

Letter Section

Two-step fourth-order P-stable methods with phase-lag of order six for $y'' = f(t, y)$

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Abstract: We present a new family of two-step fourth-order methods which when applied to the test equation: $y'' = -\lambda^2 y$, $\lambda > 0$, are at once P-stable and have a phase-lag of order H^6 ($H = \lambda h$, h is the step-size).

Keywords: Special second-order initial-value problems, two-step fourth-order methods, interval of periodicity, P-stable methods, phase-lag.

1. Recently, Thomas [6] and Chawla and Rao [4] described two-step methods for special second-order initial-value problems

$$y'' = f(t, y), \quad y(t_0) = y_0, \quad y'(t_0) = y'_0,$$

which when applied to the test equation

$$y'' = -\lambda^2 y, \quad \lambda > 0, \tag{1}$$

have phase-lag of order higher than the order of the method. But higher order phase-lag of these methods was achieved by sacrificing P-stability since the resulting methods possess only a finite interval of periodicity. It is therefore natural to ask if we can obtain methods which are at once P-stable and have a phase-lag of order higher than the order of the method. While no such methods can be found from the families of P-stable methods considered in [5,6], we show the interesting result that by a suitable modification of Noumerov's method we obtain a new family of fourth-order methods which are at once P-stable and have a phase-lag of order H^6 . However, in comparison with the method proposed in [4], the present methods can be useful in cases where a large step-size is to be used; that is, where a modest accuracy is sufficient or in case of problems where the solution consists of a slowly varying oscillation with a high-frequency oscillation superimposed, having a small amplitude.

2. Let $h > 0$ be the step-size, $t_n = t_0 + nh$, $n = 0, 1, 2, \dots$, and set $y_n = y(t_n)$, $f_n = f(t_n, y_n)$. As in [3], let

$$\begin{aligned} \bar{y}_n &= y_n - \alpha h^2(f_{n+1} - 2f_n + f_{n-1}), & \bar{f}_n &= f(t_n, \bar{y}_n), \\ \bar{\bar{y}}_n &= \bar{y}_n - \beta h^2(f_{n+1} - 2\bar{f}_n + f_{n-1}), & \bar{\bar{f}}_n &= f(t_n, \bar{\bar{y}}_n), \end{aligned} \tag{2}$$

and consider a modification of Noumerov’s method defined by

$$y_{n+1} = 2y_n - y_{n-1} + \frac{1}{12}h^2(f_{n+1} + 10\bar{\bar{f}}_n + f_{n-1}), \tag{3}$$

where α, β are free parameters. We denote these methods by $M_4(\alpha, \beta)$.

For an $M_4(\alpha, \beta)$ applied to the test equation (1), setting $H = \lambda h$ we obtain the stability polynomial

$$p(\xi) = A(H)\xi^2 - 2B(H)\xi + A(H), \tag{4}$$

where

$$\begin{aligned} A(H) &= 1 + \frac{1}{12}H^2 + \frac{5}{6}(\alpha + \beta)H^4 - \frac{5}{3}\alpha\beta H^6, \\ B(H) &= 1 - \frac{5}{12}H^2 + \frac{5}{6}(\alpha + \beta)H^4 - \frac{5}{3}\alpha\beta H^6. \end{aligned} \tag{5}$$

As in [4], $(0, H_p)$ is an interval of periodicity of $M_4(\alpha, \beta)$ if the roots of the stability polynomial (4) are of the form: $\xi_{1,2} = \exp(\pm i\theta(H))$ for all $H \in (0, H_p)$, where $\theta(H)$ is real. Following Brusa and Nigro [1] the phase-lag, denoted by $P(H)$, is the leading coefficient in the expansion of $|(\theta(H) - H)/H|$. Now, with the help of (4) and (5) we obtain

$$\theta(H) = H + \frac{5}{12}\left(\frac{1}{200} - (\alpha + \beta)\right)H^5 + \frac{5}{6}\left(\frac{1}{10800} + \alpha\beta\right)H^7 + O(h^9). \tag{6}$$

It therefore follows that all those methods of $M_4(\alpha, \beta)$ for which $\alpha + \beta = \frac{1}{200}$ will have phase-lag of order six given by

$$P(H) = \frac{5}{6}\left|\frac{1}{10800} + \alpha\beta\right|H^6. \tag{7}$$

To discuss P-stability, first note that a method of $M_4(\alpha, \beta)$ will be P-stable provided $A(H) \pm B(H) > 0$ for all $H \in (0, \infty)$. With $\alpha + \beta = \frac{1}{200}$, from (5) it is easy to see that $A(H) - B(H) > 0$ for all $H \in (0, \infty)$, while

$$A(H) + B(H) = 2 - \frac{1}{3}H^2 + \frac{1}{120}H^4 - \frac{10}{3}\alpha\beta H^6. \tag{8}$$

It can be shown (we omit details) that $A(H) + B(H) > 0$ for all $H \in (0, \infty)$ provided

$$\alpha\beta < -\frac{1}{10800}\left(\frac{13}{18} + \sqrt{\frac{1331}{1620}}\right). \tag{9}$$

Theorem 1. *All those methods of $M_4(\alpha, \beta)$ are at once P-stable and have a phase-lag of order six for which*

$$\alpha + \beta = \frac{1}{200}, \quad \alpha\beta < -\frac{1}{10800}\left(\frac{13}{18} + \sqrt{\frac{1331}{1620}}\right), \tag{10}$$

and for these methods the phase-lag is given by

$$P(H) = \frac{5}{6}\left|\frac{1}{10800} + \alpha\beta\right|H^6. \tag{11}$$

3. To numerically illustrate our new methods we consider the problem

$$y'' + \lambda^2 y = f_0, \quad y(0) = 3, \quad y'(0) = 0; \quad \lambda = 10, \quad f_0 = 2, \tag{12}$$

Table 1
Absolute errors in the computation of $y(t)$ ($h = \frac{1}{48}\pi$)

t	Present method $M_4(\frac{1}{66}, -\frac{67}{6600})$	Cash' method [2, (2.16)]
6π	6.37 (-7)	3.67 (-3)
$\frac{27}{4}\pi$	2.19 (-3)	1.66 (-1)
7π	8.68 (-7)	5.00 (-3)
$\frac{31}{4}\pi$	2.52 (-3)	1.91 (-1)

whose exact solution is $y(t) = 2.98\cos(10t) + 0.02$. We take the method $M_4(\frac{1}{66}, -\frac{67}{6600})$; the parameter values $\alpha = \frac{1}{66}$, $\beta = -\frac{67}{6600}$ have been chosen to satisfy (10) and at the same time to minimize the error constant in (11). For comparison, we first take the fourth order P-stable method of Cash [2, (2.16)]. Both Cash's method and our method require three f -evaluations per Newton-iteration (at each step). The fact that phase-lag for our method is of order six while that for Cash's method is of order four, is clearly reflected by the distinct superiority of our results over the corresponding values for Cash's method in Table 1.

Next for comparison we take the method $M_4(\frac{1}{200})$ of Chawla and Rao [4]. While the present method $M_4(\frac{1}{66}, -\frac{67}{6600})$ is P-stable and has a phase-lag of order six, the method $M_4(\frac{1}{200})$ of [4] has phase-lag of order six also but it possesses only a finite interval of periodicity of size 2.71. Since the method $M_4(\frac{1}{200})$ requires two f -evaluations per step, we adjusted the step-size for this method so that the computational cost per unit length of the integration interval for both the methods is the same. For the method $M_4(\frac{1}{200})$ with step-size $h = \frac{1}{9}\pi$, for example, the errors get blown out since H falls outside its interval of periodicity. It produces absolute errors in $y(t)$ of 9.4(+8), 1.1(+18), 1.2(+27) at endpoints $t = 3\pi, 6\pi, 9\pi$. In comparison, the present method $M_4(\frac{1}{66}, -\frac{67}{6600})$ with the larger step-size $h = \frac{1}{6}\pi$ does integrate (12) though to a modest accuracy, with absolute errors in $y(t)$ of 7.3(-2), 1.5(-1), 2.3(-1) at the same endpoints. However, for smaller step lengths the results reported in Table 2 show that the method $M_4(\frac{1}{200})$ is much more accurate for the problem (12). Therefore, in comparison with the method $M_4(\frac{1}{200})$, the new method is useful in cases where a large step-size is to be used.

In Table 2 global errors for both the methods $M_4(\frac{1}{66}, -\frac{67}{6600})$ and $M_4(\frac{1}{200})$ show a sixth-order behaviour at endpoints which are odd multiples of $\pi/(2\lambda)$, and more interestingly, superconvergence with order 12 at endpoints which are multiples of π/λ . This phenomenon can be explained as follows.

Table 2
Absolute errors in the computation of $y(t)$

t	Present method $M_4(\frac{1}{66}, -\frac{67}{6600})$		Method $M_4(\frac{1}{200})$ of [4]	
	$h = \frac{1}{24}\pi$	$h = \frac{1}{48}\pi$	$h = \frac{1}{36}\pi$	$h = \frac{1}{72}\pi$
π	5.68 (-5)	1.71 (-8)	1.96 (-6)	4.72 (-10)
$\frac{7}{4}\pi$	3.29 (-2)	5.63 (-4)	6.06 (-3)	9.34 (-5)
2π	2.38 (-4)	6.98 (-8)	8.09 (-6)	1.91 (-9)
$\frac{11}{4}\pi$	5.21 (-2)	8.89 (-4)	9.58 (-3)	1.47 (-4)

For the method $M_4(\frac{1}{66}, -\frac{67}{6600})$ applied to the problem (12) and using the exact value for y_1 , it can be shown that the numerical solution at t_n is given by

$$y_n = 2.98 \left[\cos(n\theta(H)) + \frac{\sin(n\theta(H))}{\sin \theta(H)} (\cos H - \cos \theta(H)) \right] + 0.02. \quad (13)$$

Therefore, the error $e_n(t; H) = y(t_n) - y_n$ at $t = nh$ is given by

$$e_n(t; H) = 2.98 \left[(\cos(\lambda t) - \cos(n\theta(H))) - \frac{\sin(n\theta(H))}{\sin \theta(H)} (\cos H - \cos \theta(H)) \right]. \quad (14)$$

Now, for the method $M_4(\frac{1}{66}, -\frac{67}{6600})$ the phase-lag is given by $P(H) = (\frac{37}{813120})H^6$ and we may rewrite (6) as

$$\theta(H) = H - HP(H) + O(h^9). \quad (15)$$

With $\theta(H)$ given by (15) and working out the necessary expansions for $\cos \theta(H)$, $\cos(n\theta(H))$ and $\sin(n\theta(H))$ for $h \rightarrow 0$ and $t = nh$ fixed, from (14) we obtain the following result:

If $\lambda t = m\pi$, $m = 1, 2, \dots$, then

$$e_n(t; H) = (-1)^m 1.49(\lambda t P(H))^2 + O(h^{13}), \quad (16a)$$

while if $\lambda t = (2m + 1)\pi/2$, $m = 0, 1, 2, \dots$, then

$$e_n(t; H) = (-1)^{m+1} 2.98(\lambda t P(H)) + O(h^7). \quad (16b)$$

The same results (16) hold also for the method $M_4(\frac{1}{200})$ applied to the problem (12) with $P(H)$ replaced by the phase-lag $P^*(H) = (\frac{1}{12096})H^6$ of the method $M_4(\frac{1}{200})$. The computations reported in Table 2 are as predicted by (16).

Finally we note that if the initial conditions in (12) are adjusted so that the exact solution is $C_1 \sin(\lambda t) + C_2$, then following arguments similar to those given above we can establish results which now predict only sixth-order behaviour for the global errors for both the methods and which can be confirmed numerically.

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