}^j = \begin{bmatrix}\mathbf{u}^{j}\\ \mathbf{v}^{j}\end{bmatrix}$$

and

$$\mathbf{F}^j = \begin{bmatrix}\mathbf{f}_1^{j}\\ \mathbf{f}_2^{j}\end{bmatrix}$$

and define

$$\mathbf{H}(\mathbf{P}^{j+1}) = \frac{1}{\tau}\mathbf{P}^{j+1} - \frac{1}{\tau}\mathbf{P}^{j} + \hat{\mathbf{T}}^{j+1}\mathbf{P}^{j+1} - \mathbf{F}^j, \tag{5.4}$$

where $\hat{\mathbf{T}}^{j+1}$ is the 2-by-2 block diagonal matrix with $\mathbf{T}^{j+1}$ on diagonal. We will now construct the gradient matrix. This matrix can be written in block form as follows:

$$\nabla\mathbf{H} = \begin{bmatrix} Q & R\\ W & Z\end{bmatrix},$$

where the matrices $Q$, $R$, $W$, $Z$ are given below.

$$Q_{rs} = \frac{1}{\tau}\delta_{rs} - A^{j+1}\frac{\delta_{r+1s} - 2\delta_{rs} + \delta_{r-1s}}{h^2} + 2\tau h \frac{u_{r+1}^{j+1} - 2u_r^{j+1} + u_{r-1}^{j+1}}{h^2}\,\frac{u_{s+1}^{j+1} - 2u_s^{j+1} + u_{s-1}^{j+1}}{h^2} \tag{5.5}$$

$$W_{rs} = 2\tau h \frac{v_{r+1}^{j+1} - 2v_r^{j+1} + v_{r-1}^{j+1}}{h^2}\,\frac{u_{s+1}^{j+1} - 2u_s^{j+1} + u_{s-1}^{j+1}}{h^2} \tag{5.6}$$

$$R_{rs} = 2\tau h \frac{u_{r+1}^{j+1} - 2u_r^{j+1} + u_{r-1}^{j+1}}{h^2}\,\frac{v_{s+1}^{j+1} - 2v_s^{j+1} + v_{s-1}^{j+1}}{h^2} \tag{5.7}$$

where $\delta_{rs}$ is the Kronecker delta and $Z_{rs}$ is obtained by replacing $u$ by $v$ in $Q_{rs}$. Using definition (5.4) Newton's method for the system (5.3) is given by

$$\nabla \mathbf{H}(\mathbf{P}^{j+1})\,|^{(n)}(\mathbf{P}^{j+1}\,|^{(n+1)} - \mathbf{P}^{j+1}\,|^{(n)}) = -\mathbf{H}(\mathbf{P}^{j+1})\,|^{(n)}.$$

**Theorem 5.1.** *Given the nonlinear system of equations*

$$H_i\,(P_1, \ldots, P_{2M-2}) = 0, \quad i = 1, 2, \ldots, 2M-2.$$

*If $H_i$ are three times continuously differentiable in a region containing the solution $\xi_1, \ldots, \xi_{2M-2}$ and the Jacobian does not vanish in that region, then Newton's method converges at least quadratically (see [17]).*

The Jacobian is the matrix $\nabla H$ computed above. The term $\frac{1}{\tau}$ on diagonal ensures that the Jacobian does not vanish. The differentiability is guaranteed, since $\nabla H$ is quadratic.

In our first numerical experiment (Example 1) we have chosen the right-hand side so that the exact solution is given by

$$U(x,t) = x(1-x)\sin(x+t), \qquad V(x,t) = x(1-x)\cos(x+t).$$

In this case the right-hand side is

$$f_1(x,t) = x(1-x)\cos(x+t) - \left(1 + \frac{11}{30}t\right)((-2 - x + x^2)\sin(x+t) + 2(1-2x)\cos(x+t))$$

$$f_2(x,t) = -x(1-x)\sin(x+t) - \left(1 + \frac{11}{30}t\right)((-2 - x + x^2)\cos(x+t) - 2(1-2x)\sin(x+t)).$$

The parameters used are $M = 100$ which dictates $h = 0.01$. In the next four subplots we plotted the absolute value of the difference between the numerical and exact solutions on a semi-log axis at $t = 0.5$ and $t = 1$ (Fig. 1) and it is clear that the two solutions are almost identical.

In our next experiment (Example 2) we have taken zero right-hand side and initial condition given by

$$U_0(x) = U(x,0) = x(1-x)\sin(8\pi x), \qquad V_0(x) = V(x,0) = x(1-x)\cos(4\pi x).$$

In this case, we know that the solution will decay in time [11]. The parameters $M$, $h$, $\tau$ are as before. In Fig. 2, we plotted the initial solution and in Fig. 3, we have the numerical solution at four different times. In both figures the top subplot is for $u$

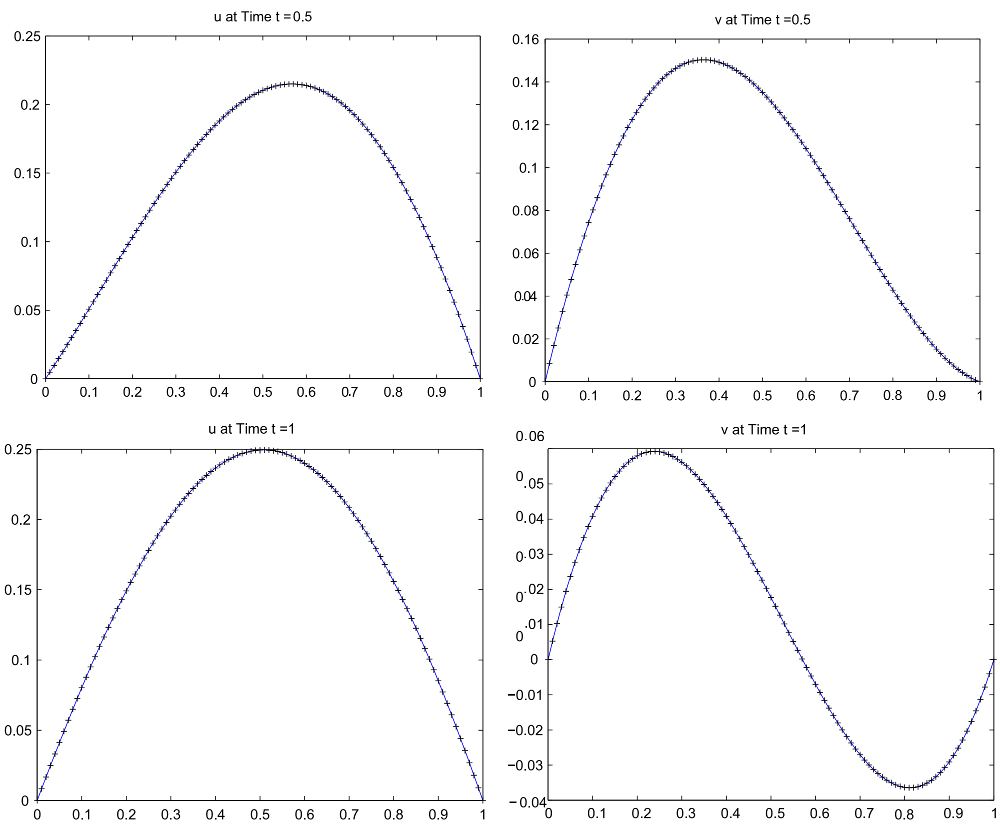


**Fig. 1.** The absolute value of the difference between the numerical and exact solutions for $u$ (left) and $v$ (right) at $t = 0.5$ (top) and $t = 1$ (bottom) on a semi-log scale.

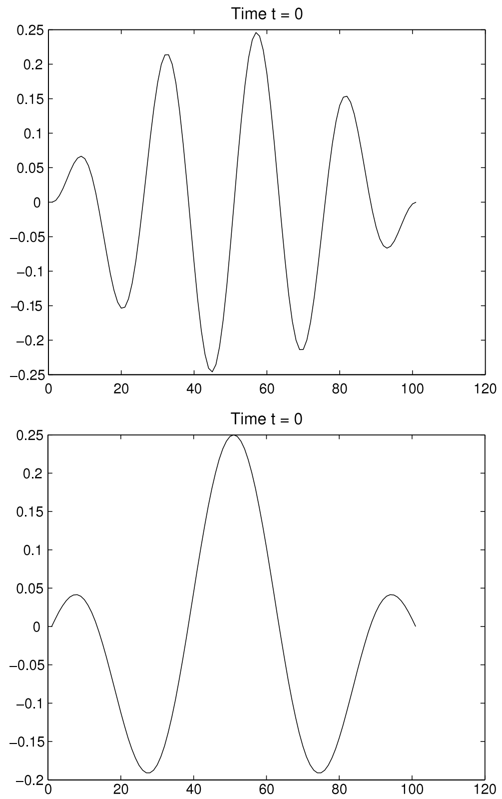


**Fig. 2.** The initial solution $U_0(x) = x(1-x)\sin(8\pi x)$ (top) and $V_0(x) = x(1-x)\cos(4\pi x)$ (bottom) for Example 2.

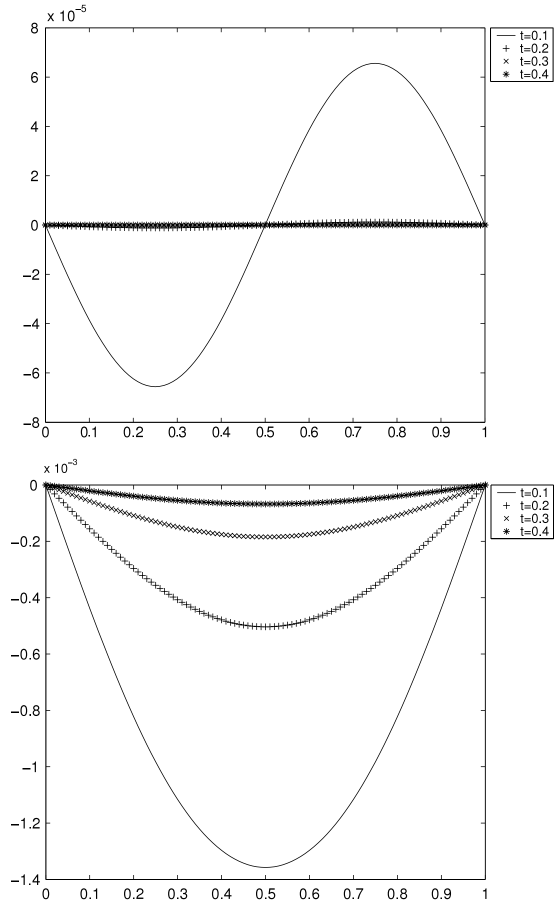


**Fig. 3.** The numerical solution at $t = 0.1, 0.2, 0.3, 0.4$ for $u$ (top) and $v$ (bottom).

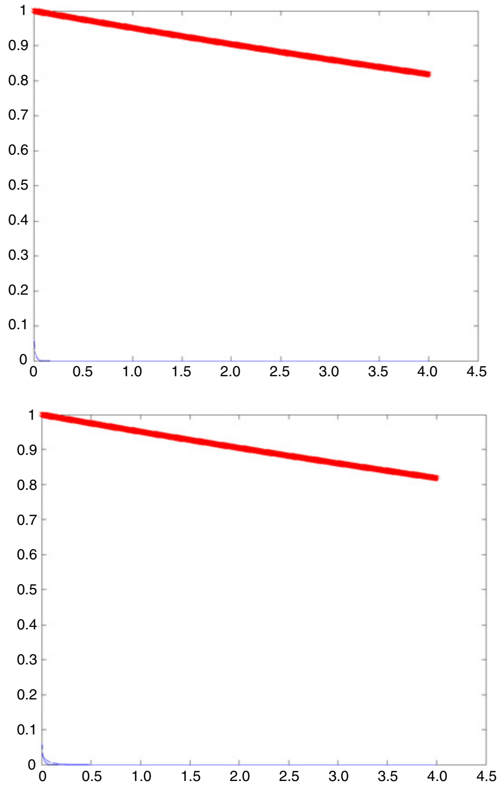


**Fig. 4.** The maximum norm of the numerical solution for $\frac{\partial U}{\partial x}$ (top) and $\frac{\partial V}{\partial x}$ (bottom) (Example 2) and $e^{-t/2}$. Solid line for $\frac{\partial U}{\partial x}$ and $\frac{\partial V}{\partial x}$ and line marked with $*$ for the exponential.

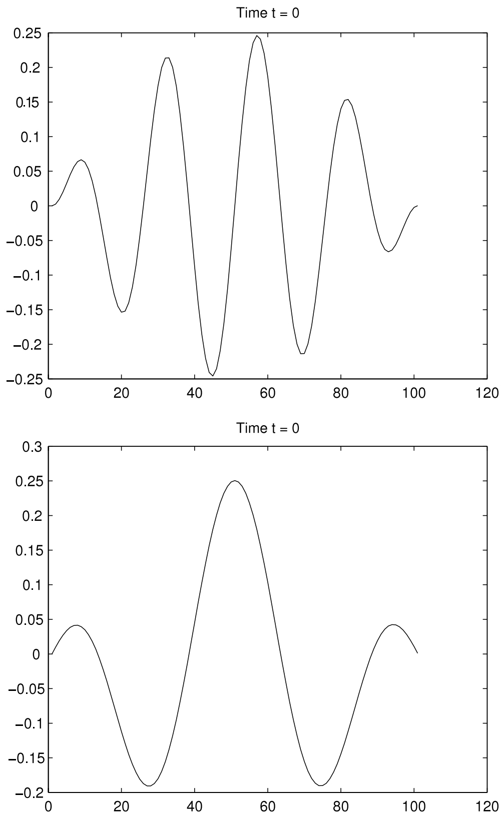


**Fig. 5.** The initial solution $U_0(x) = x(1-x)\sin(8\pi x) + 0.0002x$ (top) and $V_0(x) = x(1-x)\cos(4\pi x) + 0.001x$ (bottom) for Example 3.

and the bottom subplot is for $v$. It is clear that the numerical solution is approaching zero for all $x$. We have also plotted the maximum norm of the partial derivatives $\frac{\partial U}{\partial x}$ and $\frac{\partial V}{\partial x}$ versus the exponential $e^{-t/2}$. Fig. 4 shows that the maximum norm of

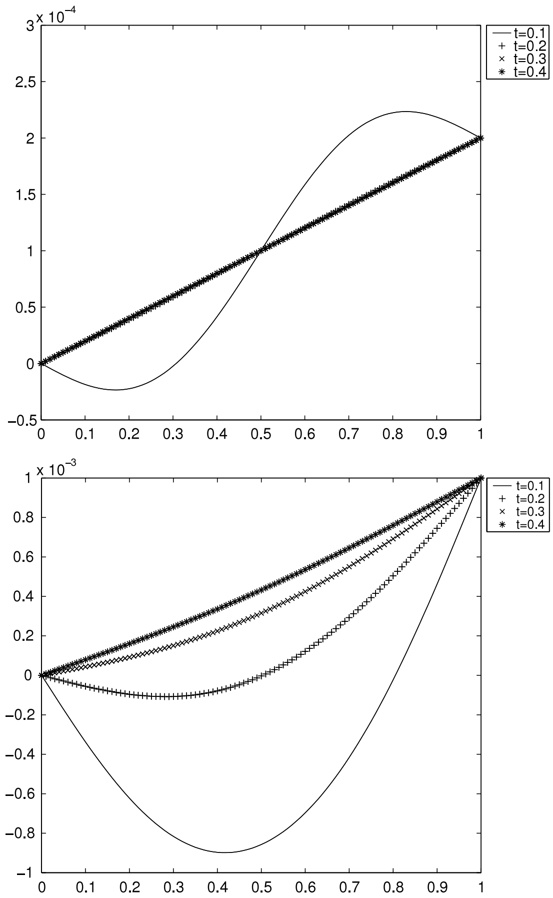


**Fig. 6.** The numerical solution at $t = 0.1, 0.2, 0.3, 0.4$ for $u$ (top) and $v$ (bottom).

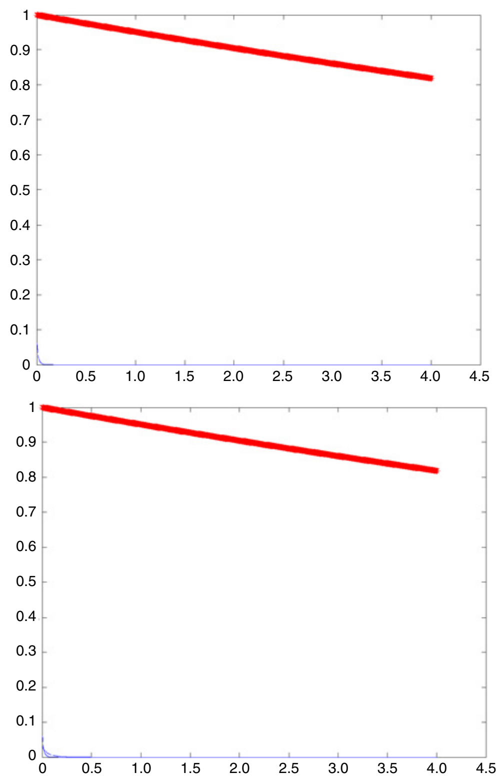


**Fig. 7.** The maximum norm of the numerical solution for $\frac{\partial U}{\partial x}$ (top) and $\frac{\partial V}{\partial x}$ (bottom) (Example 3) and $e^{-t/2}$. Solid line for $\frac{\partial U}{\partial x}$ and $\frac{\partial V}{\partial x}$ and line marked with $*$ for the exponential.

$\frac{\partial U}{\partial x}$ (top) and $\frac{\partial V}{\partial x}$ (bottom) decays faster than the exponential. Therefore the numerical approximation of the $x$-derivative of the solution of our experiment fully agrees with the theoretical results given in [11].

We have experimented with several other initial solutions, and in all cases we noticed the decay of the numerical solution as expected [11].

We have solved the problem with nonhomogeneous boundary conditions on one side of lateral boundary as well (Example 3). In this case we have taken the following initial conditions:

$$U_0(x) = U(x, 0) = x(1-x)\sin(8\pi x) + 0.0002x, \qquad V_0(x) = V(x, 0) = x(1-x)\cos(4\pi x) + 0.001x.$$

We plotted the initial solution in Fig. 5 and the numerical solution at various times in Fig. 6. Now the solution approaches the steady state solution $U(x) = 0.0002x$ and $V(x) = 0.001x$ respectively.

We have also plotted the maximum norm of the partial derivatives $\frac{\partial U}{\partial x}$ and $\frac{\partial V}{\partial x}$ versus the exponential $e^{-t/2}$. Fig. 7 shows that the maximum norm of $\frac{\partial U}{\partial x}$ (top) and $\frac{\partial V}{\partial x}$ (bottom) decays faster than the exponential. Therefore the numerical approximation of the $x$-derivative of the solution of our experiment shows exponential decay as in the homogeneous case. Theoretically we could not prove better than polynomial decay. It is possible that this faster decay happens only under special circumstances.

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