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On Satellite Umbra/Penumbra Entry and Exit Positions

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Abstract

The problem of computing Earth satellite entry and exit positions through the Earth's umbra and penumbra, for satellites in elliptical orbits, is solved without the use of a quartic equation. A condition for existence of a solution in the case of a cylindrical shadow is given. This problem is of interest in case one would like to include perturbation force resulting from solar radiation pressure. Most satellites (including geosynchronous) experience periodic eclipses behind the Earth. Of course when the satellite is eclipsed, it's not exposed to solar radiation pressure. When we need extreme accuracy, we must develop models that turn the solar radiation calculations "on" and "off," as appropriate, to account for these periods of inactivity.

Introduction

The problem of computing Earth satellite (in elliptical orbits) entry and exit positions through the Earth's umbra and penumbra is a problem dating from the earliest days of the space age, but it is still of the utmost importance to many space projects for thermal and power considerations [1]. It's also important for optical tracking of a satellite. To a lesser extent, the satellite external torque history and the sensor systems are influenced by the time the satellite spends in the Earth's shadow.

The umbra is the conical total shadow projected from the Earth on the side opposite the sun. In this region, the intensity of the solar radiation is zero. The penumbra is the partial shadow between the umbra and the full-light region (see Fig. 1). In the penumbra, the light of the sun is only partially cut off by the Earth, and the intensity is between 0 and 1. All textbooks discussing the problem (e.g. Geyling and Westerman [2] and Escobal [3]) and even the recent work by Mullins [1] suggest the use of a quartic equation analytic solution to determine umbra/penumbra boundaries. Because the quartic is a result of squaring the equation of interest, one must check all four solutions and discard the spurious ones. In this paper, we examine solving the original equation numerically. We will give a condition for the existence of a solution, discuss the initial guess for an

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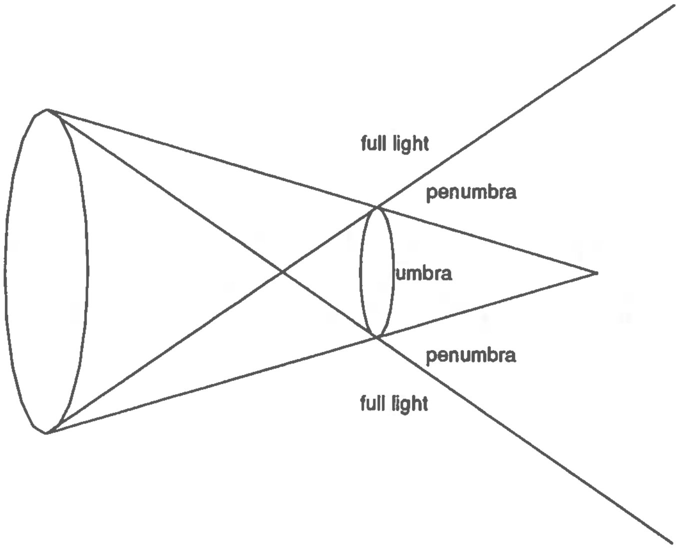

FIG. 1. Earth Umbra and Penumbra.

iterative scheme, and compare the complexity of the two schemes (our scheme versus the analytic solution of the quartic [1]).

The shadow problem has been solved in the past by assuming a cylindrical shadow behind the Earth [2], or a conical shadow which is more realistic [1, 4]. The numerical solution will be discussed for each case.

Problem Formulation

In this section, we formulate the problem using both cylindrical and conical shadow geometry. We'll see that the solution method is different in the two cases.

Cylindrical Shadow

In this case the orbital geometry is given in Fig. 2 [3, 5].

The analysis given in Escobal [3] and Vallado [5] show that the true anomaly, ν, at entrance and exit into the shadow satisfies the following equation:

$$R_{\oplus}^2(1 + e\cos\nu)^2 + p^2(\beta_1\cos\nu + \beta_2\sin\nu)^2 - p^2 = 0 \tag{1}$$

where $R_{\oplus}$ is the radius of the Earth (~6378.136 km), $r_{\odot}$ is the sun's position vector (~$1.496(10^8)$ km), p is semi-parameter, and e is the eccentricity. The remaining

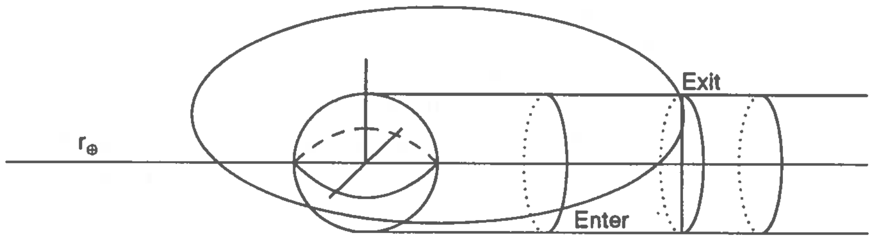

FIG. 2. Cylindrical Shadow.

classical orbital elements are inclination, i, longitude of the ascending node, Ω, and the argument of perigee, ω. The parameters β_1 and β_2 are given by

$$\beta_1 = \frac{\mathbf{r}_\odot \cdot \mathbf{P}}{\|\mathbf{r}_\odot\|}$$

$$\beta_2 = \frac{\mathbf{r}_\odot \cdot \mathbf{Q}}{\|\mathbf{r}_\odot\|}$$

The Earth-centered unit vectors $\mathbf{P}$ and $\mathbf{Q}$ are defined by

$$\mathbf{P} = \begin{bmatrix} \cos\omega\cos\Omega - \sin\omega\sin\Omega\cos i \\ \cos\omega\sin\Omega + \sin\omega\cos\Omega\cos i \\ \sin\omega\sin i \end{bmatrix}$$

$$\mathbf{Q} = \begin{bmatrix} -\sin\omega\cos\Omega - \cos\omega\sin\Omega\cos i \\ -\sin\omega\sin\Omega + \cos\omega\cos\Omega\cos i \\ \cos\omega\sin i \end{bmatrix}$$

For circular orbits and if $i = 0, \pi$, $\mathbf{P}$ should be redefined in a convenient manner [3].

Conical Shadow

In this case, one must distinguish between umbra (full shadow) and penumbra (partial shadow) regions. In the umbra case, we must solve a system of two nonlinear equations [1]. The first equation models the surface of the shadow cone

$$F(x_{sh}, y_{sh}, z_{sh}) = y_{sh}^2 + z_{sh}^2 - (d - x_{sh})^2 \tan^2\sigma = 0 \qquad (2)$$

where d is the distance from center of the Earth to apex of shadow cone ($\sim 1.3836(10^6)$ km), and σ is half-angle of that cone ($\sim 0.26412°$). The second equation describes the orbit[3]

$$G(x_0, y_0) = \left(\frac{x_0 + ae}{a}\right)^2 + \left(\frac{y_0}{b}\right)^2 - 1 = 0 \qquad (3)$$

where $b = a\sqrt{1 - e^2}$. Because the two equations are *not* in the same coordinate system, we take $\mathbf{r}_{sh}$ and rotate it to get $\mathbf{r}_0$. The transformation is given by

$$\mathbf{r}_0 = ROT3(\omega)ROT1(i)ROT3(\Omega)ROT1(-\epsilon)ROT3(\pi - L)\mathbf{r}_{sh}$$

where ϵ is the mean obliquity of the ecliptic ($\sim 23.5°$), L is the ecliptic longitude of the sun, and $ROT1(\phi)$, $ROT3(\phi)$ are rotations about the x, z axes (respectively) by ϕ. If we denote the transformation matrix by A, then

$$x_{sh} = a_{11}x_0 + a_{21}y_0 \qquad (4)$$

$$y_{sh} = a_{12}x_0 + a_{22}y_0 \qquad (5)$$

$$z_{sh} = a_{13}x_0 + a_{23}y_0 \qquad (6)$$

[3]This doesn't mean that we are interested in only two-body problems, but certainly excludes parabolic or hyperbolic orbits as previously mentioned.

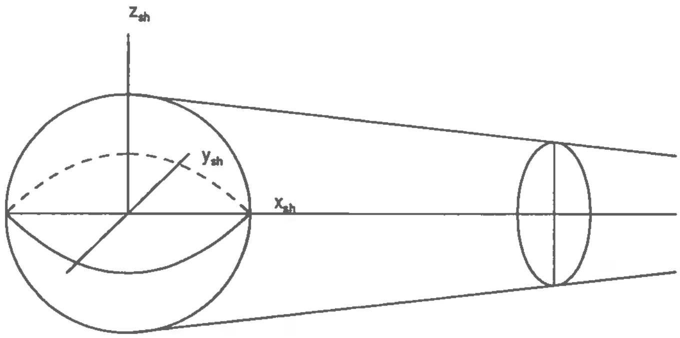

FIG. 3. Earth-Centered Coordinate System and x Along the Shadow Cone Axis.

Notice that z_0 is zero at the intersection of equations (2) and (3). Because only solutions with $x_{sh} > 0$ are acceptable (see Fig. 3), we must satisfy

$$a_{11}x_0 + a_{21}y_0 > 0 \tag{7}$$

Substituting equations (4–6) into equation (2), we get the following

$$F_1(x_0, y_0) = \alpha_0 x_0^2 + \alpha_1 y_0^2 + 2\alpha_2 x_0 y_0 + \alpha_3 x_0 + \alpha_4 y_0 - d^2 \tan^2 \sigma = 0 \tag{8}$$

where

$$\alpha_0 = a_{12}^2 + a_{13}^2 - a_{11}^2 \tan^2 \sigma$$
$$\alpha_1 = a_{22}^2 + a_{23}^2 - a_{21}^2 \tan^2 \sigma$$
$$\alpha_2 = a_{12}a_{22} + a_{13}a_{23} - a_{11}a_{21} \tan^2 \sigma$$
$$\alpha_3 = 2a_{11}d \tan^2 \sigma$$
$$\alpha_4 = 2a_{21}d \tan^2 \sigma$$

This equation should be solved with equations (3) and (7).

Mullins [1] suggests solving equation (8) subject to equations (3) and (7), using a quartic equation for x and then checking each of the four solutions with solutions of a quadratic equation for y as a function of x. Mullins admits: "The coefficients (of the quartic) are messy functions of the angles shown . . . ." In a following section, we show another way to solve the problem without going through a quartic equation and thus without computing these "messy coefficients."

In the penumbra case, Mullins [1] shows that equation (2) becomes

$$F(x_{sh}, y_{sh}, z_{sh}) = y_{sh}^2 + z_{sh}^2 - (d' + x_{sh})^2 \tan^2 \sigma' = 0$$

where d' is the distance from the center of the Earth to the apex of the cone between the sun and the Earth ($\sim 1.35849(10^6)$ km), and σ' is half angle of that cone ($\sim 0.26901°$). This leads to an equation similar to equation (8) to solve. The idea presented in a following section will be used here too.

Complexity of Quartic Solution

The problem (for cylindrical shadow) can be solved analytically using the quartic equation

$$A_0 \cos^4 \nu + A_1 \cos^3 \nu + A_2 \cos^2 \nu + A_3 \cos \nu + A_4 = 0 \tag{9}$$

and then the spurious roots can be rejected based on the following conditions. The physical solution should satisfy the original equation and

$$\beta_1 \cos \nu + \beta_2 \sin \nu < 0$$

The coefficients of the quartics are given by:

$$A_0 = \left(\frac{R_\oplus}{p}\right)^4 e^4 - 2\left(\frac{R_\oplus}{p}\right)^2 (\beta_2^2 - \beta_1^2)e^2 + (\beta_1^2 + \beta_2^2)^2$$

$$A_1 = 4\left(\frac{R_\oplus}{p}\right)^4 e^3 - 4\left(\frac{R_\oplus}{p}\right)^2 (\beta_2^2 - \beta_1^2)e$$

$$A_2 = 6\left(\frac{R_\oplus}{p}\right)^4 e^2 - 2\left(\frac{R_\oplus}{p}\right)^2 (\beta_2^2 - \beta_1^2) - 2\left(\frac{R_\oplus}{p}\right)^2 (1 - \beta_2^2)e^2 + 2(\beta_2^2 - \beta_1^2)(1 - \beta_2^2) - 4\beta_1^2\beta_2^2$$

$$A_3 = 4\left(\frac{R_\oplus}{p}\right)^4 e - 4\left(\frac{R_\oplus}{p}\right)^2 (1 - \beta_2^2)e$$

$$A_4 = \left(\frac{R_\oplus}{p}\right)^4 - 2\left(\frac{R_\oplus}{p}\right)^2 (1 - \beta_2^2) + (1 - \beta_2^2)^2$$

If the work is done economically, one finds that the number of multiplications and divisions required to compute the coefficients of the quartic is 38. To find the solution of the quartic requires 64 multiplications/divisions, 5 square roots, 4 cube roots, 1 arccosine, and 3 cosine evaluations. The cosine and arccosine evaluations are required only if the discriminant is negative [6].

Numerical Solution for Cylindrical Shadow

To solve the shadow equation (1) numerically, we can use either bracketing or fixed-point type methods. Neta [7] has collected algorithms[4] of both types and compared those from the point of view of efficiency. Reference to available software is given. In the following, we describe only Newton's and Halley's methods which are of fixed-point type. It is first suggested to check the existence of a solution. First, rewrite equation (1) as

$$f(x) = Ax^2 + Bx + Cx\sqrt{1 - x^2} + D = 0 \tag{10}$$

[4]This monograph is available from the first author.

where $x = \cos \nu$ and the coefficients are given by:

$$A = e^2 + \frac{p^2}{R_\oplus^2}(\beta_1^2 - \beta_2^2)$$
$$B = 2e$$
$$C = 2\frac{p^2}{R_\oplus^2}\beta_1\beta_2$$
$$D = 1 + \frac{p^2}{R_\oplus^2}\beta_2^2 - \frac{p^2}{R_\oplus^2}$$

In order to have a solution, the function must change sign in the interval (which is $-1 \leq x \leq 1$) and thus we must have

$$f(-1)f(1) \leq 0 \tag{11}$$

Clearly equality means that either $f(1) = 0$ or $f(-1) = 0$, i.e. $\cos \nu = \pm 1$. The strict inequality in equation (11) is equivalent to

$$1 - \left(\frac{R_\oplus}{a(1+e)}\right)^2 > \beta_1^2 > 1 - \left(\frac{R_\oplus}{a(1-e)}\right)^2$$

Note that there is *no* condition on β_2.

Remark: Close analysis of equation (10) shows that the use of this transformation will not be useful in the numerical solution (see Neta and Vallado [8]). This result is consistent with Junkins's talk [9]. Therefore, we rewrite equation (1) as follows

$$A\cos^2 \nu + B\cos \nu + C\cos \nu \sin \nu + D = 0, \qquad 0 \leq \nu \leq 2\pi \tag{12}$$

where A, B, C, and D are the same as in equation (10). This is the equation we will solve numerically.

Newton's Method

To solve a nonlinear equation $f(x) = 0$ via Newton's method, we require an initial guess x_0. Then an iterative procedure can be followed to construct a sequence of estimates x_n, by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad n = 0, 1, \ldots$$

The iterative process converges if either

$$|f(x_n)| < \mathrm{Tol}_f$$

or

$$|x_{n+1} - x_n| < \mathrm{Tol}_x$$

for given tolerances. In either case we take x_{n+1} as the root. The convergence rate is quadratic. If the iterative process doesn't converge in a certain number of iterations, we stop. In this case we suggest bracketing methods. Newton's method will diverge if we hit a point where $f'(x)$ is very small. It is advocated by many to use so called hybrid methods (see Neta [7]). Such methods combine a bracketing step whenever the iteration starts to diverge.

Halley's Method

Halley's method converges faster (third-order compare to second-order for Newton). The iterative process is

$$x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)}{2[f'(x_n)]^2 - f(x_n)f''(x_n)}, \qquad n = 0, 1, \ldots$$

or

$$x_{n+1} = x_n - \frac{\dfrac{f(x_n)}{f'(x_n)}}{1 - \dfrac{f''(x_n)}{2f'(x_n)}\dfrac{f(x_n)}{f'(x_n)}}, \qquad n = 0, 1, \ldots$$

Bracketing Methods

In general, bracketing methods are slower, but they are safer in the sense that convergence is guaranteed. For example, the bisection method starts with an initial interval, $[a_0, b_0]$, containing the root. The process halves the interval at every step. After n iterations, the length of the interval containing the root is $(b_0 - a_0)/2^n$. Therefore, the number of iterations required depends on the length of the initial interval and the tolerance.

This simplistic method can be modified by using Regula Falsi (solving a linear equation at every step) or modified Regula Falsi (which is useful when the curvature of f is large enough.)

For example, we have solved equation (10) with $A = 1$, $B = -2$, $C = 1$, and $D = 1$. Newton's method required 6 iterations for convergence to 10^{-8}, Halley's method required 5 iterations, and the bisection methods used 38 iterations, for *both* roots.

Convergence

We have already mentioned that bisection and other bracketing methods are guaranteed to converge if one starts with an interval containing the root. For Newton's and Halley's methods which are of fixed-point type, the convergence results are given here. Let's recall that a fixed-point iterative procedure is given by

$$x_{n+1} = g(x_n)$$

The function $g(x)$ for Newton's method is

$$g(x) = x - \frac{f(x)}{f'(x)}$$

and for Halley's method is

$$g(x) = x - \frac{\dfrac{f(x)}{f'(x)}}{1 - \dfrac{f''(x)}{2f'(x)}\dfrac{f(x)}{f'(x)}}$$

We now quote the results for convergence of fixed-point methods (see any numerical analysis book). The first theorem gives nonlocal convergence results.

Existence Theorem: If $g(x)$ maps the interval $[a, b]$ into itself and $g(x)$ is continuous, then $g(x)$ has *at least* one fixed point in the interval.

Uniqueness Theorem: Under the above conditions and

$$|g'(x)| \leq L < 1 \quad \text{for all } x \in [a, b]$$

then there exists *exactly one* fixed point in the interval.

Convergence Theorem: Under the conditions of the last theorem, the error $e_n \equiv x_n - \xi$ satisfies

$$|e_n| \leq \frac{L^n}{1 - L} |x_1 - x_0|$$

Note that the theorem ascertains convergence of the fixed-point algorithm for *any* $x_0 \in [a, b]$ and thus is called a global convergence theorem. It is generally possible to prove only a *local* result. Note also that when L is close to unity, the convergence is slow.

Local Convergence Theorem for Newton's Method: Let f'' be continuous and $f'(x) \neq 0$ in some open interval containing the root ξ. Then there exists an $\epsilon > 0$ such that Newton's method is quadratically convergent whenever $|x_0 - \xi| < \epsilon$.

Note that quadratic convergence means that the error e_n at the nth step satisfies

$$e_n \leq K e_{n-1}^2$$

As we mentioned earlier, Halley's method is of order 3, i.e.

$$e_n \leq C e_{n-1}^3$$

It is important to note that one can get quadratic (Newton) or cubic (Halley) convergence if the initial guess is close enough. This may not be easy for the first computation of the entry/exit, but it is true at subsequent crossings.

In the later examples, one can see the rate at which the value of f at the nth iterate approaching zero, one gains several digits of accuracy at every step.

Initial Guess

Because the problem is to solve for $\cos \nu$, we know that the solution, if it exists, must lie in the interval $[-1, 1]$. For bracketing methods we suggest using this interval, and for Newton's and Halley's method, we take the midpoint of the interval, i.e. $x_0 = 0$. If one uses equation (12) instead of equation (10), the initial guess could be π which is the midpoint of $[0, 2\pi]$.

For subsequent crossings through the shadow, we can take x_0 to be the previous solution.

Complexity of Numerical Solution

All iterative procedures require function evaluations, and some will require the evaluation of the first and perhaps second derivative. The evaluation of the function requires five multiplications/divisions (using nested multiplication) and two trigonometric function evaluations. The evaluation of the first derivative is accomplished by five multiplications/divisions. The second derivative requires four multiplications/divisions. For one iteration of Halley's method we need 14 multiplications/divisions and two trigonometric function evaluations. For one iteration of Newton's method we need 10 multiplications/divisions and two trigonometric function evaluations. For the bisection method we need five multiplications/divisions and two trigonometric function evaluations. If we multiply the number of iterations (for *both* roots) by the cost per iteration we find that Newton's method is the cheapest with 60 multiplications/divisions and 12 trigonometric function evaluations, then Halley's method with 70 multiplications/divisions and 10 trigonometric function evaluations, then (after the quartic) the bisection method with 190 multiplications/divisions and 76 trigonometric function evaluations. To each of these, we add 11 multiplications/divisions for computing the coefficients. In comparison, Newton's method is cheaper than solving the quartic and it doesn't require checking for spurious roots. Even Halley's method is competitive with the analytic solution of the quartic. We summarize the results in Table 1.

Numerical Solution for Conical Shadow

In this section, we describe a numerical method to solve equation (8) and equation (3) subject to equation (7). We suggest guessing an initial approximation $x_0 = 0$ and use equation (3) to get the corresponding y_0

$$y_0 = \pm b\sqrt{1 - e^2} \tag{13}$$

Because equation (3) is quadratic, we offer here the correct sign to satisfy equation (7). Note that equation (7) describes a half-plane whose boundary is a line in Figs. 4 and 5. The equation of the line is:

$$y_0 = -\frac{a_{11}}{a_{21}} x_0 \tag{14}$$

If we have $a_{21} > 0$ and $a_{11} > 0$, then the half-plane includes the first quadrant (Fig. 4). Since we take $x_0 = 0$, then y_0 must be positive to get a point in the half-plane

$$y_0 > -\frac{a_{11}}{a_{21}} x_0$$

TABLE 1. Operation Count

Operation	Number of mult./div.	Number of square roots	Number of cubic roots	Number of trig. func.	Number of iterations
Newton	71	0	0	12	6
Halley	81	0	0	10	5
Bisection	201	0	0	76	38
Quartic	102	5	4	2[1]	0

[1]This doesn't include checking for spurious roots.

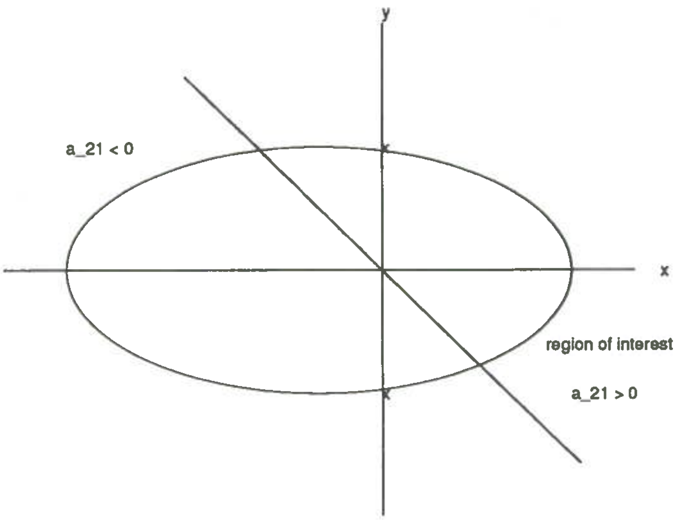

FIG. 4. a_{11} and a_{21} Have the Same Sign.

If we now consider the case $a_{21} < 0$ and $a_{11} < 0$, then the half-plane

$$y_0 < -\frac{a_{11}}{a_{21}} x_0$$

includes the third quadrant and with $x_0 = 0$ we must have $y_0 < 0$. Therefore the sign of y_0 is the same as the sign of a_{21}.

The other two cases when a_{21} and a_{11} have different signs are shown in Fig. 5. Again one can see that with $x_0 = 0$, y_0 and a_{21} must have the same sign.

Therefore the sign of the radical in equation (13) is the same as the sign of a_{21}.

We now rewrite equation (8) as

$$F_1(x, y) = Ax^2 + By(x)^2 + Cxy(x) + Dx + Ey(x) + F$$

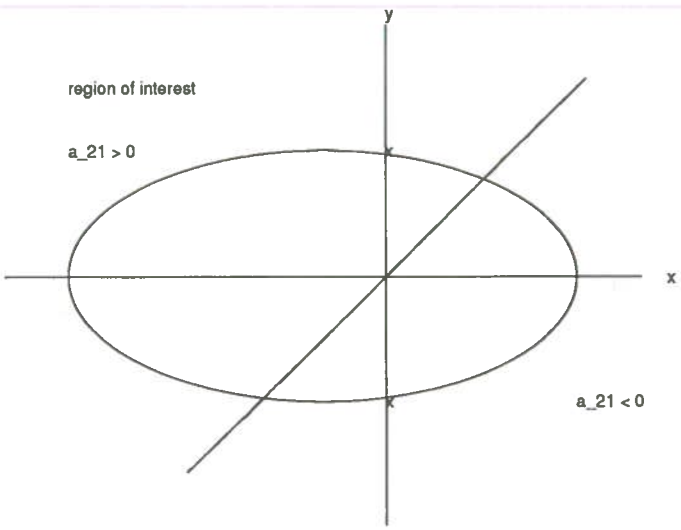

FIG. 5. a_{11} and a_{21} Have Opposite Signs.

with [using equation (3)]

$$y(x) = \pm\sqrt{1 - e^2}\sqrt{a^2 - (x + ae)^2}$$

For Newton's method, we need F_1' and y' which are given by

$$F_1'(x, y) = 2Ax + 2By(x)y'(x) + Cy(x) + Cxy'(x) + D + Ey'(x)$$

and

$$y'(x) = \mp\frac{(1 - e^2)(x + ae)}{y}$$

Now the iterative procedure is as follows

$$x_{n+1} = x_n - \frac{F_1(x_n, y_n)}{F_1'(x_n, y_n)} \qquad n = 1, 2, \ldots$$

$$y_{n+1} = \pm\sqrt{1 - e^2}\sqrt{a^2 - (x_{n+1} + ae)^2}$$

Note that the appropriate sign must be chosen.

Cylindrical Shadow Example

In this section we give an example of the results using the numerical methods discussed. The orbit parameters are eccentricity, $e = 0.002$; inclination, $i = 63.4°$; semi-parameter, $p = 1.029R_\oplus$; and all other parameters are zero. The Julian date is 2458866.5 which translates to January 18, 2025, 00:00:00. Computing β_1 and β_2 we get

$$\beta_1 = 0.459588$$

$$\beta_2 = -0.6807135\,.$$

When computing the coefficients of equation (12), we note that we can divide through by $R_\oplus$. Thus

$$A = 0.22365359961987$$

$$B = -0.48663607239723$$

$$C = -0.66251206398010$$

and

$$D = -0.55219149589539$$

To compare our answer with the solution of the quartic, we have used MAPLE (a software for symbolic as well as numeric manipulation). The solution of the quartic is given by MAPLE as

$$0.9515384807,\ 0.6383876663,\ -0.6284006779,\ -0.9573391655$$

We have used the initial guesses $\pi/2$ and π (this last one is the midpoint for the interval for ν) for both Halley's method and Newton's. The results are summarized in Tables 2 and 3. We haven't listed the results for bisection to save space, but it was found that 19 iterations are required for each root.

If we lump the roots and trigonometric functions evaluations, both Newton's and Halley's algorithms are competitive with the quartic. Clearly in the next entry/exit, we have a better starting value and the number of iterations will drop. In such a case, we believe that even the bisection is competitive.

TABLE 2. Halley's Method

Index	x_n	f_n
0	3.1415927410126	0.15809811820307
1	3.3458677144367	7.1754595942126E-3
2	3.3548687370636	3.7933632368592E-7
0	1.5707963705063	−0.55219144566452
1	2.0102227850806	−4.9619495511140E-2
2	2.0592174428168	−7.9155781539941E-5
3	2.0592977378196	−2.2241000152690E-9

Conical Shadow Example

In this section we utilize two cases, one is a Molniya orbit and the other is Topex. In both cases the Julian date is 2451059.5, which is September 3, 1998, 0 hrs UTC. The orbit parameters for the Molniya are eccentricity, $e = 0.7310151$; inclination, $i = 63.7771°$; semi-major axis, $a = 26573.92$ km, and all other parameters are zero. The orbit parameters for Topex are eccentricity, $e = 0.0007391$; inclination, $i = 66.0424°$; semi-major axis, $a = 7714.39$ km, and all other parameters are zero.

We have used the initial guess $x_0 = 0$ as suggested previously for Newton's method. The results are summarized in Table 4.

Newton's method required four iterations (in the case of Topex, one more iteration is required to reduce the value of F_1 to 10^{-8}). Again with a better choice of initial guess (for subsequent crossings), one can reduce the number of iterations.

Conclusions

In this paper, we suggest the use of iterative techniques to compute the entry and exit positions through the Earth's umbra and penumbra. We also show how to choose the initial guess for the first and subsequent crossings. Several iterative methods for the solution of the problem are compared to the current method.

TABLE 3. Newton's Method

Index	x_n	f_n
0	3.1415927410126	0.15809811820307
1	3.3802270351102	−2.0352665051308E-2
2	3.3549832208569	−9.1037138606143E-5
3	3.3548692425999	−3.3682949540648E-8
0	1.5707963705063	−0.55219144566452
1	2.0513187842744	−7.8994163839551E-3
2	2.0592637093607	−3.3538132013122E-5
3	2.0592977304045	−2.2241000152690E-9

TABLE 4. Newton's Method for Molniya (Top) and Topex (Bottom)

Index	x_n	y_n	f_n
0	0.0	−12373.266877648	—
1	4488.2707072959	−7907.2661725674	3205757.601269
2	4644.5415889873	−7683.3182279388	17470.353126
3	4645.4027483515	−7682.0619334299	0.564417
4	4645.4027761748	−7682.0618928360	2.23E-8
0	0.0	−7714.3857858694	13188317.066076
1	2941.3435563094	−7129.2853450404	−5828386.245817
2	2219.7272005566	−7386.4233192724	−231728.769016
3	2188.4458400588	−7395.7758038092	−507.821106
4	2188.3769864374	−7395.7962306511	−2.474E-3
5	2188.3769861019	−7395.7962307506	1.490E-8

Newton's method converges fast especially at subsequent crossings because the initial guess is good.

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