Adaptive Method for the Numerical Solution of Fredholm Integral Equations of the Second Kind Part II. Singular Kernels

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Abstract. An adaptive method based on a product integration rule for the numerical solution of Fredholm integral equations of the second kind with singular kernel is developed. We discuss two types of singular kernels, i.e. $\log |x-y|$ and $|x-y|^{-\alpha}$, $\alpha < 1$. The choice of mesh points is made automatically so as to equidistribute both the change in the discrete solution and its gradient. Some numerical experiments with this method are presented.

1 Introduction

Consider a Fredholm integral equation of the second kind, which is to say the problem of finding a function $f(x)$ such that

$$f(x) = \int_a^b k(x, y) f(y)\, dy + g(x), \; x \epsilon [a, b], \tag{1}$$

for a given function $g(x)$ and a given singular kernel $k(x, y)$. In the present Part II we consider two most common types of singularities

$$\text{i.} \qquad k(x, y) = R(x, y) \log |x-y| , \tag{2}$$

$$\text{ii.} \qquad k(x, y) = R(x, y) |x-y|^{-\alpha}, \; \alpha < 1, \tag{3}$$

where R(x, y) is non singular. Without loss of generality we can let the interval be [0, 1].

Fredholm integral equations of the second kind appear in many applications, e.g. transport theory (Case and Zweifel [2], Wing [8]), potential theory (Stakgold [7], Sneddon [5]), fracture mechanics and elasticity (Gerasoulis and Strivastav [3], Sneddon and Lowengrub [6]).

In this part we develop an adaptive method based on a product integration rule for obtaining the numerical solution of (1) with kernel (2) or (3). The idea is to start with a given number of equally spaced points on [0, 1] or a given mesh. The solution at this stage is obtained by solving a linear system of algebraic equations. The program then decides if the mesh need to be refined and where. This is done in such a way that both the change in the approximate solution and its gradient are equidistributed. We'll show how the idea in Part I is modified to accommodate the singularities. The idea of equidistribution of the solution and its gradient for boundary value problems was used before (See Neta and Nelson [4] and references there).

The matrix at every stage is not recomputed but only the necessary rows and columns are computed. The new system is solved iteratively as explained in [4].

In the next section we describe the method in detail for kernels with logarithmic singularity. Section 3 will discuss the other type of singularity. The input required in both cases is described in Section 4. The last section will be devoted to numerical experiments with the method.

2 Development of the Method - Logarithmic Singularity

Given a function $g(x)$ and a kernel $k(x, y)$, find a function $f(x)$ defined on [0, 1] and satisfying (1). In this section we discuss the case that $k(x, y)$ is given by (2). This equation can be rewritten in the form

$$f(x) = \int_0^1 R(x, y) \log |x - y| f(y)\, dy + g(x). \tag{4}$$

Let $0 = x_1 < x_2 < \ldots < x_N = 1$ be a subdivision of [0, 1] with $h_i = x_{i+1} - x_i$. Using a product integration rule (See Atkinson [1]),

$$R(x, y) f(y) = \frac{R_i(x)(x_{i+1} - y) f_i + (y - x_i) R_{i+1}(x) f_{i+1}}{x_{i+1} - x_i} \tag{5}$$

where

$$R_i(x) = R(x, x_i), \tag{6}$$

$$f_i = f(x_i), \tag{7}$$

we have

$$f(x) = \sum_{i=1}^{N-1} \left(R_i(x) f_i \phi^1_{i\,i+1}(x) + R_{i+1}(x) f_{i+1} \phi^2_{i\,i+1}(x)\right) + g(x), \tag{8}$$

where

$$\phi^1_{i\,i+1}(x) = \frac{1}{x_{i+1} - x_i} \int_{x_i}^{x_{i+1}} (x_{i+1} - y) \log |x - y|\, dy, \tag{9}$$

$$\phi^2_{i\,i+1}(x) = \frac{1}{x_{i+1} - x_i} \int_{x_i}^{x_{i+1}} (y - x_i) \log|x - y|\, dy. \tag{10}$$

Substituting $x = x_j$ in (8) and combining the two sums one has the following system of equations

$$f_j = \sum_{i=1}^{N} R_{ji}\, f_i\, (\phi^1_{i\,i+1\,j} + \phi^2_{i-1\,i\,j}) + g_j\,,\ j = 1, 2, \ldots, N, \tag{11}$$

where

$$\phi^\ell_{i\,j\,k} = \phi^\ell_{i\,j}(x_k),\ \ell = 1, 2, \tag{12}$$

and

$$g_j = g(x_j). \tag{13}$$

In (11) we assume that

$$\phi^1_{N\,N+1\,j} = \phi^2_{0\,1\,j} = 0\,, \text{for all } j. \tag{14}$$

The integrals in (9) - (10) are evaluated exactly and the values of $\phi^\ell_{i\,j\,k}$ are computed separately for the cases $k = i$ and $k = j$. It can be shown that

$$\phi^1_{i\,i+1\,i} = \frac{1}{2}(x_{i+1} - x_i)\left(\log(x_{i+1} - x_i) - \frac{3}{2}\right), \tag{15}$$

$$\phi^1_{i\,i+1\,i+1} = \frac{1}{2}(x_{i+1} - x_i)\left\{\log(x_{i+1} - x_i) - \frac{1}{2}\right\}, \tag{16}$$

$$\begin{aligned}\phi^1_{i\,i+1\,j} = {} & \frac{1}{2}\left[\frac{(x_{i+1} - x_i)^2 - (x_{i+1} - x_j)^2}{x_{i+1} - x_i}\right] \log|x_i - x_j| \\ & + \frac{1}{2}\,\frac{(x_{i+1} - x_j)^2}{x_{i+1} - x_i} \log|x_{i+1} - x_j| \\ & - \frac{1}{2}(x_{i+1} - x_j) - \frac{1}{4}(x_{i+1} - x_i),\ j \neq i,\ i + 1,\end{aligned} \tag{17}$$

$$\phi^2_{i-1\,i\,i-1} = \frac{1}{2}(x_i - x_{i-1})\left\{\log(x_i - x_{i-1}) - \frac{1}{2}\right\}, \tag{18}$$

$$\phi^2_{i-1\,i\,i} = \frac{1}{2}(x_i - x_{i-1})\left\{\log(x_i - x_{i-1}) - \frac{3}{2}\right\}, \tag{19}$$

$$\phi^2_{i-2\,i\,j} = \frac{1}{2}\left[\frac{(x_i - x_{i-1})^2 - (x_j - x_{i-1})^2}{x_i - x_{i-1}}\right] \log|x_j - x_i|$$

$$+\frac{1}{2}\frac{(x_j - x_{i-1})^2}{x_i - x_{i-1}}\log|x_j - x_{i-1}|$$

$$-\frac{1}{2}(x_j - x_{i-1}) - \frac{1}{4}(x_i - x_{i-1}),\ j \neq i-1,\ i. \tag{20}$$

The system (11) can be written in matrix form

$$\vec{F} = K\vec{F} + \vec{G}, \tag{21}$$

where $\vec{F}$ and $\vec{G}$ are vectors whose components are f_i and g_i respectively and

$$K_{ij} = R_{ij}(\phi^1_{j\,j+1\,i} + \phi^2_{j-1\,j\,i}). \tag{22}$$

Note that the diagonal matrix D we had in the regular case is not present but the matrix K is more complicated. Thus we save on storage and increase the CPU required to evaluate the entries of K. The sytem (21) is solved iteratively using Gauss-Seidel. Initial vector $\vec{F}^0$ is obtained from the solution at the previous stage as discussed in [4].

Now, we turn to the criteria used by the computer to subdivide an interval. The program will half any interval $[x_j, x_{j+1}]$ for which any of the following is not satisfied.

i. $$\int_{x_j}^{x_{j+1}} |f'(x)|\,dx \le \delta(\max|F|),$$

$$j = 1, 2, \ldots, N-1, \gamma < 1 \quad \text{given}, \tag{23}$$

ii. $$\int_{x_j}^{x_{j+1}} |f'(x)|\,dx \le \delta(\max|\frac{df}{dx}|),$$

$$j = 1, 2, \ldots, N-1, \gamma < 1 \quad \text{given}, \tag{24}$$

iii. If the ratio $(x_{j+1} - x_j)/H$ is not greater than C given, where H is the length of the smallest interval.

These criteria were used in [4]. $f'(x)$ and $f''(x)$ in (23) - (24) are obtained from (1) by differentiation.

$$f'(x) = \int_0^1 \left[R_x(x,y)\log|x-y| + \frac{R(x-y)}{x-y}\right] f(y)\,dy + g'(x), \tag{25}$$

$$f''(x) = \int_0^1 \left[R_{xx}(x,y)\log|x-y| + \frac{2R_x(x,y)}{x-y} - \frac{R(x,y)}{(x-y)^2}\right] f(y)\,dy + g''(x). \tag{26}$$

The integrals in (23) and (24) can be estimated by using the midpoint formula which is of the same order as the trapezoidal rule. This way we avoid the necessity of evaluating the logarithmic function and $\dfrac{1}{(x-y)^{\ell}}$ at zero.

Thus

$$f'(x) \;=\; \sum_{\ell=}^{N} \zeta_{\,\ell} \left[R_x\,(x,\, x_\ell)\, \log\, |\, x - x_\ell\,| \,+\, \frac{R\,(x,\, x_\ell)}{x - x_\ell} \right] f_\ell \,+\, g'(x), \tag{27}$$

$$f''(x) \;=\; \sum_{\ell=}^{N} \zeta_{\,\ell} \left[R_{xx}\,(x,\, x_\ell)\, \log\, |\, x - x_\ell\,| \,+\, \frac{2R_x\,(x,\, x_\ell)}{x - x_\ell} \,-\, \frac{R\,(x,\, x_\ell)}{(x - x_\ell)^2} \right] f_\ell \,+\, g''(x), \tag{28}$$

where

$$\zeta_{\,\ell} \;=\; \begin{cases} \dfrac{x_2 - x_1}{2}, & \ell \,=\, 1, \\[2ex] \dfrac{x_N - x_{N-1}}{2}, & \ell \,=\, N, \\[2ex] \dfrac{x_{\ell+1} - x_{\ell-1}}{2}, & \text{otherwise} \end{cases}$$

Using the midpoint rule to approximate the integrals in (23) - (24) and combining the results with (27) -(28) we have

$$\int_{x_j}^{x_{j+1}} |\, f'(x)\,|\, dx \;=\; |\, f' \left(\frac{x_j \,+\, x_{j+1}}{2} \right) |\, (x_{j+1} \,-\, x_j)$$

$$= \; (x_{j+1} - x_j)\,|\, \sum_{\ell=1}^{N} \zeta_{\,\ell} \left[R_x\,(x_{j+\frac{1}{2}}\,, x_\ell)\, \log\, |\, x_{j+\frac{1}{2}} \,-\, x_\ell\,| \right.$$

$$\left. +\, \frac{R(x_{j+\frac{1}{2}}\,, x_\ell)}{x_{j+\frac{1}{2}} - x_\ell} \right] f_\ell \,+\, g'(x_{j+\frac{1}{2}})\,|, \tag{29}$$

and

$$\int_{x_j}^{x_{j+1}} |\, f''(x)\,|\, dx \;=\; (x_{j+1} \,-\, x_j)|\, \sum_{\ell=1}^{N} \zeta_{\,\ell} \left[R_{xx}\,(x_{j+\frac{1}{2}}\,, x_\ell)\, \log\, |\, x_{j+\frac{1}{2}} \,-\, x_\ell\,| \right.$$

$$\left. +\, \frac{2R_x(x_{j+\frac{1}{2}}\,,\, x_\ell)}{x_{j+\frac{1}{2}} \,-\, x_\ell} \,-\, \frac{R(x_{j+\frac{1}{2}}\,,\, x_\ell)}{(x_{j+\frac{1}{2}} \,-\, x_\ell)^2} \right] f_\ell \,+\, g''(x_{j+\frac{1}{2}})\,|, \tag{30}$$

where

$$x_{j+\frac{1}{2}} \;=\; \frac{x_{j+1} \,+\, x_j}{2}. \tag{31}$$

3 Algebraic Singularity

The integral equation is now

$$f(x) \;=\; \int_0^1 \, R(x,\, y) \, |\, x - y \,|^{-\alpha} \, f(y) \, dy \;+\; g(x), \tag{32}$$

where $R(x,\, y)$ is not singular. Using the same product integration rule (5) we now have the same system (21) to solve. The only difference is, of course, in the definition of $\phi^{\ell}_{i\,j\,k}$. We now have:

$$\phi^1_{i\,i+1}\,(x) \;=\; \frac{1}{x_{i+1} \;-\; x_i} \int_{x_i}^{x_{i+1}} (x_{i+1} \;-\; y)\,|\, x - y\,|^{-\alpha}\, dy, \tag{33}$$

$$\phi^2_{i\,-1\,i}\,(x) \;=\; \frac{1}{x_i \;-\; x_{i-1}} \int_{x_{i-1}}^{x_i} (y \;-\; x_{i-1})\,|\, x - y\,|^{-\alpha}\, dy. \tag{34}$$

One can show that

$$\phi^1_{i\,i+1\,i} \;=\; \frac{(x_{i+1} \;-\; x_i)^{1-\alpha}}{(1 \;-\; \alpha)(2 \;-\; \alpha)}\,, \tag{35}$$

$$\phi^1_{i\,i+1\,i\,+\,1} \;=\; \frac{(x_{i+1} \;-\; x_i)^{1-\alpha}}{2 \;-\; \alpha}\,, \tag{36}$$

$$\phi^1_{i\,i+1\,j} \;=\; \frac{|\, x_{i+1} \;-\; x_j\,|}{x_{i+1} \;-\; x_i}\; \frac{|\, x_{i+1} \;-\; x_j\,|^{1-\alpha} \;-\; |\, x_i \;-\; x_j\,|^{1-\alpha}}{1 \;-\; \alpha}$$

$$-\; \frac{1}{x_{i+1} \;-\; x_i}\; \frac{|\, x_{i+1} \;-\; x_j\,|^{2-\alpha} \;-\; |\, x_i \;-\; x_j\,|^{2-\alpha}}{2 - \alpha}\,,\; j \;\neq\; i,\; i+1\,, \tag{37}$$

$$\phi^2_{i-1\,i\,i-1} \;=\; \frac{(x_i \;-\; x_{i-1}\,)^{1-\alpha}}{(2 \;-\; \alpha)} \tag{38}$$

$$\phi^2_{i-1\,i\,i} \;=\; \frac{(x_i \;-\; x_{i-1}\,)^{1-\alpha}}{(1 - \alpha)\,(2 \;-\; \alpha)} \tag{39}$$

$$\phi^2_{i-1\,i\,j} \;=\; \frac{|\, x_i \;-\; x_j\,|^{2-\alpha} \;-\; |\, x_{i-1} \;-\; x_j\,|^{2-\alpha}}{(x_i \;-\; x_{i-1})\;\;(2 \;-\; \alpha)}$$

$$-\; \frac{|\, x_j \;-\; x_{i-1}\,|}{x_i \;-\; x_{i-1}}\; \frac{|\, x_i \;-\; x_j\,|^{1-\alpha} \;-\; |\, x_{i-1} \;-\; x_j\,|^{1-\alpha}}{1 - \alpha}\,,\; j \;\neq\; i, -1\,,\; i. \tag{40}$$

The criteria used to decide which subintervals should be refined are the same. The derivatives of $f(x)$ are given below

$$f'(x) \;=\; \int_0^1 [R_x(x,\, y)\,|\, x - y\,|^{-\alpha} \;-\alpha R(x,\, y)\, syn(x - y)\,|\, x - y\,|^{-\alpha\,-1}]\, f(y)\, dy \;+\; g'(x)\,, \tag{41}$$

$$f''(x) = \int_0^1 [R_{xx}(x,\, y)\,|\, x - y\,|^{-\alpha} - 2\alpha R_x(x,\, y)\,|\, x - y\,|^{-\alpha - 1}\, syn(x - y)$$

$$+ \alpha\,(\alpha + 1)\, R\,(x,\, y)\,|\, x - y\,|^{-\alpha - 2}]\, f(y)\, dy + g''(x). \tag{42}$$

As before the midpoint rule is used to approximate the integral in (23)-(24) and one has

$$\int_{x_j}^{x_{j+1}} |\, f'(x)\,|\, dx = (x_{j+1} - x_j)\,|\sum_{\ell=1}^{N} \zeta_{\,\ell} \left[R_x\,(x_{j+\frac{1}{2}}\,, x_\ell)\,|\; x_{j+\frac{1}{2}} - x_\ell\,|^{-\alpha}\right.$$

$$\left. - \alpha\, R(x_{j+\frac{1}{2}}\,, x_\ell)\,|\, x_{j+\frac{1}{2}} - x_\ell\,|^{-1-\alpha}\right] f_\ell + g'(x_{j+\frac{1}{2}})\,|\,, \tag{43}$$

$$\int_{x_j}^{x_{j+1}} |\, f''(x)\,|\, dx = (x_{j+1} - x_j)\,|\sum_{\ell=1}^{N} \zeta_{\,\ell} \left[R_{xx}\,(x_{j+\frac{1}{2}}\,, x_\ell)\,|\; x_{j+\frac{1}{2}} - x_\ell\,|^{-\alpha}\right.$$

$$- 2\alpha\, R_x(x_{j+\frac{1}{2}}\,, x_\ell)\,|\, x_{j+\frac{1}{2}} - x_\ell\,|^{-\alpha - 1}$$

$$\left. + \alpha\,(\alpha + 1)\, R(x_{j+\frac{1}{2}}\,, x_\ell)\,|\, x_{j+\frac{1}{2}} - x_\ell\,|^{-\alpha - 2}\right] f_\ell + g''(x_{j+\frac{1}{2}})\,|\,. \tag{44}$$

We now turn to description of the input to be supplied by the user and the storage required.

4 Computer Program Input

First we descibe the variables, then the vectors and matrices required.

MAXN	-	maximum number of nodes allowed (see dimension).
GAMMA	-	γ in (24)
DELTA	-	δ in (23)
C	-	the ratio between the largest and smallest interval.
NITER	-	maximum number of iterations allowed for Gauss-Seidel's method.
TOL	-	tolerance for convergence of Gauss-Seidel.
N	-	number of initial nodes (default is 9).
M	-	number of additional nodes allowed (default is MAXN-9).

Vectors:

X	-	nodes .
F	-	solution .
G	-	right hand side, values of $g(x)$.
FXI	-	values of the integral in (23).
FXXI	-	values of integral in (24).
RKX	-	values of one row of k_x or k_{xx}.

Matrices:

RK	-	values of $k(x, y)$.
RKD	-	(I-RK).

Functions to be supplied by the user:

GG(x)	-	evaluate $g(x)$.
GX(x)	-	evaluate $g'(x)$.
GXX(x)	-	evaluate $g''(x)$.
RKF(x, y)	-	evaluate $R(x, y)$.
RKFX(x, y)	-	evaluate $R_x(x, y)$.
RKFXX(x, y)	-	evaluate $R_{xx}(x, y)$.

5 Numerical Experiment

In this section we describe some of the experiments performed using our method with various kernels. In the first experiment we solve the following problem,

$$f(x) = \int_0^1 \log |x - y| f(y)\, dy + g(x), \tag{45}$$

where $g(x)$ is chosen such that the exact solution is

$$f(x) = \begin{cases} 10, & x\epsilon(0,.5), \\ -90x + 55, & x\epsilon(.5,.6), \\ 1, & x\epsilon(.6,1). \end{cases} \tag{46}$$

The results are summarized in Table 1 for various values of the parameters γ, δ and C.

Number of Nodes					
initial	allowed	used	$\gamma = \delta$	C	error
40	100	100	.9	4	.4400-2
40	100	100	.5	4	.4400-2
40	100	100	.1	4	.4403-2
40	100	100	.9	10	.4400-2
40	100	100	.5	10	.4400-2
40	100	100	.1	10	.4403-2

Table 1:

Note that the process requires more than 100 nodes for convegence even with $\gamma = \delta = .9$.

In Table 2 we have listed the maximum absolute error between the approximate and exact solution for the problem

$$f(x) = \int_0^1 \log|x - y| f(y)\,dy + g(x), \tag{47}$$

whose exact solution is

$$f(x) = \begin{cases} 10, & x\epsilon(0,.2), \\ 100x - 10, & x\epsilon(.2,.3), \\ 20, & x\epsilon(.3,.5), \\ -70x + 55, & x\epsilon(.5,.6), \\ 13, & x\epsilon(.6,1). \end{cases} \tag{48}$$

Number of Nodes					
initial	allowed	used	$\gamma = \delta$	C	error
40	100	100	.9	4	.3393-2
40	100	100	.5	4	.6364-2
40	100	100	.1	4	.6364-2
40	100	100	.9	10	.3393-2
40	100	100	.5	10	.6364-2
40	100	100	.1	10	.6364-2

Table 2:

Note that slightly better results were obtained with large γ and δ.

The next experiments involve kernels of the form $|\,x - y\,|^{-\alpha}$ for various values of α less than 1. In Table 3 we summarized the results for the problem

$$f(x) \;=\; \int_0^1 |\,x - y\,|^{-\alpha} f(y)\,dy \;+\; g(x)\,, \tag{49}$$

whose exact solution is

$$f(x) \;=\; x\,, \tag{50}$$

and

$$\alpha \;=\; \frac{1}{2}\,,\; g(x) \;=\; x \;-\; \frac{4}{3}\,x^{3/2} \;-\; \frac{2}{3}(1 \;+\; 2x)\,\sqrt{1 - x.} \tag{51}$$

Number of Nodes					
initial	allowed	used	$\gamma = \delta$	C	error
5	100	100	.9	4	.2503-13
5	100	100	.5	4	.2646-13
5	100	100	.1	4	.2297-13
5	100	100	.9	10	.2817-13
5	100	100	.5	10	.2687-13
5	100	100	.1	10	.2027-13

Table 3:

Note the excellent results independent of the values of the parameters. On the other hand, we encounter some difficulty with the convergence of the Gauss Seidel iterative process. In order to overcome the difficulty we replaced the iterative method by a direct method <u>only</u> for the results in Table 3.

Next, we solve the same problem (49) with exact solution (50) but various values of α. It turned out that for $\alpha = .1$ or $\alpha = .9$ the Gauss Seidel process converged fast. We list in Table 4 the values of α along with the number of nodes used and the error.

Number of Nodes						
initial	allowed	used	α	$\gamma = \delta$	C	error
5	100	100	.1	.5	10	.1943-15
5	100	100	.2	.5	10	.7669-12
5	100	100	.3	.5	10	.1331-9
5	100	–	.4	.5	10	Gauss Seidel Diverge
5	100	–	.5	.5	10	Gauss Seidel Diverge
5	100	–	.6	.5	10	Gauss Seidel Diverge
5	100	–	.7	.5	10	Gauss Seidel Diverge
5	100	100	.8	.5	10	.3548-12
5	100	100	.9	.5	10	.7466-14

Table 4:

Note that better results were obtained with less nodes for $\alpha = .9$ compared to the case $\alpha = .1$.

The last problem solved is,

$$f(x) = \int_0^1 |x-y|^{-\alpha} f(y)\,dy + g(x), \tag{52}$$

The exact solution is patched from constants and linear functions with different slopes,

$$f(x) = \begin{cases} 1/3, & x\epsilon(0,1/4), \\ 4x - 2/3, & x\epsilon(1/4,1/3), \\ 2/3, & x\epsilon(1/3,1/2), \\ \dfrac{104}{3}x - \dfrac{50}{3}, & x\epsilon(1/2,5/8), \\ 5, & x\epsilon(5/8,1). \end{cases} \tag{53}$$

Here, again, we encountered difficulty with the Gauss Seidel iterative process. Note that the results are better for smaller α.

We conclude that one has to establish a theoretical foundation for the method. Here, we encountered difficulties with Gauss Seidel iterative method. We were unable to obtain good enough results (except in Tables 3, 4) with few nodes as in Part I.

Number of Nodes						
initial	allowed	used	α	$\gamma = \delta$	C	error
40	100	100	.1	.1	4	.3281-2
40	100	100	.2	.1	4	.1963-2
40	100	100	.3	.1	4	.7016-3
40	100	100	.4	.1	4	.2066-2
40	100	100	.5	.1	4	.5750-2
40	100	100	.6	.1	4	.6020-2
40	100	100	.7	.1	4	.4527-1
40	100	100	.8	.1	4	.4426-1
40	100	100	.9	.1	4	.3154

Table 5:

References

1. K. E. Atkinson, A Survey of Numerical Methods for the Solution of Fredholm Integral Equations of the Second Kind, SIAM, Philadelphia, 1976.

2. K. M. Case, and P. F. Zweifel, Linear Transport Theory, Addison Wesley, 1967.

3. A. Gerasoulis, and R. P. Srivastav, A Griggiths crack problem for a nonhomogeneous medium, *International Journal of Engineering Science*, **18**, (1980), 239-247.

4. B. Neta, and P. Nelson, Adaptive method for the numerical solution of Fredholm integral equations of the second kind. Part I: regular kernel, Institute for Numerical Transport Theory Technical Report, INTT-27, Texas Tech University, Department of Mathematics, Lubbock, February, 1984. Also appeared in *Applied Mathematics and Computation*, **21**, (1987), 171-184.

5. I. N. Sneddon, Mixed Boundary Value Problems in Potential Theory, North Holland, 1966.

6. I. N. Sneddon, and M. Lowengrub, Crack Problems in the Classical Theory of Elasticity, John Wiley, New York, 1969.

7. I. Stakgold, Boundary Value Problems of Mathematical Physics, MacMillan, New York, 1967.

8. G. M. Wing, An Introduction to Transport Theory, John Wiley, 1962.