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Constructing a family of optimal eighth-order modified Newton-type multiple-zero finders along with the dynamics behind their purely imaginary extraneous fixed points

Young Hee Geum [a], Young Ik Kim [a,*], Beny Neta [b]

[a] Department of Mathematics, Dankook University, Cheonan, 330-714, Republic of Korea
[b] Naval Postgraduate School, Department of Applied Mathematics, Monterey, CA93943, United States

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ABSTRACT

An optimal family of eighth-order multiple-zero finders and the dynamics behind their basins of attraction are proposed by considering modified Newton-type methods with multivariate weight functions. Extensive investigation of purely imaginary extraneous fixed points of the proposed iterative methods is carried out for the study of the dynamics associated with corresponding basins of attraction. Numerical experiments strongly support the underlying theory pursued in this paper. An exploration of the relevant dynamics of the proposed methods is presented along with illustrative basins of attraction for various polynomials.

1. Introduction

The classical second-order Newton's method is most popular to locate an approximate root of a nonlinear equation. It is, however, only linearly convergent to find the repeated roots for a nonlinear equation under consideration. In order to efficiently find approximate repeated roots of a nonlinear equation in the form $f(x) = 0$, we usually employ modified Newton's method [1,2] with quadratic-order convergence, given the multiplicity $m \geq 1$, as follows:

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}, n = 0, 1, 2, \ldots \tag{1.1}$$

Note that numerical scheme (1.1) is a second-order one-point optimal [3] method which is supported by Kung–Traub's conjecture [3] that any multipoint method [4] without memory can reach its convergence order of at most 2^{r-1} for r functional evaluations. We can find other higher-order multiple-zero finders in the literature, e.g., [5–15].

Assuming a known multiplicity of $m \geq 1$, we propose in this paper a family of eighth-order modified Newton-type multiple-zero finders in the form of:

* Corresponding author.
E-mail addresses: conpana@empas.com (Y.H. Geum), yikbell79@yahoo.com (Y.I. Kim), bneta@nps.edu (B. Neta).

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$$\begin{cases} y_n = x_n - m \cdot \dfrac{f(x_n)}{f'(x_n)}, \\ w_n = x_n - m \cdot L_f(x_n) \cdot \dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - m \cdot H_f(x_n) \cdot \dfrac{f(x_n)}{f'(x_n)}, \end{cases} \tag{1.2}$$

where the desired generic forms of weight functions L_f and H_f will be extensively studied for eighth-order of convergence in Section 3. To the best of our knowledge, there is no other generic eighth-order method for multiple roots.

In what follows, we briefly organize the remaining portion of the paper as follows. Section 2 introduces existing studies on multiple-zero finders. Fully investigated in Section 3 is methodology and convergence analysis for newly proposed multiple-zero finders. A main theorem on the properties of the family of proposed methods (1.2) is derived to discover eighth-order convergence as well as to induce asymptotic error constants and error equations by use of a family of weight functions L_f and H_f dependent on two function-to-function ratios. In Section 4, special cases of rational weight functions are considered. Section 5 extensively investigates the extraneous fixed points and relevant dynamics underlying the basins of attraction. Tabulated in Section 6 are computational results for a variety of numerical examples. Table 6 compares the magnitudes of $e_n = x_n - \alpha$ of the proposed methods with those of existing sixth-order multiple-zero finders. Dynamical characteristics of the proposed methods are illustrated at great length by means of their basins of attraction with various test equations. Overall conclusions are stated at the end along with comments on future development of higher-order methods extending the current approach.

2. Review of existing sixth-order multiple-zero finders

In the literature as claimed at the end of the first paragraph of Section 1, we rarely find multiple-zero finders with convergence order higher than 4. Recently Geum–Kim–Neta [16,17] have developed two families of sixth-order multiple-zero finders with extensive analysis of their relevant dynamics behind the basins of attraction from the viewpoint of the extraneous fixed points.

Let a function $f : \mathbb{C} \to \mathbb{C}$ have a repeated zero α with integer multiplicity $m \geq 1$ and be analytic [18] in a small neighborhood of α. Then the following two members of the aforementioned Geum–Kim–Neta's family are of sixth-order convergence and described below by (2.1) and (2.2).

$$\begin{cases} y_n = x_n - m\dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n - \dfrac{m + a_1 s}{1 + b_1 s + b_2 s^2} \times \dfrac{1}{1 + 2(m-1)t} \cdot \dfrac{f(y_n)}{f'(y_n)}, \ s = \left[\dfrac{f(y_n)}{f(x_n)}\right]^{\frac{1}{m}}, \ t = \left[\dfrac{f'(y_n)}{f'(x_n)}\right]^{\frac{1}{m-1}}, \quad \text{for } m \geq 2, \end{cases} \tag{2.1}$$

where $a_1 = \frac{2m(4m^4-16m^3+31m^2-30m+13)}{(m-1)(4m^2-8m+7)}$, $b_1 = \frac{4(2m^2-4m+3)}{(m-1)(4m^2-8m+7)}$ and $b_2 = -\frac{4m^2-8m+3}{4m^2-8m+7}$.

$$\begin{cases} y_n = x_n - m\dfrac{f(x_n)}{f'(x_n)}, \\ w_n = x_n - m\dfrac{(s-2)(2s-1)}{(s-1)(5s-2)} \cdot \dfrac{f(x_n)}{f'(x_n)} \\ x_{n+1} = x_n - m\dfrac{(s-2)(2s-1)}{(5s-2)(s+v-1)} \cdot \dfrac{f(x_n)}{f'(x_n)}, \ s = \left[\dfrac{f(y_n)}{f(x_n)}\right]^{\frac{1}{m}}, \ v = \left[\dfrac{f(w_n)}{f(x_n)}\right]^{\frac{1}{m}}, \quad \text{for } m \geq 1. \end{cases} \tag{2.2}$$

These two members will be compared with another family of optimal eighth-order multiple-zero finders to be developed in the next section.

3. Methodology and convergence analysis

Let a function $f : \mathbb{C} \to \mathbb{C}$ possess a repeated zero α with integer multiplicity $m \geq 1$ and be analytic in a small neighborhood of α. Then new three-point iterative methods locating an approximate zero α of multiplicity m are proposed below to have optimal eighth-order convergence: for a given initial guess x_0 sufficiently close to α,

$$\begin{cases} y_n = x_n - m \cdot \dfrac{f(x_n)}{f'(x_n)}, \\ w_n = x_n - m \cdot L_f(s) \cdot \dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - m \cdot H_f(s,u) \cdot \dfrac{f(x_n)}{f'(x_n)} = x_n - m \cdot [L_f(s) + K_f(s,u)] \cdot \dfrac{f(x_n)}{f'(x_n)}, \end{cases} \tag{3.1}$$

where $n = 0, 1, 2, \ldots,$

$$s = \left[\frac{f(y_n)}{f(x_n)}\right]^{\frac{1}{m}}, \tag{3.2}$$

$$u = \left[\frac{f(w_n)}{f(y_n)}\right]^{\frac{1}{m}}, \tag{3.3}$$

and where $L_f : \mathbb{C} \to \mathbb{C}$ is analytic in a neighborhood of 0 and $K_f : \mathbb{C}^2 \to \mathbb{C}$ is holomorphic [19,20] in a neighborhood of (0, 0). Note that H_f in (1.2) is expressed as the sum of L_f and K_f. Since s and u are respectively one-to-m multiple-valued functions, we consider their principal analytic branches [18]. Hence, it is convenient to treat s as a principal root given by $s = \exp[\frac{1}{m}\mathrm{Log}(\frac{f(y_n)}{f(x_n)})]$, with $\mathrm{Log}(\frac{f(y_n)}{f(x_n)}) = \mathrm{Log}\left|\frac{f(y_n)}{f(x_n)}\right| + i\,\mathrm{Arg}(\frac{f(y_n)}{f(x_n)})$ for $-\pi < \mathrm{Arg}(\frac{f(y_n)}{f(x_n)}) \le \pi$; this convention of $\mathrm{Arg}(z)$ for $z \in \mathbb{C}$ agrees with that of Log[z] command of Mathematica [21] to be employed later in numerical experiments of Section 6. By means of further inspection of s, we find that $s = \left|\frac{f(y_n)}{f(x_n)}\right|^{\frac{1}{m}} \cdot \exp[\frac{i}{m}\,\mathrm{Arg}(\frac{f(y_n)}{f(x_n)})] = O(e_n)$. Similarly we treat $u = \left|\frac{f(w_n)}{f(y_n)}\right|^{\frac{1}{m}} \cdot \exp[\frac{i}{m}\,\mathrm{Arg}(\frac{f(w_n)}{f(y_n)})] = O(e_n^2)$. In addition, we find that $O(\frac{f(x_n)}{f'(x_n)}) = O(e_n)$.

Definition 1. (*Error Equation, Asymptotic Error Constant, Order of Convergence*)
Let $x_0, x_1, \ldots, x_n, \ldots$ be a sequence converging to α and $e_n = x_n - \alpha$ be the nth iterate error. If there exist real numbers $p \in \mathbb{R}$ and $b \in \mathbb{R} - \{0\}$ such that the following error equation holds

$$e_{n+1} = b\,{e_n}^p + O(e_n^{p+1}), \tag{3.4}$$

then b or $|b|$ is called the asymptotic error constant and p is called the order of convergence [22].

Recently a special case of (3.1) has been treated in [23] with weight functions of the form

$$L_f(s) = 1 + \frac{s \cdot Q(h)}{m}, \quad K_f(s, u) = \frac{1}{m} s \cdot u \cdot G(h, u), \quad h = \frac{s}{a_1 + a_2 s}, \tag{3.5}$$

where $Q : \mathbb{C} \to \mathbb{C}$ and $G : \mathbb{C}^2 \to \mathbb{C}$ are analytic in a neighborhood of (0) and (0, 0), respectively; a_1 and a_2 are non-zero complex parameters.

Our primary aim is to investigate a more generic family of optimal eighth-order methods (3.1). A main theorem will be first established for the optimal convergence. On the basis of the results of the main theorem, we will construct weight functions H_f and K_f for a multiparametric family of eighth-order multiple-zero finders. To this end, we observe that it suffices to consider both weight functions H_f and K_f up to the seventh-order terms in e_n due to the fact that $O(\frac{f(x_n)}{f'(x_n)}) = O(e_n)$, which leads us to the development of a more generic family of optimal eighth-order multiple-zero finders.

Applying the Taylor's series expansion of f about α, we get the following relations:

$$f(x_n) = \frac{f^{(m)}(\alpha)}{m!}{e_n}^m\left[1 + \theta_2 e_n + \theta_3 e_n^2 + \theta_4 e_n^3 + \theta_5 e_n^4 + \theta_6 e_n^5 + \theta_7 e_n^6 + \theta_8 e_n^7 + \theta_9 e_n^8 + O(e_n^9)\right], \tag{3.6}$$

$$\begin{aligned} f'(x_n) = \frac{f^{(m)}(\alpha)}{(m-1)!}{e_n}^{m-1}\Bigg[1 &+ \frac{m+1}{m}\theta_2 e_n + \frac{m+2}{m}\theta_3 e_n^2 + \frac{m+3}{m}\theta_4 e_n^3 + \frac{m+4}{m}\theta_5 e_n^4 \\ &+ \frac{m+5}{m}\theta_6 e_n^5 + \frac{m+6}{m}\theta_7 e_n^6 + \frac{m+7}{m}\theta_8 e_n^7 + O(e_n^8)\Bigg], \end{aligned} \tag{3.7}$$

where $\theta_k = \frac{m!}{(m-1+k)!}\frac{f^{(m-1+k)}(\alpha)}{f^{(m)}(\alpha)}$ for $k \in \mathbb{N} - \{1\}$. For convenience, we drop the subscript n from e_n whenever required to do so. Dividing (3.6) by (3.7), we have

$$\frac{f(x_n)}{f'(x_n)} = \frac{e}{m} - \frac{\theta_2 e^2}{m^2} + \frac{Y_3 e^3}{m^3} + \frac{Y_4 e^4}{m^4} + \frac{Y_5 e^5}{m^5} + \frac{Y_6 e^6}{m^6} + \frac{Y_7 e^7}{m^7} + \frac{Y_8 e^8}{m^8} + O(e^9), \tag{3.8}$$

where $Y_3 = (1+m)\theta_2^2 - 2m\theta_3$, $Y_4 = -(1+m)^2\theta_2^3 + m(4+3m)\theta_2\theta_3 - 3m^2\theta_4$ and $Y_i = Y_i(m, \theta_2, \theta_3, \ldots, \theta_9)$ for $5 \le i \le 8$.

Thus, from relation (3.8), we obtain

$$y_n = \alpha + \frac{\theta_2 e^2}{m} - \frac{Y_3 e^3}{m^2} - \frac{Y_4 e^4}{m^3} - \frac{Y_5 e^5}{m^4} - \frac{Y_6 e^6}{m^5} - \frac{Y_7 e^7}{m^6} - \frac{Y_8 e^8}{m^7} + O(e^9). \tag{3.9}$$

$$\begin{aligned} f(y_n) = \frac{f^{(m)}(\alpha)}{m!}\left(\frac{\theta_2}{m}\right)^m e^{2m}\Big\{1 &- \frac{Y_3}{\theta_2}e + \frac{(m-1)Y_3^2 - 2Y_4\theta_2 + 2\theta_2^4}{2m\theta_2^2}e^2 \\ &- \frac{(m-1)(m-2)Y_3^3 + 6Y_5\theta_2^2 + 6Y_3\theta_2(Y_4 - mY_4 + (m+1)\theta_2^3)}{6m^2\theta_2^3}e^3 + \sum_{i=4}^{7} J_i e^i + O(e^8)\Big\}, \end{aligned} \tag{3.10}$$

where $J_i = J_i(m, \theta_2, \theta_3, \ldots, \theta_9, Y_3, Y_4, \ldots, Y_8)$ for $4 \le i \le 7$.

By Taylor's expansion or multinomial expansion, we get an expression s in (3.2) as follows:

$$s = \frac{\theta_2}{m}e - \frac{Y_3 + \theta_2^2}{m^2}e^2 + \frac{-2Y_4 + \theta_2[2Y_3 + (m+3)\theta_2^2 - 2m\theta_3]}{2m^3}e^3 + \sum_{i=4}^{8} W_i e^i + O(e^9), \tag{3.11}$$

where $W_i = W_i(m, \theta_2, \theta_3, \ldots, \theta_9, Y_3, Y_4, \ldots, Y_8)$ for $4 \le i \le 8$.

With the use of s in (3.11), expanding Taylor series of $L_f(s)$ about 0 up to seventh-order terms we find:

$$L_f(s) = L_0 + L_1 s + L_2 s^2 + L_3 s^3 + L_4 s^4 + L_5 s^5 + L_6 s^6 + L_7 s^7 + O(e^8), \tag{3.12}$$

where $L_j = \frac{L_f^{(j)}(0)}{j!}$ for $0 \le 0 \le 7$.

Hence by substituting (3.6)–(3.12) into w_n in (3.1) with explicit use of $Y_j (3 \le j \le 8)$ from relation (3.8), we find:

$$w_n = \alpha + (1 - L_0)e + \frac{(L_0 - L_1)}{m}\theta_2 e^2 + Z_3 e^3 + Z_4 e^4 + Z_5 e^5 + Z_6 e^6 + Z_7 e^7 + Z_8 e^8 + O(e^9), \tag{3.13}$$

where $Z_i = Z_i(\theta_2, \theta_3 \cdots, \theta_9, L_0, L_1, \ldots, L_7)$ for $3 \le i \le 8$. By selecting $L_0 = 1, L_1 = 1, L_2 = 2$, we have

$$w_n = \alpha + \frac{(m + 9 - 2L_3)\theta_2^3 - 2m\theta_2\theta_3}{2m^3}e^4 + Z_5 e^5 + Z_6 e^6 + Z_7 e^7 + Z_8 e^8 + O(e^9), \tag{3.14}$$

Hence, we obtain $f(w_n)$ as follows:

$$\begin{aligned} f(w_n) = \frac{f^{(m)}(\alpha)\delta^m}{m! 2^m m^{3m}} e^{4m}\Big[1 + \frac{Z_5}{3\delta}e + \frac{2(m-1)Z_5^2 + Z_6\delta}{36m\delta^2}e^2 \\ + \frac{10(m-1)(m-2)Z_5^3 + \nu_1}{1620m^2\delta^3}e^3 + \frac{20(m-1)(m-2)(m-3)Z_5^4 + \nu_2}{38880m^3\delta^4}e^4 + O(e^5)\Big], \end{aligned} \tag{3.15}$$

where $\delta = (9 + m - 2L_3)\theta_2^3 - 2m\theta_2\theta_3$, $\nu_1 = 45(m-1)Z_5Z_6\delta + 27\, Z_7\delta^2$ and $\nu_2 = 180(m-1)(m-2)Z_5^2Z_6\delta + 27(m-1)(5Z_6^2 + 8Z_5Z_7)\delta^2 + 108Z_8\delta^3 + 19440\delta^5\theta_2$.

With the use of (3.10) and (3.15), we get an expression u in (3.3) after Taylor's expansion or multinomial expansion as follows:

$$u = \frac{((9 + m - 2L_3)\theta_2^2 - 2m\theta_3)}{2m^2}e^2 + \beta_3 e^3 + \sum_{i=4}^{6} \beta_i e^i + O(e^7), \tag{3.16}$$

where $\beta_3 = \frac{Z_5 + 3((m+1)\theta_2^2 - 2m\theta_3)((9+m-2L_3)\theta_2^2 - 2m\theta_3)}{6m^3\theta_2}f$ and $\beta_i = \beta_i(m, \theta_2, \theta_3, \ldots, \theta_9, Z_5, Z_6, \ldots, Z_8)$ for $4 \le i \le 6$. Using s in (3.11) and u in (3.16) and expanding Taylor series of $K_f(s, u)$ about $(0, 0)$ up to seventh-order terms we find:

$$\begin{aligned} K_f(s, u) = 1 + s + 2s^2 + L_3 s^3 + L_4 s^4 + L_5 s^5 + L_6 s^6 + K_{70} s^7 \\ + [s + 2s^2 + (1 + L_3)s^3 + (2L_3 + L_4 - 4)s^4 + K_{51}s^5]u + (s + 4s^2 + K_{32}s^3)u^2 + K_{13}su^3 + O(e^8), \end{aligned} \tag{3.17}$$

where $K_{ij} = \frac{1}{i!j!}\frac{\partial^{i+j}}{\partial s^i \partial u^j}K_f(s, u)|_{(s=0,u=0)}$ for $0 \le i \le 7, 0 \le j \le 3$.

Hence by substituting (3.6)–(3.17) into the proposed method (3.1) with explicit uses of $Y_3, \ldots, Y_8, Z_5, \ldots, Z_8$, we obtain the error equation as

$$x_{n+1} - \alpha = x_n - \alpha - m[L_f(s) + K_f(s, u)] \cdot \frac{f(x_n)}{f'(x_n)} = T_1 e + \sum_{i=2}^{8} T_i e^i + O(e^9), \tag{3.18}$$

where $T_1 = -K_{00}$ and the coefficients $T_i (2 \le i \le 8)$ generally depend on m, $\theta_j (2 \le j \le 9)$, $L_j (3 \le j \le 7)$ and $K_{jk} (0 \le j \le 7, 0 \le k \le 3)$. Solving $T_1 = 0$ for K_{00}, we get

$$K_{00} = 0. \tag{3.19}$$

Substituting $K_{00} = 0$ into $T_2 = 0$ and simplifying, we obtain $\frac{K_{10}}{m}\theta_2 = 0$, from which

$$K_{10} = 0 \tag{3.20}$$

follows independently of θ_2. Continuing in this manner at the ith stage with $3 \le i \le 7$, we substitute such K_{jk} found from $T_\ell = 0$ at the preceding stages for $1 \le \ell \le (i - 1)$ into $T_i = 0$ and solve $T_i = 0$ for remaining K_{jk} to find:

$$K_{01} = 0,\ K_{20} = 0, K_{11} = 1,\ K_{30} = 0, K_{40} = 0,\ K_{21} = 2,\ K_{02} = 0, K_{50} = 0,$$

$$K_{31} = 1 + L_3, K_{12} = 1, K_{60} = 0, K_{41} = 2L_3 + L_4 - 4, K_{22} = 4, K_{03} = 0, \tag{3.21}$$

independently of θ_2 and θ_3.

Substituting (3.18)–(3.20) into $T_8 = 0$ and simplifying, we get:

$$T_8 = \frac{\theta_2}{24m^7}\left[\phi_1\theta_2^6 + \phi_2\theta_2^4\theta_3 + \phi_3\theta_2^2\theta_3^2 + \phi_4\theta_2^3\theta_4 + 24(K_{13} - 1)m^3\theta_3^3 - 24m^3\theta_2\theta_3\theta_4\right], \tag{3.22}$$

where

$$\phi_1 = 3879 - 108K_{51} - 24K_{70} - 1780L_3 + 216L_4 + 12L_3(2K_{51} + 17L_3 - 4L_4 - 2L_5) + 108L_5 + 1349m$$
$$+ 12(-K_{51} + (-30 + L_3)L_3 + 2L_4 + L_5)m + 5(33 - 4L_3)m^2 + 7m^3 - 6K_{32}(9 - 2L_3 + m)^2 - 3K_{13}(9 - 2L_3 + m)^3,$$
$$\phi_2 = 2m[-1349 + 12K_{51} - 12(-30 + L_3)L_3 - 24L_4 - 12L_5 - 312m + 36L_3m - 19m^2$$
$$+ 12K_{32}(9 - 2L_3 + m) + 9K_{13}(9 - 2L_3 + m)^2],$$
$$\phi_3 = -12m^2[-43 + 2K_{32} + 4L_3 - 5m + 3K_{13}(9 - 2L_3 + m)],$$
$$\phi_4 = 12m^2(m + 9 - 2L_3).$$

The consequence of the analysis carried out thus far immediately leads us to the following theorem.

Theorem 3.1. *Let $m \in \mathbb{N}$ be given. Let $f : \mathbb{C} \to \mathbb{C}$ have a zero α of multiplicity m and be analytic in a small neighborhood of α. Let $k \in \mathbb{N}$ be given. Let $\theta_j = \frac{m!}{(m-1+j)!} \cdot \frac{f^{(m-1+j)}(\alpha)}{f^{(m)}(\alpha)}$ for $j \in \mathbb{N} - \{1\}$. Let x_0 be an initial guess chosen in a sufficiently small neighborhood of α. Let $L_f : \mathbb{C} \to \mathbb{C}$ be analytic in a neighborhood of 0 and let $K_f : \mathbb{C}^2 \to \mathbb{C}$ be holomorphic in a neighborhood of (0, 0). Let $L_k = \frac{1}{k!}\frac{d^k}{ds^k}L_f(s)|_{s=0}$ for $0 \le k \le 7$ and $K_{ij} = \frac{1}{i!j!}\frac{\partial^{i+j}}{\partial s^i \partial u^j}K_f(s,u)|_{(s=0,u=0)}$ for $0 \le i \le 7, 0 \le j \le 3$. Suppose that $L_0 = L_1 = 1, L_2 = 2, K_{00} = K_{10} = K_{01} = K_{20} = K_{02} = K_{03} = K_{30} = K_{40} = K_{50} = K_{60} = 0, K_{11} = K_{12} = 1, K_{21} = 2, K_{22} = 4, K_{31} = 1 + L_3, K_{41} = 2L_3 + L_4 - 4$ hold. Then iterative methods (3.1) are of eighth-order and possess the following error equation:*

$$e_{n+1} = \frac{\theta_2}{24m^7}\left[\phi_1\theta_2^6 + \phi_2\theta_2^4\theta_3 + \phi_3\theta_2^2\theta_3^2 + \phi_4\theta_2^3\theta_4 + 24(K_{13} - 1)m^3\theta_3^3 - 24m^3\theta_2\theta_3\theta_4\right]e_n^8 + O(e_n^9), \tag{3.23}$$

where $\phi_i (1 \le i \le 4)$ is given in (3.22).

4. Special cases of weight functions

According to Theorem 3.1, we are able to find $L_f(s)$ and $K_f(s, u)$ in the form of Taylor polynomials below:

$$\begin{cases} L_f(s) &= 1 + s + 2s^2 + L_3s^3 + L_4s^4 + L_5s^5 + L_6s^6 + L_7s^7 + O(e^8), \\ K_f(s,u) &= su\left[1 + u + K_{13}u^2 + 2s(1 + 2u) + s^2(1 + L_3 + K_{32}u) + s^3(-4 + 2L_3 + L_4) + K_{51}s^4\right] \\ & \quad + \ K_{70}s^7 + O(e^8), \end{cases} \tag{4.1}$$

where $L_3, L_4, L_5, L_6, L_7, K_{13}, K_{32}, K_{51}$ and K_{70} are free parameters.

It is evident that various forms of weight functions $L_f(s)$ and $K_f(s, u)$ are available to design a family of optimal multiple-zero finders. In the existing studies by [16,24], we have noticed that either weight function L_f or K_f is of polynomial type has empirically shown poor convergence. Consequently, taking into account the fact that $s = O(e), u = O(e^2)$ and $\frac{f(x_n)}{f'(x_n)} = O(e)$, we shall establish eighth-order convergence by restricting ourselves to considering $L_f(s)$ as a family of second-order univariate rational functions and $K_f(s, u)$ as a family of fifth-order bivariate rational functions with real coefficients in the form below.

$$\begin{cases} L_f(s) &= \dfrac{b_0 + b_1s + b_2s^2}{1 + a_1s}, \\ K_f(s,u) &= \dfrac{q_0 + q_1s + q_2s^2 + q_3s^3 + q_4s^4 + (q_5 + q_6s + q_7s^2 + q_8s^3 + q_9s^4)u}{1 + r_1s + r_2s^2 + r_3s^3 + r_4s^4 + (r_5 + r_6s + r_7s^2 + r_8s^3 + r_9s^4)u}, \end{cases} \tag{4.2}$$

where $a_i, b_i, q_i, r_i \in \mathbb{R}$ are to be determined for optimal eighth-order convergence. Note that $K_f(s, u)$ adopts the only linear u reducing the complexities that may be strengthened from the presence of nonlinear u.

We let (4.2) satisfy the constraints described by hypotheses of Theorem 3.1, which give us the following coefficients: with $b = b_2$

$$\begin{cases} b_0 = 1, b_1 = -1 + b, a_1 = -2 + b, \\ q_0 = q_1 = q_2 = q_3 = q_4 = q_5 = 0, q_6 = 1, q_7 = 2 + r_1, q_8 = \dfrac{1}{2}\big[-2 - 2b^2 + q_9 - r_1 + 2b(4 + r_1) - r_3\big], \\ r_2 = -5 + 2b + q_8 - 2r_1, r_5 = -1, r_6 = -2 - r_1. \end{cases} \tag{4.3}$$

As a result, the reduced form of the desired weight functions is found to be:

$$\begin{cases} L_f(s) \quad = \dfrac{1 + (b-1)s + bs^2}{1 + (b-2)s}, \\ K_f(s,u) = \dfrac{su\big[2 + 2(2 + r_1)s - (2 + 2b^2 - q_9 + r_1 - 2b(4 + r_1) + r_3)s^2 + 2q_9 s^3\big]}{2 + 2r_1 s + [-12 - 2b(b-6) + q_9 + (2b-5)r_1 - r_3]s^2 + 2r_3 s^3 + 2r_4 s^4 + 2[-1 - (2 + r_1)s + r_7 s^2 + r_8 s^3 + r_9 s^4]u}, \end{cases} \tag{4.4}$$

where $b, r_1, r_3, r_4, r_7, r_8, r_9, q_9 \in \mathbb{R}$ are 8 free parameters.

In view of the fact that $s = O(e)$ and $u = O(e^2)$, we find $K_f(s,u) = O(e^7)$ from (3.17), according to which the last sub-step iterative scheme of (3.1) should give rise to an optimal convergence order of eight with a suitable choice of parameters.

Although numerous cases of weight functions satisfying Theorem 3.1 can be constructed in this paper, we are especially interested in special cases with $b = 0$ for which all of the extraneous fixed points of the proposed scheme (3.1) are purely imaginary. The notion of an extraneous fixed point and its preference for being purely imaginary will be fully discussed in Section 5. From (5.7) of Section 5, we desire the governing equation of the extraneous fixed points to take the form of

$$H(z) = \frac{1}{2(1+t)} \cdot \frac{G(t)}{\Omega(t)}, \ t = z^2, \tag{4.5}$$

where $G(t) = t^{\gamma_1}(1+t)^{\gamma_2}(1+3t)^{\gamma_3} \cdot g(t)$ for $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{N}$. In addition, $g(t)$ and $\Omega(t)$ are polynomials of degree at most 2 and 6, respectively, with $\gamma_1 + \gamma_2 + \gamma_3 = 5$. Observe that $G(t)$ and $\Omega(t)$ have common factors, which further simplify the resulting expressions of $H(z)$. The remaining task is again for us to determine appropriate parameters of weight functions in such a way that all the roots of $H(z)$ should be located on imaginary axis of the complex plane.

In Section 5, we shall give an extensive investigation with an appropriate selection of free parameters leading us to purely imaginary extraneous fixed points. To this end, we will seek feasible relationships among the free parameters by imposing some constraints on simplifying the numerator of the resulting expression $G(t)$ to be described in (5.8). The following cases **A–J** are of our main interest whose values of $(\gamma_1, \gamma_2, \gamma_3)$ and 7 parameters $q_9, r_1, r_3, r_4, r_7, r_8, r_9 \in \mathbb{R}$ for each case with $\lambda = r_8 + r_9$ are discussed in Section 5. We remark that the cases under consideration form a biparametric family of methods dependent upon two parameters λ and r_9.

Case A: $(\gamma_1, \gamma_2, \gamma_3) = (0, 2, 3), \ -\frac{11}{4} < \lambda \le 21 - 10\sqrt{3}$,

$q_9 = -\frac{3}{5}(4 + \lambda), r_1 = \frac{1}{5}(-29 - \lambda), r_3 = -9 - 2\lambda, r_4 = \frac{1}{20}(68 - 5r_9 + 22\lambda)$,

$r_7 = \frac{1}{20}(5r_9 - 18(4 + \lambda)), r_8 = -r_9 + \lambda$.

Case B: $(\gamma_1, \gamma_2, \gamma_3) = (0, 3, 2), \ 1 < \lambda \le \frac{65}{16}$,

$q_9 = -1 - \lambda, r_1 = -\frac{13}{2}, r_3 = -\frac{11}{2} - 3\lambda, r_4 = \frac{1}{4}(-6 - r_9 + 10\lambda)$,

$r_7 = \frac{1}{4}(r_9 - 2(10 + \lambda)), r_8 = -r_9 + \lambda$.

Case C: $(\gamma_1, \gamma_2, \gamma_3) = (0, 4, 1), \ -5 + 4\sqrt{5} \le \lambda < 7.$,

$q_9 = -5, r_1 = \frac{1}{2}(-9 - \lambda), r_3 = \frac{1}{2}(-31 - \lambda), r_4 = \frac{1}{4}(66 - r_9 - 8\lambda)$,

$r_7 = \frac{1}{4}(-4 + r_9 - 6\lambda), r_8 = -r_9 + \lambda$.

Case D: $(\gamma_1, \gamma_2, \gamma_3) = (0, 5, 0), \ \frac{1}{3}(-2 + 6\sqrt{6}) \le \lambda < \frac{71}{6}$,

$q_9 = -\frac{2}{3} - \lambda, r_1 = \frac{1}{9}(-47 - 3\lambda), r_3 = -\frac{55}{9} - \frac{8\lambda}{3}, r_4 = \frac{1}{12}(68 - 3r_9 + 6\lambda)$,

$r_7 = \frac{1}{36}(-88 + 9r_9 - 42\lambda), r_8 = -r_9 + \lambda$.

Case E: $(\gamma_1, \gamma_2, \gamma_3) = (1, 2, 2), \ -6 < \lambda < 4$,

$q_9 = \frac{1}{3}(-5 - \lambda), r_1 = \frac{1}{6}(-37 - 2\lambda), r_3 = \frac{1}{6}(-43 - 8\lambda), r_4 = \frac{1}{12}(-3r_9 + 2(5 + \lambda))$,

$r_7 = \frac{1}{12}(-52 + 3r_9 - 14\lambda), r_8 = -r_9 + \lambda$.

Case F: $(\gamma_1, \gamma_2, \gamma_3) = (1, 3, 1), \ -\frac{11}{3} < \lambda < 7$,

$q_9 = \frac{1}{2}(-3 - \lambda), r_1 = \frac{1}{4}(-25 - \lambda), r_3 = \frac{1}{4}(-27 - 7\lambda), r_4 = \frac{1}{4}(3 - r_9 + \lambda)$,

$r_7 = \frac{1}{4}(-18 + r_9 - 4\lambda), r_8 = -r_9 + \lambda$.

Case G: $(\gamma_1, \gamma_2, \gamma_3) = (1, 4, 0), \ 1 < \lambda < 7$,

$q_9 = 2 - \lambda, r_1 = \frac{1}{3}(-17 - \lambda), r_3 = \frac{1}{3}(-1 - 8\lambda), r_4 = \frac{1}{4}(-4 - r_9 + 2\lambda)$,

$r_7 = \frac{1}{12}(-40 + 3r_9 - 14\lambda), r_8 = -r_9 + \lambda$.

Case H: $(\gamma_1, \gamma_2, \gamma_3) = (2, 2, 1)$, $-\frac{73}{9} < \lambda < -1$,
$q_9 = \frac{1}{3}$, $r_1 = \frac{1}{6}(-43 - 3\lambda)$, $r_3 = \frac{1}{6}(-13 - 3\lambda)$, $r_4 = -\frac{1}{6} - \frac{r_9}{4}$,
$r_7 = \frac{1}{12}(-76 + 3r_9 - 18\lambda)$, $r_8 = -r_9 + \lambda$.
Case I: $(\gamma_1, \gamma_2, \gamma_3) = (2, 3, 0)$, $-\frac{19}{2} < \lambda < 22 - 14\sqrt{2}$,
$q_9 = \frac{6+\lambda}{7}$, $r_1 = \frac{1}{7}(-41 - \lambda)$, $r_3 = \frac{1}{7}(-17 - 4\lambda)$, $r_4 = \frac{1}{28}(-7r_9 - 2(6 + \lambda))$,
$r_7 = \frac{1}{28}(-104 + 7r_9 - 22\lambda)$, $r_8 = -r_9 + \lambda$.
Case J: $(\gamma_1, \gamma_2, \gamma_3) = (3, 2, 0)$, $-\frac{31}{3} < \lambda < -7$,
$q_9 = \frac{1}{5}(26 + 3\lambda)$, $r_1 = 5 + \lambda$, $r_3 = \frac{1}{5}(-23 - 4\lambda)$, $r_4 = \frac{1}{20}(-52 - 5r_9 - 6\lambda)$,
$r_7 = \frac{1}{4}(72 + r_9 + 6\lambda)$, $r_8 = -r_9 + \lambda$.

5. Extraneous fixed points and their dynamics

Understanding the dynamics of iterative map (3.1) requires the knowledge of its extraneous fixed points [25] as well as relevant basins of attraction. The dynamics underlying basins of attraction was initiated by Stewart [26] and followed by works of Amat et al. e.g. [27,28], Andreu et al. [29], Argyros–Magreñan [30], Chicharro et al. [31], Chun et al. [32], Chun–Neta [33], Cordero et al. [34], Geum et al. [16,35], Magreñan [36], Magreñán et al. [37], Neta et al. [38–40] and Scott et al. [41].

An approximate zero α of a nonlinear equation $f(x) = 0$ is usually sought by means of a fixed point ξ of iterative methods of the form

$$x_{n+1} = R_f(x_n),\ n = 0, 1, \ldots, \tag{5.1}$$

where R_f is the iteration function under consideration. In general, R_f might possess other fixed points $\xi \neq \alpha$, being called the *extraneous fixed points* of the iteration function R_f. Such extraneous fixed points may induce attractive, indifferent or repulsive cycles as well as other periodic orbits or chaotic attractors [42] influencing the dynamics underlying the basins of attraction. Exploration of the dynamics and discovery of its complicated behavior give us a valuable motivation of the current analysis. In view of proposed family of methods (3.1), we establish a weighted family of modified Newton-like iterative maps (5.1) as follows:

$$x_{n+1} = R_f(x_n) = x_n - m\frac{f(x_n)}{f'(x_n)}H_f(x_n), \tag{5.2}$$

where $H_f(x_n) = L_f(s) + K_f(s, u)$ can be regarded as a weight function of the classical Newton's method. It is obvious that α is a fixed point of R_f. The points $\xi \neq \alpha$ for which $H_f(\xi) = 0$ are extraneous fixed points of R_f.

For convenience of analysis of the relevant dynamics, we only consider combinations of weight functions $L_f(s)$ and $K_f(s, u)$ in the form of univariate and bivariate rational functions as described by (4.2). A special attention will be paid to some selected cases to be shown later in this section in order to pursue further properties of their extraneous fixed points and relevant dynamics associated with their basins of attraction. The existence of such extraneous fixed points would affect the global iteration dynamics, which was demonstrated for simple zeros via König functions and Schröder functions [25] applied to a family of functions $\{f_k(x) = x^k - 1, k \geq 2\}$ according to the joint work of Vrscay and Gilbert [25] published in 1988. Especially the presence of attractive cycles induced by the extraneous fixed points of R_f may alter the basins of attraction due to the trapped sequence $\{x_n\}$. Even in the case of repulsive or indifferent fixed points, an initial value x_0 chosen near a desired root may converge to another unwanted remote root. Indeed, these aspects of the Schröder functions were observed in an application to the same family of functions $\{f_k(x) = x^k - 1, k \geq 2\}$.

For simplified dynamics related to the extraneous fixed points of iterative maps (5.2), we first choose a simple quadratic polynomial from the family of functions $\{f_k(x) = x^k - 1, k \geq 2\}$. By closely following the works of Chun et al. [24,43] and Neta et al. [38,40,44], we then construct $H_f(x_n) = L_f(s) + K_f(s, u)$ in (5.2). We now take the multiplicity m of zero α into account and apply a prototype quadratic polynomial $f(z) = (z^2 - 1)^m$ to $H_f(x_n)$ in order to construct $H(z)$, with a change of a variable $t = z^2$, in the form of

$$H(z) = \frac{\mathcal{N}(t)}{\mathcal{D}(t)}, \tag{5.3}$$

where both $\mathcal{D}(t)$ and $\mathcal{N}(t)$ are polynomial functions of t with no common factors. Since H is a rational function, it would be preferable for us to deal with the underlying dynamics of iterative map (5.2) on the Riemann sphere [45] where points "0(zero)" and "∞" can be treated as the desired extraneous fixed points. If such points arise, we are interested in only the finite extraneous fixed point 0 under which the relevant dynamics can be described in a region containing the origin by investigating the attractor basins associated with iterative map (5.2).

The extraneous fixed points ξ of R_f in (5.2) can be directly found from the roots t of $H(z)$ with $z = t^{1/2}$ via relation below:

$$\xi = \begin{cases} t^{\frac{1}{2}}, & \text{if } t \neq 0, \\ 0(\text{double root}), & \text{if } t = 0. \end{cases} \tag{5.4}$$

5.1. Purely imaginary extraneous fixed points

It is clear that the boundary of two basins of attraction of two roots for the prototype quadratic polynomial $f(z)=(z^2-1)^m$ is the imaginary axis of the complex plane. Indeed, the imaginary axis symmetrically divides the whole complex plane into two half planes. Since we want to display the convergence behavior in the dynamical planes through the basins of attraction in a square region centered at the origin, the resulting dynamics behind the extraneous fixed points on the symmetry (imaginary) axis would be less influenced by the presence of the possible periodic or chaotic attractors. This motivates our exploration of the extraneous fixed points on the imaginary axis influencing the convergence behavior of iterative map (5.2).

Our important task is to construct a possible combination of weight functions L_f and K_f leading to purely imaginary extraneous fixed points, whose investigation was first done by Chun et al. [43]. As a preliminary task, we first describe the following lemma regarding the negative real roots of a quadratic equation, which would play a role in determining the desired purely imaginary extraneous fixed points.

We now introduce the following lemma shown in [17] regarding the negative real roots of a quadratic equation for later use to characterize the equation $g(t)$ described by (5.11).

Lemma 5.1. *Let $q(x)=ax^2+bx+c$ be a quadratic equation with real coefficients $a\neq 0$, b, c satisfying $b^2-4ac\geq 0$. Let t_1 and t_2 be the two roots of $q(x)=0$. Then both roots $t_1<0$ and $t_2<0$ hold if and only if all three coefficients a, b, c have the same sign.*

Employing weight function $L_f(s)$ with parameter $b=0$ in (4.4) applied to $f(z)=(z^2-1)^m$, we find:

$$\begin{cases} s &= \dfrac{1}{4}(1-\dfrac{1}{z^2}),\\ L_f &= \dfrac{1}{2}(\dfrac{3z^2+1}{z^2+1}). \end{cases} \tag{5.5}$$

In addition, we are able to express $K_f(s,u)$ in terms of z and free parameters $q_9, r_1, r_3, r_4, r_7, r_8, r_9$ with the use of

$$u=\frac{1}{4}\cdot\frac{(z^2-1)^2}{(z^2+1)^2}. \tag{5.6}$$

Although such lengthy expression of K_f is not explicitly shown here, the simplified second-order form of L_f will greatly reduce the complexity of K_f as well as the desired $H_f=L_f+K_f$ given by (5.3). Consequently, the explicit form of the relevant $H(z)$ given by (5.3) takes the form of

$$H(z)=\frac{1}{2(1+t)}\cdot\frac{G(t;\beta_0,\beta_1,\ldots,\beta_7)}{\Omega(t;\omega_0,\omega_1,\ldots,\omega_6)}, \tag{5.7}$$

where $G(t;\beta_0,\beta_1,\ldots,\beta_7)$ and $\Omega(t;\omega_0,\omega_1,\ldots,\omega_6)$ are concisely denoted by $G(t)$ and $\Omega(t)$, respectively, as below:

$$G(t)=\sum_{i=0}^{7}\beta_i\,t^i, \tag{5.8}$$

with $\beta_0=2q_9+4r_4+r_9$, $\beta_1=8-14q_9+4r_1-12r_3+4r_4-4r_8-3r_9$, $\beta_2=-288+64q_8+34q_9-112r_1-48r_3-28r_4+16r_7+8r_8-3r_9$, $\beta_3=-1112+192q_8-30q_9-652r_1+100r_3+4r_4-16r_7+20r_8+25r_9$, $\beta_4=1792-128q_8-10q_9-704r_1+64r_3+44r_4-96r_7-80r_8-45r_9$, $\beta_5=6104-384q_8+38q_9-84r_1-132r_3-20r_4+224r_7+100r_8+39r_9$, $\beta_6=8736+64q_8-26q_9+1328r_1-16r_3-20r_4-176r_7-56r_8-17r_9$, $\beta_7=1144+192q_8+6q_9+220r_1+44r_3+12r_4+48r_7+12r_8+3r_9$, and

$$\Omega(t)=\sum_{i=0}^{6}\omega_i\,t^i, \tag{5.9}$$

with $\omega_0=4r_4+r_9$, $\omega_1=-2(8r_3+4r_4+2r_8+3r_9)$, $\omega_2=-320+64q_8-128r_1+16r_3-4r_4+16r_7+20r_8+15r_9$, $\omega_3=-4(-32+48r_1-8r_3-4r_4+16r_7+10r_8+5r_9)$, $\omega_4=1024-128q_8-192r_1-32r_3-4r_4+96r_7+40r_8+15r_9$, $\omega_5=2(1472+224r_1-8r_3-4r_4-32r_7-10r_8-3r_9)$, $\omega_6=320+64q_8+64r_1+16r_3+4r_4+16r_7+4r_8+r_9$.

Observe that the weight function $L_f(z)=\frac{1}{2}(\frac{1+3t}{1+t})$ with $t=z^2$ contains two factors $(1+3t)$ and $(1+t)$. In view of this observation, we naturally consider a special case of $H(z)$ in the form of a simplified rational function possibly with such two factors. To this end, we construct

$$H_f=L_f+K_f=\frac{1}{2(1+t)}\frac{G(t)}{\Omega(t)}, \tag{5.10}$$

where $G(t)$ and $\Omega(t)$ may involve some of such factors in addition to a factor t corresponding to the origin (considered as purely imaginary) of the complex plane, as shown below:

$$\begin{cases} G(t) &= t^{\gamma_1}(1+t)^{\gamma_2}(1+3t)^{\gamma_3}\cdot g(t)\ \text{ for } \gamma_1,\gamma_2,\gamma_3\in\mathbb{N}\bigcup\{0\},\ \gamma_1+\gamma_2+\gamma_3=5\\ \Omega(t) &= \mu(t), \end{cases} \tag{5.11}$$

where $g(t)$ and $\mu(t)$ are polynomials of degree at most 2 and 6, respectively. The expression of $H(z)$ in (5.7) will be further simplified as:

$$H(z) = \frac{1}{2}\cdot t^{\gamma_1}(1+t)^{\gamma_2-1}(1+3t)^{\gamma_3-1}\cdot\frac{g(t)}{\mu(t)}\ \text{ with } t = z^2. \tag{5.12}$$

If we further restrict with $\gamma_2 \geq 2$, then all possible 10 combinations of $(\gamma_1, \gamma_2, \gamma_3)$ are listed by {(0, 2, 3), (0, 3, 2), (0, 4, 1), (0, 5, 0), (1, 2, 2), (1, 3, 1), (1, 4, 0), (2, 2, 1), (2, 3, 0), (3, 2, 0)}. For convenience, we assign ten case letters **A, B, C, D, F, G, H, I, J** to those 10 combinations in order.

For each case, we should let $H(z)$ have all purely imaginary extraneous fixed points. To do so, we further require that all the roots of $g(t)$ should be negative. Let $g(t) = g_0 + g_1t + g_2t^2$ and $\mu(t) = p_0 + p_1t + p_2t^2 + p_3t^3 + p_4t^4 + p_5t^5 + p_6t^6$. Then the roots of $g(t) = 0$ would contribute to the desired extraneous fixed points. In view of the fact that $\gamma_1 + \gamma_2 + \gamma_3 = 5$, the forms of (5.11) would require a set of five constraints

$$0 = G(0) = G'(0) = \cdots G^{(\gamma_1-1)}(0) = G(-1) = G'(-1) = \cdots G^{(\gamma_2-1)}(-1) = G(-\frac{1}{3}) = G'(-\frac{1}{3}) = \cdots G^{(\gamma_3-1)}(-\frac{1}{3}) \tag{5.13}$$

Since $G(-1) = 128(32+8r_1-4r_7-2r_8-r_9)$, $\Omega(-1) = -64(32+8r_1-4r_7-2r_8-r_9)$, we find that $G(-1) = -2\Omega(-1)$, from which $G(-1) = 0$ implies $\Omega(-1) = 0$. Consequently, we find $\mu(t)$=$(1+t)w(t)$ with $w(t) = d_0 + d_1t + d_2t^2 + d_3t^3 + d_4t^4 + d_5t^5$ provided that $G(-1) = 0$. For any of cases, we can solve these 5 constraints for 5 parameters q_9, r_1, r_3, r_4, r_7 in terms of at most 2 remaining parameters r_8 and r_9. If we substitute these 5 parameters back into $G(t)$ and $\Omega(t)$ in (5.11), the explicit forms of $g(t)$ and $w(t)$ with their coefficients in terms of at most 2 remaining parameters r_8, r_9 for a given combination of $(\gamma_1, \gamma_2, \gamma_3)$. If a new parameter $\lambda = r_8 + r_9$ is conveniently introduced, then for all 10 **Cases A**, **B**,…, **J**, we can express 6 parameters $q_9, r_1, r_3, r_4, r_7, r_8 \in \mathbb{R}$ in terms of two parameters λ and r_9. After a tedious algebra, the resulting parameters for all 10 cases are already described at the end of Section 3. The following proposition plays an important role in analyzing both computational and dynamical aspects of proposed family of methods (3.1) .

Proposition 5.1. *For each case, all coefficients of $g(t)$ and $w(t)$ can be expressed as an affine combination of λ.*

Proof. Since one proof is similar to another, it suffices to consider a typical case **A** with $(\gamma_1, \gamma_2, \gamma_3) = (0, 2, 3)$ and $\lambda = r_8 + r_9$. Solving the 6 constraints, we obtain $q_9 = -\frac{3}{5}(4+\lambda)$, $r_1 = \frac{1}{5}(-29-\lambda)$, $r_3 = -9-2\lambda$, $r_4 = \frac{1}{20}(68-5r_9+22\lambda)$, $r_7 = \frac{1}{20}(5r_9-18(4+\lambda))$, $r_8 = -r_9+\lambda$. Substituting these coefficients into $G(t)$ and $\Omega(t)$, we find:

$$\begin{cases} g(t) &= \dfrac{4}{5}[11+4\lambda+t(54-4\lambda)+15t^2] \\ w(t) &= \dfrac{2}{5}[34+11\lambda+t(258+37\lambda)+78t^2(14+\lambda)-6t^3(9\lambda-334)+t^4(1498-73\lambda)+t^5(234+\lambda)], \end{cases}$$

completing the proof. □

We now seek the possible extraneous fixed points from the roots of the quadratic equation

$$g(t) = g_0 + g_1t + g_2t^2 \tag{5.14}$$

with $g_i = g_i(\lambda)$, $(0 \leq i \leq 2)$, being dependent on parameter λ. Let $\mathcal{D}$ be the discriminant of $g(t)$ to be expressed in terms of parameter λ. We denote a set

$$\Delta = \{\lambda \in \mathbb{R} : \mathcal{D} \geq 0\}. \tag{5.15}$$

We further denote a set

$$\mathbf{P} = \{\lambda \in \mathbb{R} : g_0g_1 > 0 \text{ and } g_0g_2 > 0\} \tag{5.16}$$

whose elements make all three coefficients g_0, g_1, g_2 have the same sign. We now use Lemma 5.1 to locate all two negative roots of $g(t) = 0$ for purely imaginary extraneous fixed points. After a lengthy algebra, we are able to find the desired sets Δ, **P** and $\Delta \cap \mathbf{P}$ containing λ-values for which purely imaginary extraneous fixed points can be located.

Notice that extraneous fixed point zeros $\xi = 0$ (being considered as purely imaginary) may be found on the boundary of **P**. Let $\bar{\mathbf{P}}$ denote the closure of **P**. According to interesting values of $\lambda \in \Delta \cap \bar{\mathbf{P}}$, we classify the subcases of each case from **Cases A**, **B**, …, **J** by appending sequential Arabic numerals such as **Cases A1**, **A2**, …, **B1**, **B2**, …, **J1**, **J2**, ….

Presented below are values of $(\gamma_1, \gamma_2, \gamma_3)$, λ, $g(t)$, Δ, **P** and $\Delta \cap \mathbf{P}$ for each case under consideration with $\lambda = r_8 + r_9$.

Case A: $(\gamma_1, \gamma_2, \gamma_3) = (0, 2, 3)$.

(1) $g(t) = 11 + 4\lambda + t(54 - 4\lambda) + 15t^2$.

(2) $\Delta = \{\lambda : \lambda \leq 21 - 10\sqrt{3} \text{ or } \lambda \geq 21 + 10\sqrt{3}\}$, $\mathbf{P} = \{\lambda : -\frac{11}{4} < \lambda < \frac{27}{2}\}$.

(3) $\Delta \cap \mathbf{P} = \{\lambda : -\frac{11}{4} < \lambda < 21 - 10\sqrt{3}\}$.

The five subcases **A1**, **A2**, …, **A5** are identified with $\lambda \in \{-\frac{11}{4}, -2, -\frac{3}{2}, 1, 2\}$ in order.

Case B: $(\gamma_1, \gamma_2, \gamma_3) = (0, 3, 2)$.

(1) $g(t) = -1 + \lambda - 2t(\lambda - 8) + t^2(1 + \lambda)$.

(2) $\Delta = \{\lambda : \lambda \le \frac{65}{16}\}$, $\mathbf{P} = \{\lambda : 1 < \lambda < 8\}$.

(3) $\Delta \cap \mathbf{P} = \{\lambda : 1 < \lambda \le \frac{65}{16}\}$.

The five subcases **B1**, **B2**, …, **B5** are identified with $\lambda \in \{1, \frac{3}{2}, 2, 4, \frac{65}{16}\}$ in order.

Case C: $(\gamma_1, \gamma_2, \gamma_3) = (0, 4, 1)$.

(1) $g(t) = -7 + \lambda + t(10 - 6\lambda) + 5t^2(\lambda - 7)$.

(2) $\Delta = \{\lambda : \lambda \le -5 - 4\sqrt{5}$ or $\lambda \ge -5 + 4\sqrt{5}\}$, $\mathbf{P} = \{\lambda : \frac{5}{3} < \lambda < 7\}$.

(3) $\Delta \cap \mathbf{P} = \{\lambda : -5 + 4\sqrt{5} \le \lambda < 7\}$.

The five subcases **C1**, **C2**, …, **C5** are identified with $\lambda \in \{4, \frac{17}{3}, 6, \frac{125}{18}, 7\}$ in order.

Case D: $(\gamma_1, \gamma_2, \gamma_3) = (0, 5, 0)$.

(1) $g(t) = -3 + t(2 - 6\lambda) + t^2(6\lambda - 71)$.

(2) $\Delta = \{\lambda : \lambda \le \frac{1}{3}(-2 - 6\sqrt{6})$ or $\lambda \ge \frac{1}{3}(-2 + 6\sqrt{6})\}$, $\mathbf{P} = \{\lambda : \frac{1}{3} < \lambda < \frac{71}{6}\}$.

(3) $\Delta \cap \mathbf{P} = \{\lambda : \frac{1}{3}(-2 + 6\sqrt{6}) \le \lambda < \frac{71}{6}\}$.

The five subcases **D1**, **D2**, …, **D5** are identified with $\lambda \in \{5, \frac{22}{3}, 9, \frac{670}{57}, \frac{71}{6}\}$ in order.

Case E: $(\gamma_1, \gamma_2, \gamma_3) = (1, 2, 2)$.

(1) $g(t) = -3(6 + \lambda) + 2t(\lambda - 13) + t^2(\lambda - 4)$.

(2) $\Delta = \mathbb{R}$, $\mathbf{P} = \{\lambda : -6 < \lambda < 4\}$.

(3) $\Delta \cap \mathbf{P} = \{\lambda : -6 < \lambda < 4\}$.

The seven subcases **E1**, **E2**, …, **E7** are identified with $\lambda \in \{-6, -5, -3, -1, 1, \frac{27}{7}, 4\}$ in order.

Case F: $(\gamma_1, \gamma_2, \gamma_3) = (1, 3, 1)$.

(1) $g(t) = -11 - 3\lambda + 2t(\lambda - 23) + t^2(\lambda - 7)$.

(2) $\Delta = \mathbb{R}$, $\mathbf{P} = \{\lambda : -\frac{11}{3} < \lambda < 7\}$.

(3) $\Delta \cap \mathbf{P} = \{\lambda : -\frac{11}{3} < \lambda < 7\}$.

The six subcases **F1**, **F2**, …, **F6** are identified with $\lambda \in \{-\frac{11}{3}, -3, 1, 3, \frac{47}{7}, 7\}$ in order.

Case G: $(\gamma_1, \gamma_2, \gamma_3) = (1, 4, 0)$.

(1) $g(t) = 1 - \lambda - 18t + t^2(\lambda - 7)$.

(2) $\Delta = \mathbb{R}$, $\mathbf{P} = \{\lambda : 1 < \lambda < 7\}$.

(3) $\Delta \cap \mathbf{P} = \{\lambda : 1 < \lambda < 7\}$.

The five subcases **G1**, **G2**, …, **G5** are identified with $\lambda \in \{1, 2, 4, \frac{130}{19}, 7\}$ in order.

Case H: $(\gamma_1, \gamma_2, \gamma_3) = (2, 2, 1)$.

(1) $g(t) = -73 - 9\lambda + t(6\lambda - 26) + 3t^2(1 + \lambda)$.

(2) $\Delta = \mathbb{R}$, $\mathbf{P} = \{\lambda : -\frac{73}{9} < \lambda < -1\}$.

(3) $\Delta \cap \mathbf{P} = \{\lambda : -\frac{73}{9} < \lambda < -1\}$.

The seven subcases **H1**, **H2**, …, **H7** are identified with $\lambda \in \{-\frac{73}{9}, -6, -5, -3, -2, -\frac{15}{14}, -1\}$ in order.

Case I: $(\gamma_1, \gamma_2, \gamma_3) = (2, 3, 0)$.

(1) $g(t) = 19 + 2\lambda - 2t(\lambda - 15) + 7t^2$.

(2) $\Delta = \{\lambda : \lambda \le 22 - 14\sqrt{2}$ or $\lambda \ge 22 + 14\sqrt{2}\}$, $\mathbf{P} = \{\lambda : -\frac{19}{2} < \lambda < 15\}$.

(3) $\Delta \cap \mathbf{P} = \{\lambda : -\frac{19}{2} < \lambda \le 22 - 14\sqrt{2}\}$.

The five subcases **I1**, **I2**, …, **I5** are identified with $\lambda \in \{-\frac{19}{2}, -7, -6, -1, 1\}$ in order.

Case J: $(\gamma_1, \gamma_2, \gamma_3) = (3, 2, 0)$.

(1) $g(t) = -7(7 + \lambda) + t(58 + 4\lambda) + t^2(31 + 3\lambda)$.

(2) $\Delta = \mathbb{R}$, $\mathbf{P} = \{\lambda : -\frac{31}{3} < \lambda < -7\}$.

(3) $\Delta \cap \mathbf{P} = \{\lambda : -\frac{31}{3} < \lambda < -7\}$.

The five subcases **J1**, **J2**, …, **J5** are identified with $\lambda \in \{-\frac{31}{3}, -\frac{2334}{227}, -9, -\frac{26}{3}, -7\}$ in order.

Although rich subcases are available as considered thus far, in Table 1, we preferably present 32 sub-subcases **A3Z, A4X**, …, **I5Z, J4X, J5Y** based on simplified forms of their corresponding weight functions $K_f(s, u)$ along with parameter values of $q_9, r_1, r_3, r_4, r_7, r_8, r_9$, λ and $H(z)$. The sub-subcase numbers ending in letters X, Y, Z correspond to the values of $r_4 = 0, r_7 = 0, r_9 = 0$, respectively. Indeed, Table 2 lists the extraneous fixed points for the specially selected 18 sub-subcases.

The analysis done so far and a thorough inspection of Table 1 yield the following useful remark.

Remark 5.2. (i) Once λ is chosen, we have freedom to select parameter r_9. Note that $H(z)$ can be obtained without specifying parameter values of r_9 for all selected cases. (ii) Three cases **(A3X, B3Z, E2Y)** (highlighted in yellow) give the

Table 1
$H(z)$ for the selected values of λ and $q_9, r_1, r_3, r_4, r_7, r_8, r_9$.

Case	$H(z),\ t=z^2$	(q_9,r_1,r_3,r_4,r_7)	(r_8,r_9)	λ
A3Z	$\frac{2(1+3t)^2(1+12t+3t^2)}{7+60t+210t^2+204t^3+31t^4}$	$(-\frac{3}{2},-\frac{11}{2},-6,\frac{7}{4},-\frac{9}{4})$	$(-\frac{3}{2},0)$	$\frac{3}{2}$
A4X	$\frac{(3+t)(1+3t)^4}{9+59t+234t^2+390t^3+285t^4+47t^5}$	$(-3,-6,-11,0,0)$	$(-17,18)$	1
B2X	$\frac{2(1+t)(1+3t)(1+26t+5t^2)}{9+100t+238t^2+148t^3+17t^4}$	$(-\frac{5}{2},-\frac{13}{2},-10,0,-\frac{7}{2})$	$(-\frac{15}{2},9)$	$\frac{3}{2}$
B3Y	$\frac{2(1+3t)^2(1+12t+3t^2)}{7+60t+210t^2+204t^3+31t^4}$	$(-3,-\frac{13}{2},-\frac{23}{2},-\frac{5}{2},0)$	$(-22,24)$	2
B4Z	$\frac{2(1+t)^2(1+3t)^2(3+5t)}{17+81t+274t^2+346t^3+253t^4+53t^5}$	$(-5,-\frac{13}{2},-\frac{35}{2},\frac{17}{2},-7)$	$(4,0)$	4
C2Z	$\frac{8(1+t)^2(1+18t+5t^2)}{31+220t+322t^2+172t^3+23t^4}$	$(-5,-\frac{22}{3},-\frac{55}{3},\frac{31}{6},-\frac{19}{2})$	$(\frac{17}{3},0)$	$\frac{17}{3}$
C3Z	$\frac{2(1+t)(1+3t)(1+26t+5t^2)}{9+100t+238t^2+148t^3+17t^4}$	$(-5,-\frac{15}{2},-\frac{37}{2},\frac{9}{2},-10)$	$(6,0)$	6
C5X	$\frac{64t(1+t)^2(1+3t)}{5+123t+370t^2+406t^3+121t^4-t^5}$	$(-5,-8,-19,0,-9)$	$(-3,10)$	7
D2Z	$\frac{4(1+t)^3(1+14t+9t^2)}{(1+3t)(14+59t+71t^2+41t^3+7t^4)}$	$(-8,-\frac{23}{3},-\frac{77}{3},\frac{28}{3},-11)$	$(\frac{22}{3},0)$	$\frac{22}{3}$
D3X	$\frac{16(1+t)^3(3+t)(1+17t)}{183+1457t+3206t^2+2978t^3+1235t^4+157t^5}$	$(-\frac{29}{3},-\frac{74}{9},-\frac{271}{9},0,-\frac{25}{9})$	$(-\frac{95}{3},\frac{122}{3})$	9
D4Z	$\frac{4(1+t)^3(19+434t+3t^2)}{329+2948t+5678t^2+4532t^3+1105t^4}$	$(-\frac{236}{19},-\frac{521}{57},-\frac{2135}{57},\frac{658}{57},-\frac{307}{19})$	$(\frac{670}{57},0)$	$\frac{670}{57}$
D5Y	$\frac{32(1+t)^3(1+23t)}{139+1249t+2398t^2+1906t^3+455t^4-3t^5}$	$(-\frac{25}{2},-\frac{55}{6},-\frac{113}{3},-\frac{14}{3},0)$	$(-\frac{319}{6},65)$	$\frac{71}{6}$
E2Y	$\frac{2(1+3t)^2(1+12t+3t^2)}{7+60t+210t^2+204t^3+31t^4}$	$(0,-\frac{9}{2},-\frac{1}{2},\frac{3}{2},0)$	$(1,-6)$	-5
E5X	$\frac{2t(1+t)(7+t)(1+3t)^2}{1+30t+134t^2+224t^3+113t^4+10t^5}$	$(-2,-\frac{13}{2},-\frac{17}{2},0,-\frac{9}{2})$	$(-3,4)$	1
F2X	$\frac{2(1+t)(1+3t)(1+26t+5t^2)}{9+100t+238t^2+148t^3+17t^4}$	$(0,-\frac{11}{2},-\frac{3}{2},0,-\frac{3}{2})$	$(-3,0)$	-3
F2Y	$\frac{2(1+t)(1+3t)(1+26t+5t^2)}{9+100t+238t^2+148t^3+17t^4}$	$(0,-\frac{11}{2},-\frac{3}{2},-\frac{3}{2},0)$	$(-9,6)$	-3
F3Y	$\frac{2t(1+t)(7+t)(1+3t)^2}{1+30t+134t^2+224t^3+113t^4+10t^5}$	$(-2,-\frac{13}{2},-\frac{17}{2},-\frac{9}{2},0)$	$(-21,22)$	1
F4X	$\frac{8t(1+t)(1+3t)(5+10t+t^2)}{3+81t+302t^2+434t^3+191t^4+13t^5}$	$(-3,-7,-12,0,-6)$	$(-3,6)$	3
F6Y	$\frac{64t(1+t)^2(1+3t)}{5+123t+370t^2+406t^3+121t^4-t^5}$	$(-5,-8,-19,-9,0)$	$(-39,46)$	7
G1X	$\frac{64t^2(1+t)^2(3+t)}{-1+25t+278t^2+434t^3+251t^4+37t^5}$	$(1,-6,-3,0,-5)$	$(3,-2)$	1
G2X	$\frac{8(1+t)^2(1+18t+5t^2)}{31+220t+322t^2+172t^3+23t^4}$	$(0,-\frac{19}{3},-\frac{17}{3},0,-\frac{17}{3})$	$(2,0)$	2
G3X	$\frac{16t(1+t)(1+6t+t^2)}{(1+3t)(1+33t+27t^2+3t^3)}$	$(2,-7,-11,0,-7)$	$(0,4)$	4
G5Z	$\frac{64t(1+t)^2(1+3t)}{5+123t+370t^2+406t^3+121t^4-t^5}$	$(-5,-8,-19,\frac{5}{2},-\frac{23}{2})$	$(7,0)$	7
H2X	$\frac{8t^2(1+3t)^2(19+5t)}{-1+19t+286t^2+1246t^3+1315t^4+207t^5}$	$(\frac{1}{3},-\frac{25}{6},\frac{5}{6},0,\frac{5}{2})$	$(-\frac{16}{3},-\frac{2}{3})$	-6
H3Y	$\frac{32t^2(7+14t+3t^2)}{-1+28t+322t^2+364t^3+55t^4}$	$(\frac{1}{3},-\frac{14}{3},\frac{1}{3},1,0)$	$(-\frac{1}{3},-\frac{14}{3})$	-5
H4X	$\frac{16t^2(1+3t)(23+22t+3t^2)}{-1+37t+646t^2+1498t^3+811t^4+81t^5}$	$(\frac{1}{3},-\frac{17}{3},-\frac{2}{3},0,-2)$	$(-\frac{7}{3},-\frac{2}{3})$	-3
H7X	$\frac{256t^2(2+t)(1+3t)}{-1+49t+886t^2+1666t^3+475t^4-3t^5}$	$(\frac{1}{3},-\frac{20}{3},-\frac{5}{3},0,-5)$	$(-\frac{1}{3},-\frac{2}{3})$	-1
I3X	$\frac{16t(1+t)(1+6t+t^2)}{(1+3t)(1+33t+27t^2+3t^3)}$	$(0,-5,1,0,1)$	$(-6,0)$	-6
I3Y	$\frac{16t(1+t)(1+6t+t^2)}{(1+3t)(1+33t+27t^2+3t^3)}$	$(0,-5,1,1,0)$	$(-2,-4)$	-6
I5Y	$\frac{64t^2(1+t)^2(3+t)}{-1+25t+278t^2+434t^3+251t^4+37t^5}$	$(1,-6,-3,-5,0)$	$(-17,18)$	1
J4X	$\frac{32t^2(7+14t+3t^2)}{-1+28t+322t^2+364t^3+55t^4}$	$(0,-\frac{11}{3},\frac{7}{3},0,5)$	$(-\frac{26}{3},0)$	$-\frac{26}{3}$
J5X	$\frac{256t^4(3+t)}{-1+9t-42t^2+210t^3+699t^4+149t^5}$	$(1,-2,1,0,7)$	$(-5,-2)$	-7

same $H(z) = \frac{2(1+3t)^2(1+12t+3t^2)}{7+60t+210t^2+204t^3+31t^4}$, four cases **(B2X, C3Z, F2X, F2Y)** (highlighted in light gray) give the same $H(z) = \frac{2(1+t)(1+3t)(1+26t+5t^2)}{9+100t+238t^2+148t^3+17t^4}$, three cases **(C5X, F6Y, G5Z)** (highlighted in green) give the same $H(z) = \frac{64t(1+t)(1+3t)}{5+123t+370t^2+406t^3+121t^4-t^5}$, two cases **(C2Z, G2X)** (highlighted in magenta) give the same $H(z) = \frac{8(1+t)^2(1+18t+5t^2)}{31+220t+322t^2+172t^3+23t^4}$, three cases **(G3X, I3X, I3Y)** (highlighted in violet) give the same $H(z) = \frac{16t(1+t)(1+6t+t^2)}{(1+3t)(1+33t+27t^2+3t^3)}$, two cases **(H3Y, J4X)** (highlighted in light green) give the same $H(z) = \frac{32t^2(7+14t+3t^2)}{-1+28t+322t^2+364t^3+55t^4}$, and two cases **(E5X, F3Y)** (highlighted in gray) give the same $H(z) = \frac{2t(1+t)(7+t)(1+3t)^2}{1+30t+134t^2+224t^3+113t^4+10t^5}$.

5.2. Stability of extraneous fixed points

After locating the roots of $H(z)$ investigated thus far for $f(z) = (z^2 - 1)^m$, we list in Table 2 the desired purely imaginary extraneous fixed points in typical subcases. By computing the absolute values of multipliers $R_f'(\xi)$ for iterative map (5.2) with $f(z) = (z^2 - 1)^m$, we claim that all of the purely imaginary extraneous fixed points ξ of H in each of the listed cases in Table 2 are indifferent except for extraneous fixed point double 0. The extraneous fixed point double 0 for each of Cases **G3X** and **I3X** is found to be repulsive and highlighted by a framed-value. Interestingly attractive extraneous fixed points have not been found in any of the selected cases. Stabilities of the multipliers for all cases **A**, **B**, ..., **J** are well described in the following proposition.

Proposition 5.3. *Let ξ be the extraneous fixed points obtained from the expression $H(z)$ with $t = z^2$ in (5.12) and let $\lambda = r_8 + r_9$ be as described earlier in* Section 5.1. *Then stabilities of the possible extraneous fixed points ξ for the* 10 *cases* **A**, **B**, ..., **J** *are characterized by the following:*

(1) The nonzero extraneous fixed points for the 10 *cases are all found to be indifferent.*

(2) The multipliers of the extraneous fixed point double 0 *for cases* **E, F** *and* **G** *are respectively given by $-\frac{(36+5\lambda)}{(6+\lambda)}$ and $-\frac{(32+3\lambda)}{(8+\lambda)}$. They are found to be repulsive respectively for $-7 < \lambda \le -3(\lambda \ne -6)$ and $-10 < \lambda < -\frac{14}{3}(\lambda \ne -8)$.*

(3) The extraneous fixed point (quadruple, sextuple, octuple) 0 *is found to be indifferent for cases* **E–J**.

Proof. (1): It suffices to show for Case **A** for the nonzero extraneous fixed points. Proofs for the remaining cases can be similarly treated.

The corresponding $H(z)$ for **Case A** found to be:

$$H(z) = \frac{(1+3t)^3(11+4\lambda+t(54-4\lambda)+15t^2)}{34+11\lambda+t(258+37\lambda)+78t^2(14+\lambda)-6t^3(-334+9\lambda)+t^4(1498-73\lambda)+t^5(234+\lambda)}, \tag{5.17}$$

where $t = z^2$. Hence the extraneous fixed points ξ are given by $\pm\frac{i}{\sqrt{3}}$(triple) and $\pm\tau$ with $t = \tau^2$ for which $11 + 4\lambda + t(54 - 4\lambda) + 15t^2 = 0$. Besides, the corresponding derivative of the iterative map R_f in (5.2) is given by

$$R_f'(z) = \frac{(t-1)[11+4\lambda+t(107+18\lambda)+t^2(542+48\lambda)+t^3(1046-36\lambda)+t^4(791-36\lambda)+t^5(63+2\lambda)]}{2t[34+11\lambda+t(258+37\lambda)+78t^2(14+\lambda)-6t^3(-334+9\lambda)+t^4(1498-73\lambda)+t^5(234+\lambda)]}. \tag{5.18}$$

By direct substitution of the extraneous fixed points $z = \pm\frac{i}{\sqrt{3}}$ (triple), i.e., $t = -\frac{1}{3}$ into $R_f'(z)$, we immediately find $R_f'(\pm\frac{i}{\sqrt{3}}) = 1$. We now let the extraneous fixed points $\pm\tau$ satisfy

$$11 + 4\lambda + t(54 - 4\lambda) + 15t^2 = 0$$

with $t = \tau^2$. For brevity, we first denote the left side of the above equation by $d_\lambda(t) = 11 + 4\lambda + t(54 - 4\lambda) + 15t^2$. Then the second factors of the numerator and the denominator of (5.18) are respectively given by

$$11+4\lambda+t(107+18\lambda)+t^2(542+48\lambda)+t^3(1046-36\lambda)+t^4(791-36\lambda)+t^5(63+2\lambda)$$

$$= q_{1\lambda}(t)\cdot d_\lambda(t) + r_{1\lambda}(t) = r_{1\lambda}(t),$$

$$34+11\lambda+t(258+37\lambda)+78t^2(14+\lambda)-6t^3(-334+9\lambda)+t^4(1498-73\lambda)+t^5(234+\lambda)$$

$$= q_{2\lambda}(t)\cdot d_\lambda(t) + r_{2\lambda}(t) = r_{2\lambda}(t),$$

where

$$q_{1\lambda}(t) = \frac{\delta_0 + 15t(-232047+43026\lambda-2136\lambda^2+32\lambda^3)+225t^2(8463-396\lambda+8\lambda^2)+3375t^3(63+2\lambda)}{50625},$$

$$q_{2\lambda}(t) = \frac{\delta_1 + 15t(-118746+24483\lambda-1128\lambda^2+16\lambda^3)+225t^2(9834-213\lambda+4\lambda^2)+3375t^3(234+\lambda)}{50625},$$

$$r_{1\lambda}(t) = \frac{128[\delta_3 + t(-5127489+1955610\lambda-258525\lambda^2+15180\lambda^3-405\lambda^4+4\lambda^5)]}{50625},$$

and

$$r_{2\lambda}(t) = \frac{256[\delta_4 + t(-1660176+592380\lambda-73710\lambda^2+4110\lambda^3-105\lambda^4+\lambda^5)]}{50625},$$

with $\delta_0 = 12963393 - 3532032\lambda + 309888\lambda^2 - 10752\lambda^3 + 128\lambda^4$, $\delta_1 = 8475174 - 2088711\lambda + 170964\lambda^2 - 5616\lambda^3 + 64\lambda^4$, $\delta_3 = -1109691 - 99990\lambda + 83745\lambda^2 - 8760\lambda^3 + 325\lambda^4 - 4\lambda^5$, and $\delta_4 = -357444 - 40500\lambda + 25290\lambda^2 - 2430\lambda^3 + 85\lambda^4 - \lambda^5$. Hence (5.18) at this extraneous fixed points $\pm\tau$ with $t = \tau^2$ becomes

$$R_f'(z) = \frac{(t-1)r_{1\lambda}(t)}{2t\, r_{2\lambda}(t)}. \tag{5.19}$$

Since $(t-1)r_{1\lambda}(t)-2t\,r_{2\lambda}(t)=d_\lambda(t)\cdot(100881-27594\lambda+2421\lambda^2-84\lambda^3+\lambda^4)=0$ in view of the fact $d_\lambda(t)=0$, we find that $R'_f(\pm\tau)=1$. Further, in case of $\lambda=\frac{3}{2}$, $d_\lambda(t)$ reduces to a degenerated first-degree polynomial equation. The corresponding extraneous fixed points τ are found to be $\pm\frac{i}{\sqrt{7}}$ and $R'_f(\pm\frac{i}{\sqrt{7}})=1$. The proofs for all other remaining cases can be similarly made.

By direct substitution of the extraneous fixed points $z=0$ (double), i.e., $t=0$ into $R'_f(z)$, we immediately find $R'_f(0)=-7$, implying repulsive fixed points 0.

(2)(i): Case **E** for extraneous fixed points 0 (double).

The corresponding $H(z)$ and $R'_f(z)$ are found to be:

$$\begin{cases} H(z)=-\dfrac{4t(1+3t)^2[-3(6+\lambda)+2t(\lambda-13)+t^2(\lambda-4)]}{5+\lambda+t(157+23\lambda)+t^2(730+74\lambda)+14t^3(95+\lambda)+t^4(769-91\lambda)+t^5(81-21\lambda)}, \\ R'_f(z)=-\dfrac{(t-1)[-31-5\lambda-2t(89+10\lambda)-6t^2(60+\lambda)+2t^3(-95+14\lambda)+3t^4(-3+\lambda)]}{5+\lambda+t(157+23\lambda)+t^2(730+74\lambda)+14t^3(95+\lambda)+t^4(769-91\lambda)+t^5(81-21\lambda)}. \end{cases} \tag{5.20}$$

By direct substitution of the extraneous fixed points 0 (double), i.e., $t=0$, into $R'_f(z)$, we immediately find $R'_f(0)=-\frac{(31+5\lambda)}{(5+\lambda)}$. Thus the extraneous fixed points 0 are repulsive for $-6<\lambda\le 4(\lambda\ne -5)$.

(ii): Case **F** for extraneous fixed points 0 (double).

The corresponding $H(z)$ and $R'_f(z)$ are found to be:

$$\begin{cases} H(z)=\dfrac{4t(1+t)(1+3t)[-11-3\lambda+2t(\lambda-23)+t^2(\lambda-7)]}{-3-\lambda-3t(33+7\lambda)-2t^2(251+17\lambda)+14t^3(-65+\lambda)+t^4(-487+35\lambda)+t^5(-47+7\lambda)}, \\ R'_f(z)=\dfrac{(t-1)[-19-5\lambda-2t(61+5\lambda)+4t^2(\lambda-62)+2t^3(5\lambda-59)+t^4(\lambda-5)]}{-3-\lambda-3t(33+7\lambda)-2t^2(251+17\lambda)+14t^3(\lambda-65)+t^4(35\lambda-487)+t^5(7\lambda-47)}. \end{cases} \tag{5.21}$$

By direct substitution of the extraneous fixed points 0 (double), i.e., $t=0$, into $R'_f(z)$, we immediately find $R'_f(0)=-\frac{(19+5\lambda)}{(3+\lambda)}$. Thus the extraneous fixed points 0 are repulsive for $-\frac{11}{3}<\lambda\le 7(\lambda\ne -3)$.

(iii): Case **G** for extraneous fixed points 0 (double).

The corresponding $H(z)$ and $R'_f(z)$ are found to be:

$$\begin{cases} H(z)=\dfrac{32t(1+t)^2[1-\lambda-18t+t^2(-7+\lambda)]}{6-3\lambda-t(26+49\lambda)-2t^2(394+23\lambda)+14t^3(\lambda-94)+t^4(65\lambda-818)+t^5(19\lambda-130)}, \\ R'_f(z)=\dfrac{(t-1)[10-13\lambda-12t(17+\lambda)+2t^2(\lambda-172)+4t^3(5\lambda-53)+3t^4(\lambda-6)]}{6-3\lambda-t(26+49\lambda)-2t^2(394+23\lambda)+14t^3(\lambda-94)+t^4(65\lambda-818)+t^5(19\lambda-130)}. \end{cases} \tag{5.22}$$

By direct substitution of the extraneous fixed points 0 (double), i.e., $t=0$, into $R'_f(z)$, we immediately find $R'_f(0)=\frac{10-13\lambda}{3(\lambda-2)}$. Thus the extraneous fixed points 0 are repulsive for $1<\lambda\le 7(\lambda\ne 2)$.

(3) Cases **E–J** for extraneous fixed points 0.

The corresponding $H(z)$ for each of cases **H–J** is found to be:

$$H(z)=\begin{cases} \dfrac{8t^2(1+3t)[-73-9\lambda+t(6\lambda-26)+3t^2(1+\lambda)]}{1-t(55+6\lambda)-2t^2(503+60\lambda)-14t^3(125+6\lambda)+t^4(-307+168\lambda)+t^5(45+42\lambda)} & \text{for case } \mathbf{H}, \\ \dfrac{64t^2(1+t)(19+2\lambda-2t(\lambda-15)+7t^2)}{-6-\lambda+7t(22+3\lambda)+2t^2(906+67\lambda)+t^3(3108-70\lambda)+t^4(1842-85\lambda)+t^5(258+\lambda)} & \text{for case } \mathbf{I}, \\ \dfrac{128t^3[-7(7+\lambda)+t(58+4\lambda)+t^2(31+3\lambda)]}{-26-3\lambda+t(262+31\lambda)-14t^2(134+17\lambda)-14t^3(254+47\lambda)+t^4(7982+641\lambda)+t^5(2334+227\lambda)} & \text{for case } \mathbf{J}. \end{cases} \tag{5.23}$$

In addition, subcases **E1X–E1Z, F1X–F1Z, G1X–G1Z** possess extraneous fixed points 0 (quadruple), i.e., each $H(z)$ of them contains a factor t^2whose explicit expression is not shown here. Note that each $H(z)$ of the selected cases **E–J** has the factor t^k for $k\ge 2$ in the numerator for some λ. Hence, we find that the corresponding fixed points 0 are found from the repeated roots of t^k, stating $R'_f(0)=1$. □

In case that $f(z)$ is a generic polynomial rather than $(z^2-1)^m$, it would be certainly interesting to investigate the dynamics underlying the relevant extraneous fixed points. However, due to the increased algebraic complexity, we resort to an effective way of exploring such dynamics through a variety of basins of attraction under iterative map (5.2) with $f(z)$ as a generic polynomial. We will illustrate the basins of attraction to explore the dynamics of the iterative map R_p of the form

$$z_{n+1}=R_p(z_n)=z_n-\frac{p(z_n)}{p'(z_n)}H_p(z_n), \tag{5.24}$$

for a generic polynomial $p(z_n)$ and a weight function $H_p(z_n)$. In fact, basins of attraction for the fixed points or the extraneous fixed points will be illustrated throughout various polynomials in the latter part of Section 6.

We now prefix the iterative maps in Table 2 corresponding to cases **A3Z, A4X, B2X, B3Y, B4Z, C2Z, C3Z, D2Z, E2Y, F2X, G1X, G2X, G3X, H2X, H3Y, I3X, J4X, J5X** with **W** for later use in describing their relevant dynamics. In addition, we identify map **GKN6A** and **GKN6B**, respectively for methods (2.1) and (2.2).

Table 2
Extraneous fixed points ξ and their stability for selected cases.

Case	ξ	No. of ξ
A3Z	$\pm i/\sqrt{3}(double), \pm 1.9786i, \pm 0.291798i$	8
A4X	$\pm\sqrt{3}i, \pm i/\sqrt{3}(quadruple)$	10
B2X	$\pm i, \pm i/\sqrt{3}, \pm 2.27184i, \pm 0.196851i$	8
B3Y	$\pm i/\sqrt{3}(double), \pm 1.9786i, \pm 0.291798i$	8
B4Z	$\pm i(double), \pm i/\sqrt{3}(double), \pm 0.774597i$	10
C2Z	$\pm i(double), \pm 0.237572i, \pm 1.88243i$	8
C3Z	$\pm i, \pm i/\sqrt{3}, \pm 0.196851i, \pm 2.27184i$	8
D2Z	$\pm i(triple), \pm 0.273951i, \pm 1.21676i$	10
E2Y	$\pm i/\sqrt{3}(double), \pm 0.291798i, \pm 1.9786i$	8
F2X	$\pm i, \pm i/\sqrt{3}, \pm 0.196851i, \pm 2.27184i$	8
G1X	$0(quadruple), \pm i(double), \pm \sqrt{3}i$	10
G2X	$\pm i(double), \pm 0.237572i, \pm 1.88243i$	8
G3X	$0(double), \pm i, \pm 0.414214i, \pm 2.41421i$	8
H2X	$0(quadruple), \pm i/\sqrt{3}(double), \pm 1.94936i$	10
H3Y	$0(quadruple), \pm 0.754652i, \pm 2.02415i$	8
I3X	$0(double), \pm i, \pm 2.41421i, \pm 414214i$	8
J4X	$0(quadruple), \pm 2.02415i, \pm 0.754652i$	8
J5X	$0(octuple), \pm \sqrt{3}i$	10

In the above table, all nonzero extraneous fixed points are indifferent, while boxed-values of zero extraneous fixed points are repulsive. Interestingly, no attractive extraneous fixed points exist for the selected cases.

6. Numerical experiments and complex dynamics

We first analyze computational aspects of proposed family of methods (3.1) for a number of test functions along with existing sixth-order methods **GKN6A** given by (2.1) and **GKN6B** given by (2.2); then we investigate the dynamics underlying purely imaginary extraneous fixed points based on iterative maps (5.24) through their illustrative basins of attraction. In Section 5, we were able to find extraneous fixed points using λ-values without specifying parameters r_8, r_9. For numerical experiments in both computational and dynamical aspects, we need to provide the required 7 coefficients $q_9, r_1, r_3, r_4, r_7, r_8, r_9$ of $K_f(s, u)$ for a given λ. Table 3 shows the desired parameter values and $K_f(s, u)$ for the 18 selected cases **A3Z, A4X, B2X, B3Y, B4Z, C2Z, C3Z, D2Z, E2Y, F2X, G1X, G2X, G3X, H2X, H3Y, I3X, J4X, J5X**. Each case has been implemented to verify the theoretical convergence. Later on in this section, we will explore the complex dynamics with the use of illustrated basins of attraction of selected rational iterative maps **WA3Z** through **WJ5X** and existing sixth-order methods **GKN6A** and **GKN6B**.

Numerical experiments have been implemented by Mathematica programming with 160 digits of minimum number of precision, via Mathematica command $ *MinPrecision* = 160.

Definition 2 (*Computational Convergence Order*). Assume that theoretical asymptotic error constant $\eta = \lim_{n\to\infty} \frac{|e_n|}{|e_{n-1}|^p}$ and convergence order $p \geq 1$ are known. Define $p_n = \frac{\log|e_n/\eta|}{\log|e_{n-1}|}$ as the computational convergence order. Note that $\lim_{n\to\infty} p_n = p$.

Remark 6.1. Note that p_n requires knowledge at two points x_n, x_{n-1}, while the usual COC (computational order of convergence) $\frac{\log(|x_n-x_{n-1}|/|x_{n-1}-x_{n-2}|)}{\log(|x_{n-1}-x_{n-2}|/|x_{n-2}-x_{n-3}|)}$ does require knowledge at four points $x_n, x_{n-1}, x_{n-2}, x_{n-3}$. Hence p_n can be handled with a less number of working precision digits than the usual COC whose number of working precision digits is at least p times as large as that of p_n.

Computed values of x_n are accurate with up to $*MinPrecision* significant digits. If α has the same accuracy of $*MinPrecision* as that of x_n, then $e_n = x_n - \alpha$ would be nearly zero and hence computing $|e_{n+1}|/|e_n|^p$ would unfavorably break down. In case that α is not exact, we employ the approximate α found with more precision digits of Φ + $*MinPrecision*. To supply such an approximate α, a set of following Mathematica commands are used:

$$
\begin{aligned}
&sol = FindRoot[f(x), \{x, x_0\}, PrecisionGoal \to \Phi + \$MinPrecision,\\
&\quad WorkingPrecision \to 2 * \$MinPrecision];\\
&\quad \alpha = sol[[1, 2]].
\end{aligned}
$$

In the current experiment, we assign $\Phi = 16$. As a result, the numbers of significant digits of x_n and α are found to be 160 and 176, respectively. Nonetheless, we list both of them with up to 15 significant digits for proper readability. The error bound $\epsilon = \frac{1}{2} \times 10^{-128}$ is assigned to satisfy $|x_n - \alpha| < \epsilon$.

Table 3
Parameter values of $\lambda, A_6, A_7, A_8, B_2, B_3, B_4, B_5, B_6, d_1, d_2$, and $K_f(s,u)$ for selected cases.

Case	$K_f(s,u)$	$(q_9, r_1, r_3, r_4, r_7)$	(r_8, r_9)	λ
A3Z	$-\frac{2su(s-1)^2(3s-2)}{4-22s+40s^2-24s^3+7s^4+(-4+14s-9s^2-6s^3)u}$	$(-\frac{3}{2},-\frac{11}{2},-6,\frac{7}{4},-\frac{9}{4})$	$(-\frac{3}{2},0)$	$\frac{3}{2}$
A4X	$-\frac{su(s-1)(1-3s+3s^2)}{1-6s+13s^2-11s^3+(-1+4s-17s^3+18s^4)u}$	$(-3,-6,-11,0,0)$	$(-17,18)$	1
B2X	$\frac{su(s-1)^2(5s-2)}{(2s-1)(-2+9s-10s^2+(2-5s-3s^2+9s^3)u)}$	$(-\frac{5}{2},-\frac{13}{2},-10,0,-\frac{7}{2})$	$(-\frac{15}{2},9)$	$\frac{3}{2}$
B3Y	$-\frac{su(s-1)(2s-1)(3s-2)}{2-13s+29s^2-23s^3-5s^4+(-2+9s-44s^3+48s^4)u}$	$(-3,-\frac{13}{2},-\frac{23}{2},-\frac{5}{2},0)$	$(-22,24)$	2
B4Z	$-\frac{su(s-1)(2-7s+10s^2)}{2-13s+33s^2-35s^3+17s^4+(-2+9s-14s^2+8s^3)u}$	$(-5,-\frac{13}{2},-\frac{35}{2},\frac{17}{2},-7)$	$(4,0)$	4
C2Z	$-\frac{2su(s-1)(3-13s+15s^2)}{6-44s+114s^2-110s^3+31s^4+(-6+32s-57s^2+34s^3)u}$	$(-5,-\frac{22}{3},-\frac{55}{3},\frac{31}{6},-\frac{19}{2})$	$(\frac{17}{3},0)$	$\frac{17}{3}$
C3Z	$-\frac{su(s-1)(2s-1)(5s-2)}{2-15s+39s^2-37s^3+9s^4+(-2+11s-20s^2+12s^3)u}$	$(-5,-\frac{15}{2},-\frac{37}{2},\frac{9}{2},-10)$	$(6,0)$	6
D2Z	$-\frac{su(-3+17s-35s^2+24s^3)}{3-23s+66s^2-77s^3+28s^4+(-3+17s-33s^2+22s^3)u}$	$(-8,-\frac{23}{3},-\frac{77}{3},\frac{28}{3},-11)$	$(\frac{22}{3},0)$	$\frac{22}{3}$
E2Y	$-\frac{su(s-1)(3s-2)}{-2+9s-11s^2+s^3-3s^4+(2-5s-2s^3+12s^4)u}$	$(0,-\frac{9}{2},-\frac{1}{2},\frac{3}{2},0)$	$(1,-6)$	-5
F2X	$-\frac{su(s-1)(5s-2)}{-2+11s-17s^2+3s^3+(2-7s+3s^2+6s^3)u}$	$(0,-\frac{11}{2},-\frac{3}{2},0,-\frac{3}{2})$	$(-3,0)$	-3
G1X	$-\frac{su(1-4s+4s^2+s^3)}{-1+6s-11s^2+3s^3+(1-4s+5s^2-3s^3+2s^4)u}$	$(1,-6,-3,0,-5)$	$(3,-2)$	1
G2X	$\frac{su(3-13s+15s^2)}{3-19s+38s^2-17s^3+(-3+13s-17s^2+6s^3)u}$	$(0,-\frac{19}{3},-\frac{17}{3},0,-\frac{17}{3})$	$(2,0)$	2
G3X	$-\frac{su(2s-1)(1-3s+s^2)}{1-7s+16s^2-11s^3+(-1+5s-7s^2+4s^4)u}$	$(2,-7,-11,0,-7)$	$(0,4)$	4
H2X	$-\frac{su(s-1)(-6+7s+2s^2)}{-6+25s-25s^2-5s^3+(6-13s-15s^2+32s^3+4s^4)u}$	$(\frac{1}{3},-\frac{25}{6},\frac{5}{6},0,\frac{5}{2})$	$(-\frac{16}{3},-\frac{2}{3})$	-6
H3Y	$-\frac{su(s-1)(-3+5s+s^2)}{-3+14s-17s^2-s^3-3s^4+(3-8s+s^3+14s^4)u}$	$(\frac{1}{3},-\frac{14}{3},\frac{1}{3},1,0)$	$(-\frac{1}{3},-\frac{14}{3})$	-5
I3X	$-\frac{su(1-3s+s^2)}{-1+5s-6s^2-s^3+(1-3s-s^2+6s^3)u}$	$(0,-5,1,0,1)$	$(-6,0)$	-6
J4X	$-\frac{su(-3+5s+s^2)}{-3+11s-6s^2-7s^3+(3-5s-15s^2+26s^3)u}$	$(0,-\frac{11}{3},\frac{7}{3},0,5)$	$(-\frac{26}{3},0)$	$-\frac{26}{3}$
J5X	$-\frac{su(1+s)(1-s+s^2)}{-1+2s+s^2-s^3+(1-7s^2+5s^3+2s^4)u}$	$(1,-2,1,0,7)$	$(-5,-2)$	-7

Table 4
Convergence for test functions $F_1(x) - F_3(x)$ with typically selected methods **AX2, BX1, DY7**.

MT	F	n	x_n	$\|F(x_n)\|$	$\|x_n-\alpha\|$	$\|e_n/e_{n-1}^8\|$	η	p_n
		0	−2.2	0.220278	0.165263			
		1	−2.03473702902612	1.411×10^{-37}	1.027×10^{-8}	0.01846758266	0.06316552009	8.68311
WA3Z	F_1	2	−2.03473701875034	3.678×10^{-323}	7.852×10^{-66}	0.06316551487		8.00000
		3	−2.03473701875034	7.260×10^{-866}	1.145×10^{-174}			
		0	1.4	0.0160177	0.0634181			
		1	1.46341814023295	1.666×10^{-28}	1.458×10^{-10}	0.5574998039	0.8530972576	8.15425
WB2X	F_2	2	1.46341814037882	2.869×10^{-235}	1.748×10^{-79}	0.8530972568		8.00000
		3	1.46341814037882	0.0×10^{-812}	5.406×10^{-174}			
		0	1.1	0.205800	0.0585181			
		1	1.04148187080088	2.406×10^{-18}	2.165×10^{-10}	1.574812549	1.400174304	7.95859
WH3Y	F_2	2	1.04148187058433	2.351×10^{-153}	6.770×10^{-78}	1.400174305		8.00000
		3	1.04148187058433	3.678×10^{-345}	0.0×10^{-159}			

MT = method, $\binom{1.97}{-0.36}^* = 1.96-0.36i,\ i=\sqrt{-1}$.

Typical methods **WA3Z, WB2X, WH3Y** have been successfully implemented with test functions $F_1 - F_3$ below:

$$\begin{cases} \textbf{WA3Z}: F_1(x) = (\cos(\frac{2\pi}{x}) + x^2 - \pi)^5,\ \alpha \approx -2.03473701875034,\ m = 5. \\ \textbf{WB2X}: F_2(x) = [1+\cos(x^2+1) - x\log(x^2-\pi+2)]^2(x^2+1-\pi),\ \alpha = \sqrt{\pi-1},\ m = 3. \\ \textbf{WH3Y}: F_3(x) = [\sin(x-1)^{-1} + e^{x^2} - 3]^2,\ \alpha \approx 1.04148187058433,\ m = 2. \end{cases}$$

Table 4 well verifies eighth-order convergence. The values of computational asymptotic error constant agree up to 9 significant digits with η. Table 5 lists additional test functions to further confirm the convergence behavior of proposed scheme (3.1).

In Table 6, we compare numerical errors $|x_n - \alpha|$ of proposed methods **WA3Z** through **WJ5X** with those of methods **GKN6A** and **GKN6B**. The least errors within the prescribed error bound are highlighted in bold face. Although we are limited

Table 5
Additional test functions $f_i(x)$ with zeros α, initial guesses x_0, and multiplicity m.

i	$f_i(x)$	α	x_0	m
1	$x^2[x^3 - \log(1+x^2)]^2$	0	−0.05	$m = 6$
2	$[3 + \sin x - x^3 + 2x]^2$	1.99124467662365	2.15	$m = 2$
3	$[2x - \pi + \cos x \, \log(x^2+1)]^4$	$\pi/2$	1.5	$m = 4$
4	$[2x^3 + e^{-x} + \sin(x^2) - 2]^7$	0.784656783178930	0.8	$m = 7$
5	$[x - x^3\cos(\frac{\pi x}{3}) + \frac{1}{x^2+1} - \frac{301}{10}](x-3)^4$	3	2.87	$m = 5$
6	$\exp\{\frac{[(x-\frac{1}{2})^2+3]^2}{x^5+\cos[(x-\frac{1}{2})^2+3]}\} - 1$	$0.5 + \sqrt{3}i$	$0.495 + 1.72i$	$m = 2$
7	$(x-1)(x\log x - \sqrt{x} + x^4)^2$	1	1.05	$m = 3$

Here $\log z$ ($z \in \mathbb{C}$) represents a principal analytic branch with $-\pi \le Im(\log z) < \pi$.

to the selected current experiments, within two iterations, a strict comparison shows that Method **WB2X** displays slightly better convergence for three test functions f_2, f_4, f_6 and method **WF2X** for two test functions f_3, f_7, while method **WG1X** for two test functions f_1 and f_5.

In view of a close inspection of the asymptotic error constant $\eta(\theta_i, L_f, K_f) = \frac{|x_{n+1}-\alpha|}{|x_n-\alpha|^8}$, we should be aware that the local convergence is dependent on the function $f(x)$, an initial value x_0, the zero α itself as well as the weight functions L_f and K_f. Hence, we should not expect that for all given set of test functions, the convergence of one method is always better than the others.

The efficiency index [22] abbreviated by *EI* is found to be $8^{1/4} \approx 1.68179$ for the proposed family of methods (3.1), being better than classical Newton's method and any other known method for multiple roots.

Selection of good initial guesses is crucial to guarantee the convergence behavior of Newton-like iterative map (5.24) with a weight function $H_p(z)$. It is, however, not a simple task since the initial guesses need to be close to zero α and are sensitive to computational precision, error bound and the given function $f(x)$ under consideration.

We now introduce the notion of the *basin of attraction* that is the set of initial guesses leading to long-time behavior approaching the attractors (e.g., periodic, quasi-periodic or chaotic behaviors of different types) under the action of the iterative function. Hence, one effective way of selecting stable initial guesses would be directly using visual basins of attraction. Since the area of convergence can be seen on the basins of attraction, it would be reasonable to say that a method having a larger area of convergence implies a more robust method. A quantitative analysis is clearly necessary for measuring the size of area of convergence. Conveniently, convergence behavior of global character can be clearly observed on the basin of attraction. The basic topological structure of such a basin of attraction as a region can vary greatly from system to system with various forms of weight functions.

To show the performance of the listed methods, we present Tables 7–9 featuring a statistical data giving the average number of iterations per point, CPU time (in seconds) and number of points requiring 40 iterations. In the following examples, we take a 6 by 6 square centered at the origin and containing all the zeros of the given functions. We then take 601×601 equally spaced points in the square as initial points for the iterative methods. We color the point based on the root it converged to. This way we can figure out if the method converged within the maximum number of iteration allowed and if it converged to the root closer to the initial point.

We now are ready to discuss the complex dynamics of selected iterative maps in Table 2 for **A3Z, A4X, B2X, B3Y, B4Z, C2Z, C3Z, D2Z, E2Y, F2X, G1X, G2X, G3X, H2X, H3Y, I3X, J4X, J5X** and existing sixth-order multiple-zeros finders **GKN6A** and **GKN6B**, when applied to various polynomials $p_k(z)$, $(1 \le k \le 6)$.

Example 1. As a first example, we have taken a quadratic polynomial raised to the power of 2 with all real roots:

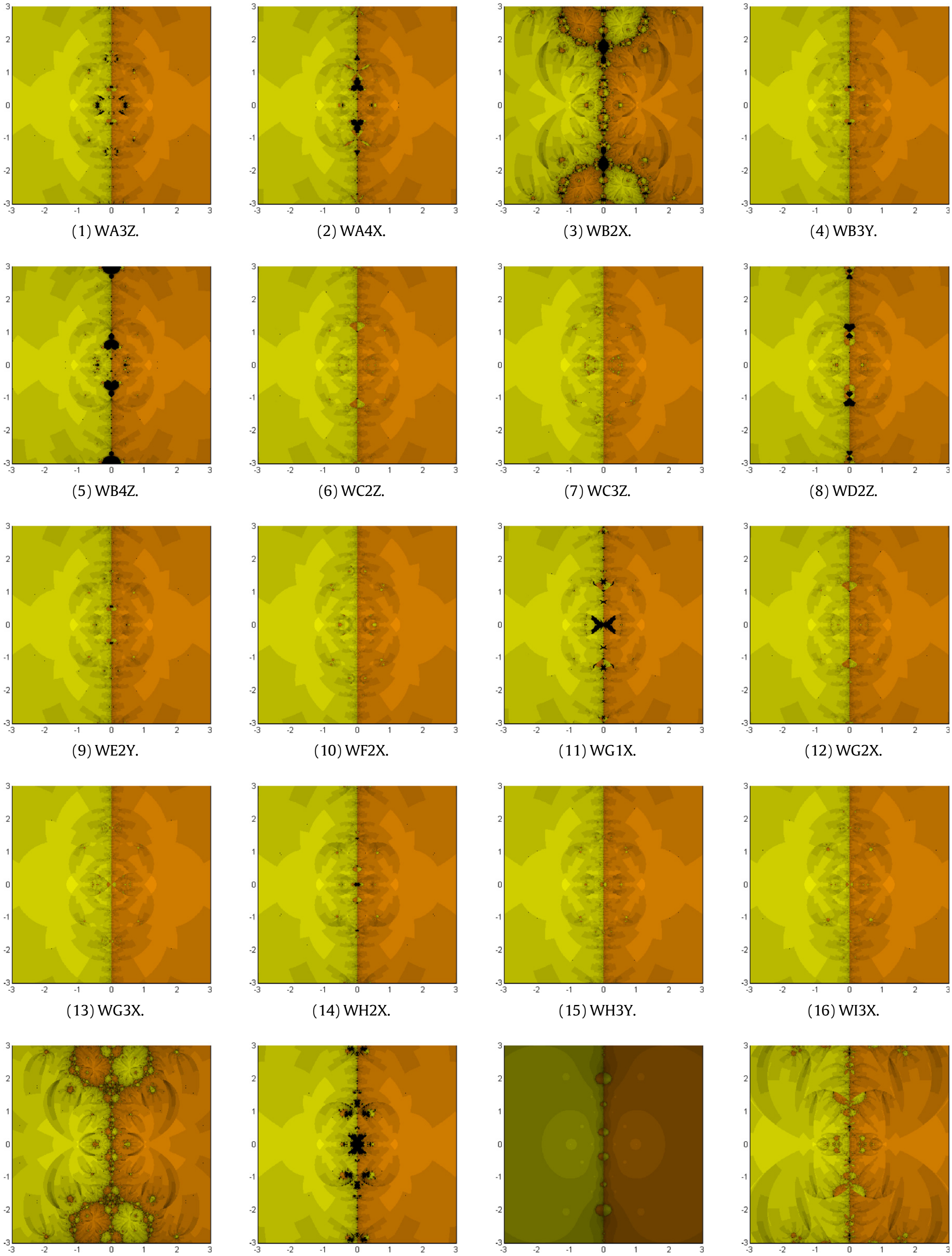
$$p_1(z) = (z^2 - 1)^2. \tag{6.1}$$

Clearly the roots are ± 1. Basins of attraction for **WA3Z** – **WJ5X**, **GKN6A** and **GKN6B** are given in Fig. 1. Consulting Tables 7–9, we find that the method **WG3X** uses the least number (2.84) of iterations per point on average (ANIP) followed by **WI3X** with 2.86 ANIP. The fastest method is **WI3X** with 528.952 s followed closely by **WG3X** with 529.467 s. The slowest are **WB2X** and **WJ4X** with 923.494 and 826.213 s, respectively. **GKN6A** has the lowest number of black points and **WJ5X** has the highest such number (7479).

Example 2. As a second example, we have taken the same quadratic polynomial now raised to the power of 5:

$$p_2(z) = (z^2 - 1)^5. \tag{6.2}$$

The basins for the best methods are plotted in Fig. 2. The worst are **WB2X**, **WB4Z**, and **WJ5X**. In terms of ANIP, the best is **WG3X** (3.70) and the worst are **WB2X** (7.23) and **WJ4X** (6.41). The fastest is **WG3X** using 1622.051 s followed by **WI3X**



(1) WA3Z. (2) WA4X. (3) WB2X. (4) WB3Y.

(5) WB4Z. (6) WC2Z. (7) WC3Z. (8) WD2Z.

(9) WE2Y. (10) WF2X. (11) WG1X. (12) WG2X.

(13) WG3X. (14) WH2X. (15) WH3Y. (16) WI3X.

(17) WJ4X. (18) WJ5X. (19) GKN6A. (20) GKN6B.

Fig. 1. The top row for **WA3Z** (left), **WA4X** (center left), **WB2X** (center right) and **WB3Y** (right). The second row for **WB4Z** (left), **WC2Z** (center left), **WC3Z** (center right) and **WD2Z** (right). The third row for **WE2Y** (left), **WF2X** (center left), **WG1X**(center right) and **WG2X** (right). The fourth row for **WG3X** (left), **WH2X** (center left), **WH3Y** (center right) and **WI3X** (right). The bottom row for **WJ4X** (left), **WJ5X** (center left), **GKN6A** (center right) and **GKN6B** (right), for the roots of the polynomial $(z^2 - 1)^2$.

Table 6
Comparison of $|x_n - \alpha|$ for selected methods applied to various test functions.

Method	$\|x_n - \alpha\|$	$f(x);\ x_0$						
		$f_1;\ -0.05$	$f_2;\ 2.15$	$f_3;\ 1.5$	$f_4;\ 0.8$	$f_5;\ 2.87$	$f_6;\ 0.495 = 1.72i$	$f_7;\ 1.05$
WA3Z	$\|x_1 - \alpha\|$	3.41e−13[a]	8.16e−9	1.13e−9	6.65e−15	8.55e−14	5.11e−16	2.30e−11
	$\|x_2 - \alpha\|$	2.37e−102	7.34e−67	7.70e−72	9.43e−114	2.32e−110	3.75e−62	6.07e−86
WA4X	$\|x_1 - \alpha\|$	7.92e−13	1.08e−8	1.37e−9	1.26e−14	9.91e−14	2.14e−16	2.39e−11
	$\|x_2 - \alpha\|$	5.01e−99	9.98e−66	4.34e−71	3.11e−111	1.05e−109	8.64e−47	8.78e−86
WB2X	$\|x_1 - \alpha\|$	3.75e−13	9.55e−10	6.44e−11	1.04e−15	8.79e−14	8.72e−17	9.45e−13
	$\|x_2 - \alpha\|$	5.69e−102	**2.68e−75**	2.29e−30	**5.14e−121**	3.09e−110	**3.38e−130**	2.16e−98
WB3Y	$\|x_1 - \alpha\|$	7.90e−13	1.02e−9	2.45e−10	4.57e−15	1.01e−13	7.42e−16	5.69e−12
	$\|x_2 - \alpha\|$	4.83e−99	7.25e−75	1.69e−28	3.16e−115	1.24e−109	3.57e−45	1.06e−33
WB4Z	$\|x_1 - \alpha\|$	4.55e−13	3.30e−8	5.18e−9	2.79e−14	8.47e−14	1.70e−15	9.58e−11
	$\|x_2 - \alpha\|$	3.37e−101	2.16e−61	6.70e−66	3.81e−108	2.13e−110	4.62e−60	2.25e−80
WC2Z	$\|x_1 - \alpha\|$	1.51e−13	1.10e−8	1.66e−9	6.69e−15	7.93e−14	9.83e−16	3.31e−11
	$\|x_2 - \alpha\|$	1.35e−105	9.79e−66	2.31e−70	9.97e−114	1.05e−110	5.15e−61	1.52e−84
WC3Z	$\|x_1 - \alpha\|$	8.93e−14	5.80e−9	8.63e−10	2.36e−15	7.82e−14	8.45e−16	1.97e−11
	$\|x_2 - \alpha\|$	9.22e−108	3.03e−68	6.30e−73	8.79e−118	9.07e−111	2.80e−61	1.43e−86
WD2Z	$\|x_1 - \alpha\|$	2.79e−13	1.82e−8	2.78e−9	1.34e−14	8.20e−14	1.13e−15	5.22e−11
	$\|x_2 - \alpha\|$	3.88e−103	9.32e−64	2.36e−68	5.38e−111	1.51e−110	9.01e−61	9.29e−83
WE2Y	$\|x_1 - \alpha\|$	3.20e−13	8.35e−9	1.17e−9	6.74e−15	8.47e−14	5.71e−16	2.41e−11
	$\|x_2 - \alpha\|$	1.31e−102	9.17e−67	1.08e−71	1.07e−113	2.11e−110	5.86e−62	9.30e−86
WF2X	$\|x_1 - \alpha\|$	2.41e−13	5.13e−9	6.25e−10	2.03e−15	8.31e−14	4.45e−16	1.24e−11
	$\|x_2 - \alpha\|$	9.59e−104	6.58e−69	**2.14e−74**	1.81e−118	1.72e−110	2.17e−62	**1.91e−88**
WF2Y	$\|x_1 - \alpha\|$	3.73e−13	1.13e−9	3.31e−11	1.05e−15	8.78e−14	9.00e−17	1.34e−12
	$\|x_2 - \alpha\|$	5.45e−102	1.08e−74	4.19e−31	5.58e−121	3.09e−110	4.36e−130	3.73e−97
WG1X	$\|x_1 - \alpha\|$	8.32e−14	1.30e−8	2.14e−9	1.00e−14	7.66e−14	1.38e−15	4.60e−11
	$\|x_2 - \alpha\|$	**6.15e−108**	5.92e−65	2.75e−69	4.05e−112	**7.25e−111**	2.02e−60	3.23e−83
WG2X	$\|x_1 - \alpha\|$	9.32e−14	9.11e−9	1.47e−9	6.49e−15	7.77e−14	1.11e−15	3.31e−11
	$\|x_2 - \alpha\|$	1.67e−107	2.30e−66	9.49e−71	8.05e−114	8.41e−111	8.55e−61	1.66e−84
WG3X	$\|x_1 - \alpha\|$	7.30e−13	7.98e−8	1.26e−8	6.32e−14	8.64e−14	3.47e−15	2.20e−10
	$\|x_2 - \alpha\|$	2.49e−99	5.51e−58	1.81e−62	5.91e−105	2.64e−110	8.03e−59	3.87e−77
WH2X	$\|x_1 - \alpha\|$	5.04e−13	8.64e−9	1.11e−9	7.85e−15	9.06e−14	1.89e−16	2.05e−11
	$\|x_2 - \alpha\|$	8.27e−101	1.12e−66	5.94e−72	4.12e−113	4.23e−110	7.03e−64	2.06e−86
WH3Y	$\|x_1 - \alpha\|$	2.91e−13	6.45e−9	9.11e−10	5.71e−15	8.41e−14	5.71e−16	2.07e−11
	$\|x_2 - \alpha\|$	5.77e−103	1.06e−67	1.29e−72	2.50e−114	1.97e−110	5.89e−62	2.59e−86
WI3X	$\|x_1 - \alpha\|$	3.17e−13	4.27e−10	1.75e−10	1.05e−15	8.63e−14	1.14e−16	2.48-12
	$\|x_2 - \alpha\|$	1.19e−102	7.76e−78	6.18e−29	1.12e−44	2.55e−110	3.84e−129	8.90e−35
WJ4X	$\|x_1 - \alpha\|$	5.88e−13	9.42e−11	2.40e−10	3.84e−15	9.47e−14	2.91e−16	9.91e−13
	$\|x_2 - \alpha\|$	3.50e−100	2.61e−30	1.59e−28	7.79e−116	6.71e−110	2.17e−46	5.62e−36
WJ5X	$\|x_1 - \alpha\|$	9.46e−13	1.83e−8	2.77e−9	2.59e−14	1.02e−13	4.42e−16	5.70e−11
	$\|x_2 - \alpha\|$	2.63e−98	1.67e−63	3.47e−68	2.15e−108	1.44e−109	2.10e−62	2.67e−82
GKN6A	$\|x_1 - \alpha\|$	4.47e−9	1.32e−5	3.16e−6	9.21e−10	1.03e−10	9.56e−11	2.10e−7
	$\|x_2 - \alpha\|$	4.21e−51	1.35e−29	6.67e−32	6.14e−53	3.09e−64	1.64e−59	2.07e−39
GKN6B	$\|x_1 - \alpha\|$	6.34e−10	4.74e−7	8.48e−8	1.64e−11	6.88e−10	3.06e−8	3.32e−9
	$\|x_2 - \alpha\|$	3.61e−57	5.10e−40	3.06e−43	2.74e−65	4.78e−59	3.93e−15	3.48e−52

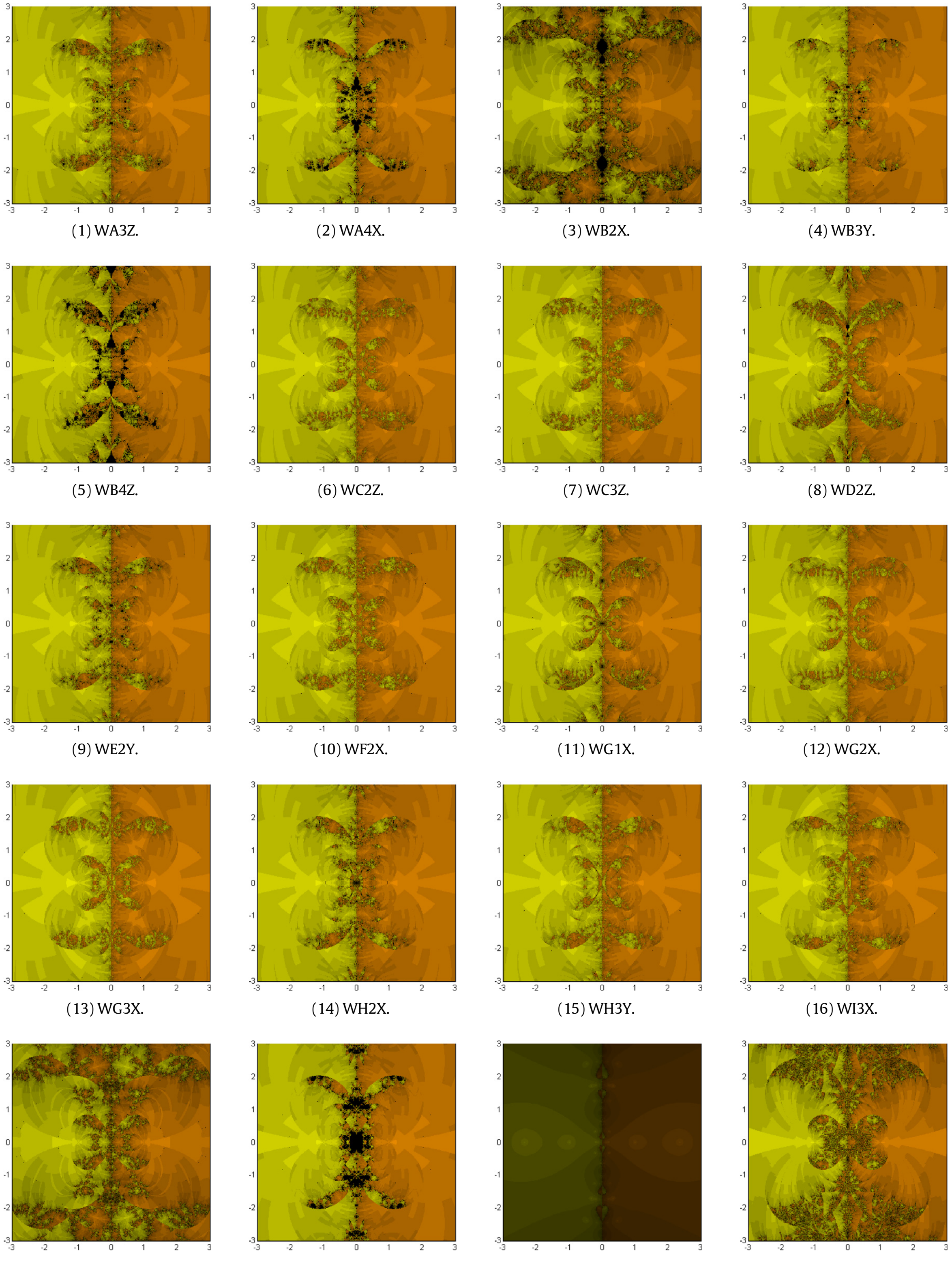
[a] 3.41e−13 ≡ 3.41×10^{-13}.

using 1633.018 s and the slowest is **GKN6A** (5183.446) preceded by **WB2X** (3128.849 s). **GKN6A** has only 7 black points. The highest number is for **WB4Z** (14 659) preceded by **WJ5X** with 14 493 and **WB2X** with 10 513 black points. We will remove these 3 methods from further consideration.

Example 3. In our third example, we have taken a cubic polynomial raised to the power of 3:

$$p_2(z) = (z^3 + 4z^2 - 10)^3. \tag{6.3}$$

Basins of attraction are given in Fig. 3. The worst is **WJ4X**. In terms of ANIP, the best is **WI3X** (3.42) followed by **WF2X** (3.68) and the worst are **GKN6A** (7.78) and **WJ4X** (6.77). The fastest is **WI3X** using 1722.109 s followed by **WF2X** using 1852.684 s and the slowest is **WJ4X** (3408.965 s). There are 2 methods with less than 10 black points. The highest number is for **WA4X** (7143) preceded by **WB3Y** (1036) and **WE2Y** (1010).



(1) WA3Z. (2) WA4X. (3) WB2X. (4) WB3Y.

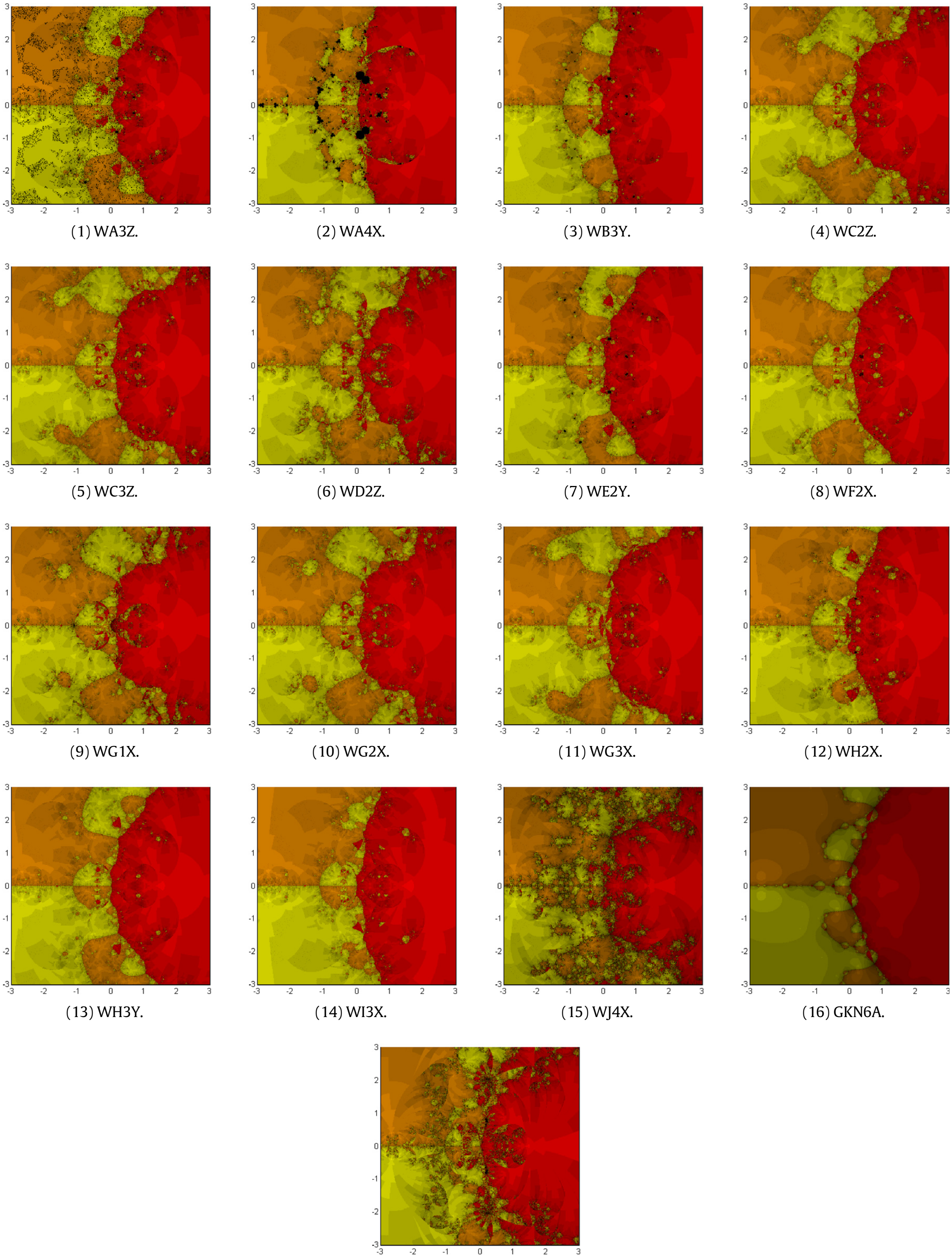
(5) WB4Z. (6) WC2Z. (7) WC3Z. (8) WD2Z.

(9) WE2Y. (10) WF2X. (11) WG1X. (12) WG2X.

(13) WG3X. (14) WH2X. (15) WH3Y. (16) WI3X.

(17) WJ4X. (18) WJ5X. (19) GKN6A. (20) GKN6B.

Fig. 2. The top row for **WA3Z** (left), **WA4X** (center left), **WB2X** (center right) and **WB3Y** (right). The second row for **WB4Z** (left), **WC2Z** (center left), **WC3Z** (center right) and **WD2Z** (right). The third row for **WE2Y** (left), **WF2X** (center left), **WG1X**(center right) and **WG2X** (right). The fourth row for **WG3X** (left), **WH2X** (center left), **WH3Y** (center right) and **WI3X** (right). The bottom row for **J4X** (left), **WJ5X** (center left), **GKN6A** (center right) and **GKN6B** (right), for the roots of the polynomial $(z^2-1)^5$.



(1) WA3Z. (2) WA4X. (3) WB3Y. (4) WC2Z.

(5) WC3Z. (6) WD2Z. (7) WE2Y. (8) WF2X.

(9) WG1X. (10) WG2X. (11) WG3X. (12) WH2X.

(13) WH3Y. (14) WI3X. (15) WJ4X. (16) GKN6A.

(17) GKN6B.

Fig. 3. The top row for **WA3Z** (left), **WA4X** (center left), **WB3Y** (center right) and **WC2Z** (right). The second row for **WC3Z** (left), **WD2Z** (center left), **WE2Y** (center right) and **WF2X** (right). The third row for **WG1X** (left), **WG2X** (center left), **WG3X**(center right) and **WH2X** (right). The fourth row for **WH3Y** (left), **WI3X** (center left), **WJ4X** (center right) and **GKN6A** (right). The bottom row for **GKN6B** (center), for the roots of the polynomial $(z^3 + 4z^2 - 10)^3$.

Table 7
Average number of iterations per point for each example (1–6).

Map	Example						
	1: m=2	2: m=5	3: m=3	4: m=4	5: m=3	6: m=5	Average
WA3Z	3.18	4.10	4.00	5.00	5.65	6.96	4.82
WA4X	3.36	4.69	4.45	5.53	4.65	6.42	4.85
WB2X	5.00	7.23	-	-	-	-	-
WB3Y	3.03	3.96	3.78	4.18	6.89	17.76	6.60
WB4Z	3.77	5.49	-	-	-	-	-
WC2Z	3.01	3.88	4.11	4.55	5.61	6.47	4.60
WC3Z	2.90	3.77	4.01	4.53	5.38	6.17	4.46
WD2Z	3.36	4.24	4.32	4.74	5.82	6.68	4.86
WE2Y	3.04	4.08	3.95	4.70	5.20	6.17	4.52
WF2X	2.89	3.77	3.68	4.29	5.10	6.01	4.29
WG1X	3.39	4.03	4.25	4.63	8.66	11.92	6.15
WG2X	3.01	3.84	4.11	4.59	5.69	6.44	4.61
WG3X	2.84	3.70	3.92	4.45	7.23	7.82	4.99
WH2X	3.08	4.09	3.84	4.60	6.11	6.93	4.77
WH3Y	3.00	3.85	3.79	4.40	5.12	5.67	4.31
WI3X	2.86	3.75	3.42	4.45	4.17	5.42	4.01
WJ4X	4.49	6.41	6.77	6.85	15.46	-	-
WJ5X	3.88	5.29	-	-	-	-	-
GKN6A	8.17	13.12	7.78	8.86	10.00	14.93	10.48
GKN6B	3.83	5.85	5.09	6.06	6.69	8.87	6.06

Example 4. As a fourth example, we have taken a different cubic polynomial raised to the power of 4:

$$p_4(z) = (z^3 - z)^4. \tag{6.4}$$

The basins are given in Fig. 4. We now see that **WJ4X** is the worst followed by **WB3Y**. In terms of ANIP, **WB3Y** is the best (4.18) followed by **WF2X** (4.29) and the worst are **GKN6A** (8.86) and **WJ4X** (6.85). The fastest is **WB3Y** (1961.51 s) and the slowest is **GKN6A** (3520.849 s) preceded by **WJ4X** (3204.385 s). Seven methods have no black point, namely **WC3Z, WD2Z, WF2X, WG2X, WH3Y, WJ4X** and **GKN6A**, **WG3x** has 4 black points and the worst being **WA4X** with 9786 points. Even though **WJ4X** has no black points, we should exclude it because of the chaotic basins. Also **WB3Y** does prefer the non zero roots and should be excluded. This is a reason why we need to view the basins as well as consulting the quantitative results. We will determine these exclusions based on the results of dynamics for remaining experiments.

Example 5. As a fifth example, we have taken a quintic polynomial raised to the power of 3:

$$p_3(z) = (z^5 - 1)^3. \tag{6.5}$$

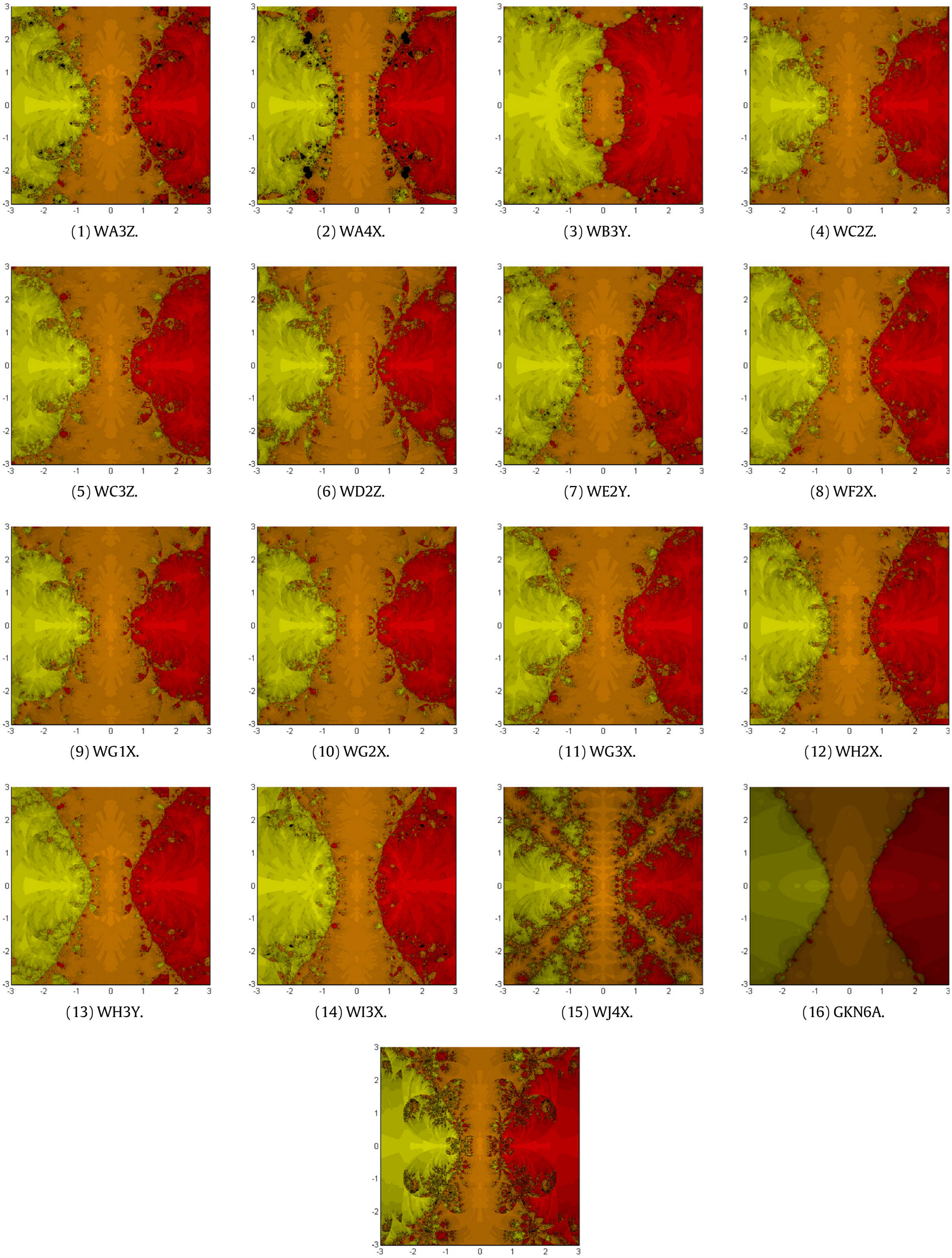
The basins for the best methods left are plotted in Fig. 5. The worst is **WJ4X** followed by **WG1X, WH2X** and **WG3X**. In terms of ANIP, the best is **WI3X** (4.17) followed by **WA4X** (4.65) and the worst are **WJ4X** (15.46) and **GKN6A** (10.00). The fastest is **WI3X** using 2066.639 s followed by **WA4X** using 2346.224 s and the slowest is **WJ4X** (7460.030 s). There are 3 methods with less than 10 black points, namely **WI3X** (1), **WC3Z** (7) and **WF2X** (9). The highest number is for **WJ4X** (68594) preceded by **WG1X** with 33592 black points. We will eliminate **WJ4X** from further consideration.

Example 6. As a sixth example, we have taken a quartic polynomial raised to the power of 5:

$$p_6(z) = (z^4 - 1)^5. \tag{6.6}$$

The basins for the best methods left are plotted in Fig. 6. It seems that most of the methods left are good except **WG1X, WG3X** and **WH2X**. Based on Table 7 we find that **WI3X** has the lowest ANIP (5.42) followed by **WH3Y** (5.67). The fastest method is **WI3X** (2644.966 s) followed by **WH3Y** (2793.229 s). The slowest is **WB3Y** (8107.528 s). The lowest number of black points is for **GKN6A** (817) and the highest number is for **WB3Y** with 129 797 black points.

In summary, we find that there is no method which is best overall. The worst in terms of the number of black points is **WB3Y** and in ANIP and CPU time is **GKN6A**. Of course this is excluding the methods eliminated along the way, namely **WB2X, WB4Z, WJ4X** and **WJ5X**. To summarize the results of the 6 examples, we have averaged the results in Tables 7–9 across examples. Based on Table 7 we find that **WI3X** uses the least number of iterations per point (4.01 on average) followed closely by **WF2X** (4.29) and **WH3Y** (4.31). All other methods use more than 4.4 iterations per point on average. The method requiring the highest number of iterations per point is **GKN6A** (10.48). The fastest method is **WI3X** (1779.93 s) followed by **WF2X** (1938.79 s). The slowest is **GKN6A** (3743.14 s). As for the number of black points (see Table 9) we find that **GKN6A** has the lowest number (425 points) followed closely by **WF2X** (447 points), **WH3Y** (449 points), and **WC3Z** (450 points). The method with the most black points is **WB3Y** (22835 points). It is clear that **WF2X** came close second in all 3 categories and **WI3X** came first in two out of the 3 categories. It is worth to observe that the existing sixth-order methods **GKN6A** and **GKN6B** are not quite comparable to many members of the proposed family in view of all 3 categories.



(1) WA3Z. (2) WA4X. (3) WB3Y. (4) WC2Z.

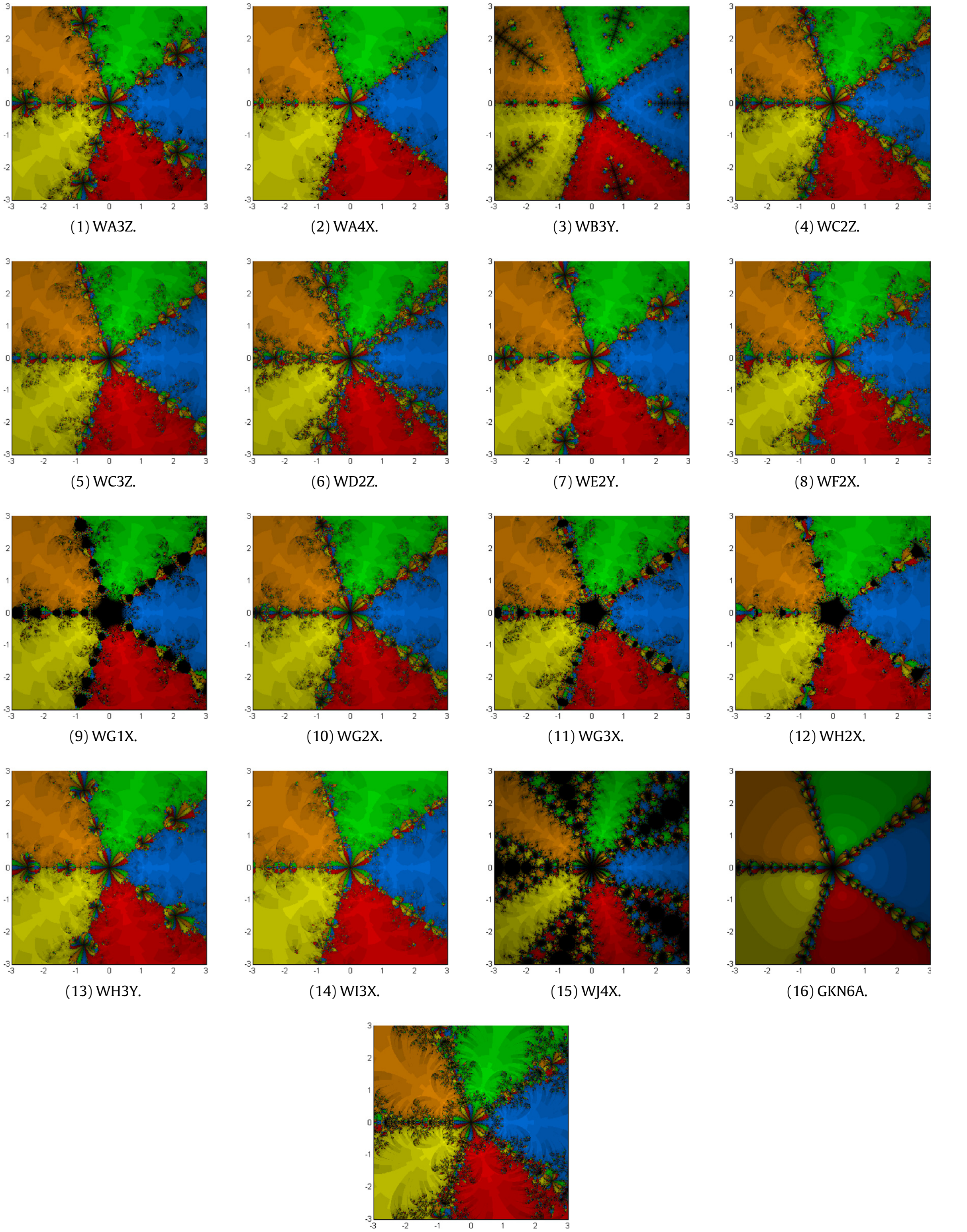
(5) WC3Z. (6) WD2Z. (7) WE2Y. (8) WF2X.

(9) WG1X. (10) WG2X. (11) WG3X. (12) WH2X.

(13) WH3Y. (14) WI3X. (15) WJ4X. (16) GKN6A.

(17) GKN6B.

Fig. 4. The top row for **WA3Z** (left), **WA4X** (center left), **WB3Y** (center right) and **WC2Z** (right). The second row for **WC3Z** (left), **WD2Z** (center left), **WE2Y** (center right) and **WF2X** (right). The third row for **WG1X** (left), **WG2X** (center left), **WG3X**(center right) and **WH2X** (right). The fourth row for **WH3Y** (left), **WI3X** (center left), **WJ4X** (center right) and **GKN6A** (right). The bottom row for **GKN6B** (center), for the roots of the polynomial $(z^3 - z)^4$.



(1) WA3Z. (2) WA4X. (3) WB3Y. (4) WC2Z.

(5) WC3Z. (6) WD2Z. (7) WE2Y. (8) WF2X.

(9) WG1X. (10) WG2X. (11) WG3X. (12) WH2X.

(13) WH3Y. (14) WI3X. (15) WJ4X. (16) GKN6A.

(17) GKN6B.

Fig. 5. The top row for **WA3Z** (left), **WA4X** (center left), **WB3Y** (center right) and **WC2Z** (right). The second row for **WC3Z** (left), **WD2Z** (center left), **WE2Y** (center right) and **WF2X** (right). The third row for **WG1X** (left), **WG2X** (center left), **WG3X**(center right) and **WH2X** (right). The fourth row for **WH3Y** (left), **WI3X** (center left), **WJ4X** (center right) and **GKN6A** (right). The bottom row for **GKN6B** (center), **WJ5X** (center left), **GKN6A** (center right) and **GKN6B** (right), for the roots of the polynomial $(z^5-1)^3$.

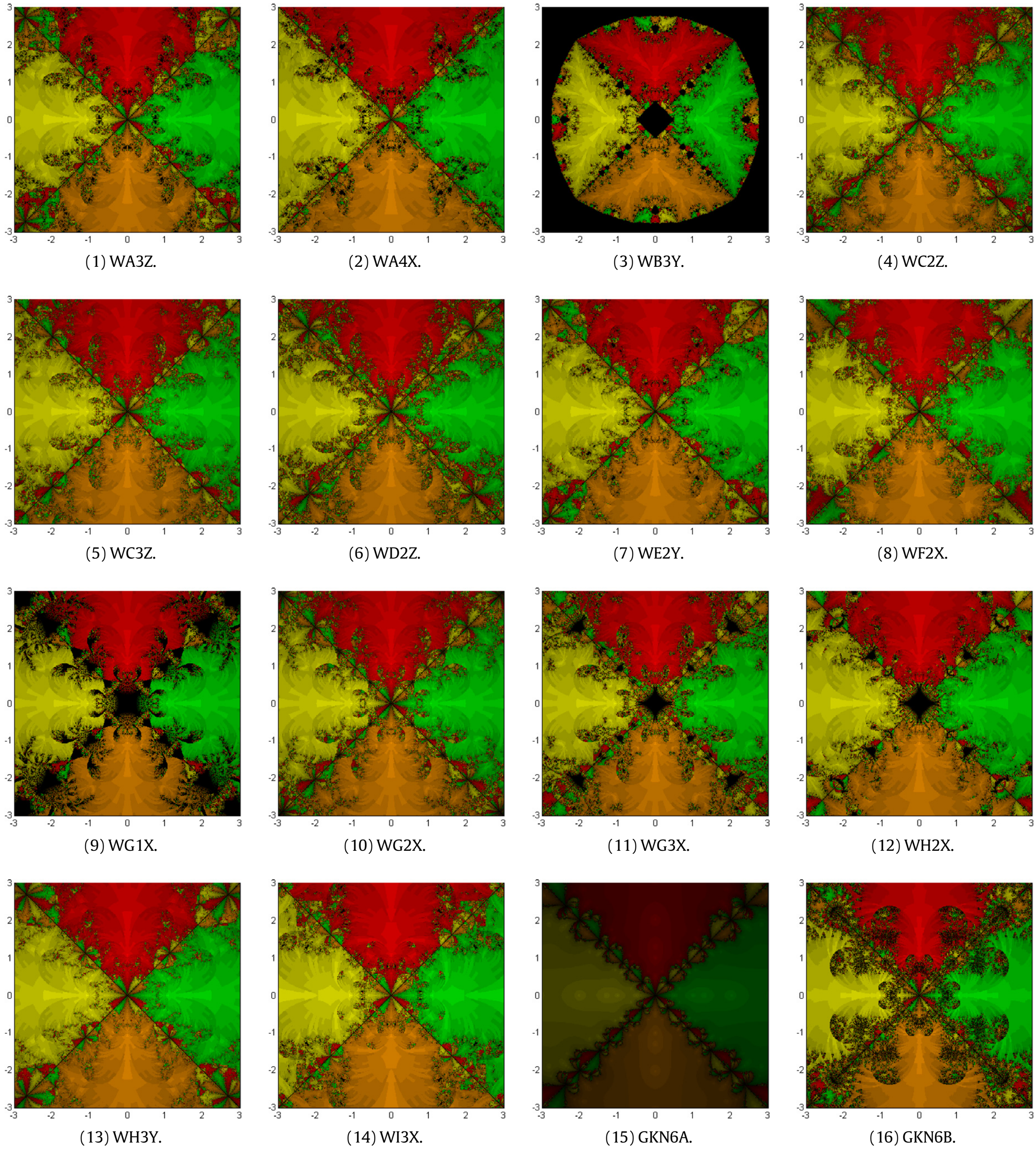


Fig. 6. The top row for **WA3Z** (left), **WA4X** (center left), **WB3Y** (center right) and **WC2Z** (right). The second row for **WC3Z** (left), **WD2Z** (center left), **WE2Y** (center right) and **WF2X** (right). The third row for **WG1X** (left), **WG2X** (center left), **WG3X**(center right) and **WH2X** (right). The bottom row for **WH3Y** (left), **WI3X** (center left), **GKN6A** (center right) and **GKN6B** (right), for the roots of the polynomial $(z^4 - 1)^5$.

We now conclude our current study as follows. Given the known multiplicity m of a zero to be sought, we have developed Theorem 3.1 to achieve optimal eighth-order convergence of proposed family of methods (3.1) by means of modified Newton-type multiple-zero finders with simple fifth-order multivariate rational weight functions. Computational aspects investigated through a number of test equations well support the developed theory underlying the convergence order as well as asymptotic error constants. We have also investigated the dynamical aspects through their basins of attraction not only with a qualitative stability analysis on purely imaginary extraneous fixed points for a prototype quadratic polynomial $f(z) = (z^2 - 1)^m$ motivated by the earlier work of Vrscay and Gilbert [25], but also with a quantitative statistical analysis for

Table 8
CPU time (in seconds) required for each example (1–6) using a Dell Multiplex-990.

Map	Example						
	1: m=2	2: m=5	3: m=3	4: m=4	5: m=3	6: m=5	Average
WA3Z	628.606	1886.894	2116.326	2429.934	2893.101	3480.772	2239.272
WA4X	632.506	2122.846	2283.137	2676.025	2346.224	3130.940	2198.613
WB2X	923.494	3128.849	-	-	-	-	-
WB3Y	608.169	1768.007	1977.001	1961.510	3553.126	8107.528	2995.890
WB4Z	715.311	2453.257	-	-	-	-	-
WC2Z	574.536	1793.434	2128.634	2170.333	2886.362	3316.956	2145.043
WC3Z	561.557	1715.122	2051.226	2165.122	2739.643	3106.869	2056.590
WD2Z	651.133	1902.401	2227.163	2217.087	2973.489	3300.217	2211.915
WE2Y	592.492	1867.784	2006.953	2259.238	2664.279	3086.511	2079.543
WF2X	537.470	1648.571	1852.684	2040.275	2561.879	2991.865	1938.791
WG1X	668.604	1835.414	2168.648	2234.106	4415.156	5815.218	2856.191
WG2X	548.656	1749.161	2125.264	2168.102	2878.359	3147.414	2102.826
WG3X	529.467	1622.051	2002.335	2070.290	3647.444	3805.535	2279.520
WH2X	581.134	1850.266	1954.692	2220.549	3049.507	3463.097	2186.541
WH3Y	580.994	1720.582	1966.096	2048.028	2569.602	2793.229	1946.422
WI3X	528.952	1633.018	1772.109	2033.894	2066.639	2644.966	1779.930
WJ4X	826.213	2834.819	3408.965	3204.385	7460.030	-	-
WJ5X	719.352	2433.927	-	-	-	-	-
GKN6A	978.267	5183.446	2914.504	3520.849	3275.100	6586.659	3743.138
GKN6B	643.909	2500.431	2410.387	2713.325	3148.443	4113.840	2588.389

Table 9
Number of points requiring 40 iterations for each example (1–6).

Map	Example						
	1: m=2	2: m=5	3: m=3	4: m=4	5: m=3	6: m=5	Average
WA3Z	2309	1877	899	4198	2959	12409	4109
WA4X	3171	8397	7143	9786	4272	12413	7530
WB2X	7263	10513	-	-	-	-	-
WB3Y	913	2061	1036	1270	1933	129797	22835
WB4Z	6307	14659	-	-	-	-	-
WC2Z	765	705	12	96	35	1201	469
WC3Z	763	715	16	0	7	1201	450
WD2Z	3039	1659	106	0	51	1209	1011
WE2Y	905	1953	1010	1632	1320	6001	2137
WF2X	731	715	29	0	9	1201	448
WG1X	4253	797	141	16	33592	58157	16159
WG2X	781	701	4	0	67	1209	460
WG3X	751	745	29	4	19568	16233	6222
WH2X	1027	1695	407	638	15983	14633	5731
WH3Y	741	715	31	0	11	1201	450
WI3X	747	813	18	1024	1	3505	1018
WJ4X	891	829	94	0	68594	-	-
WJ5X	7479	14493	-	-	-	-	-
GKN6A	601	7	2	0	1128	817	426
GKN6B	791	861	652	1642	5488	10673	3352

various polynomials $p_k(z)$. The better members of the proposed family of methods (3.1) with better convergence behavior can be directly observed from the illustrative basins of attraction.

As our future work, we will pursue an extended approach enhancing the dynamical characteristics associated with the purely imaginary extraneous fixed points of a higher-order family of simple- or multiple-zero finders by considering other types of weight functions.

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