TRAJECTORY PROPAGATION USING INFORMATION ON PERIODICITY

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Families of methods to integrate first and second order ordinary differential equations whose solution known to be periodic will be discussed. The methods can be tuned to a possibly a-priori knowledge of the user on the location of the frequencies, that are dominant in the exact solution. On the basis of such extra information the truncation error can considerably be reduced in magnitude. The paper compares these methods to well known integrators and discusses a simple mechanism to estimate the frequency during the integration process.

INTRODUCTION

Much effort has gone into the development of numerical solution of first and second order ordinary differential equations. See Bulirsch and Stoer[1], Butcher[2], Daniel and Moore[3], Gear[4,5,6], Gragg[7], Hull[8], Hull et al[9,10,11], Krogh[12,13], Herrick[14], Shanks[15], Lear[16], Fox[17], Montenbrunk[18] and many others. Some of these integrators were applied to orbital mechanics. Several researchers compared the wealth of integrators available. Hull et al[19] used the number of function evaluations, overhead cost and reliability as criteria for comparison. Recently Der[20] presented a comparative study of various trajectory propagators using high order numerical integrators.

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Panovsky et al[21,22] and Richardson et al[23,24] are developing and improving Chebyshev methods for the numerical integration of first and second order ordinary differential equations. The methods were constructed primarily for use in a variety of astrodynamics applications. Richardson et al[24] claim that "because the solution to astrodynamics equations often exhibits a periodic or quasi-periodic character, it was felt that the application of a numerical procedure based on trigonometric (Chebyshev) interpolation rather than polynomial interpolation would be more suitable." However they didn't choose to develop or use methods based on trigonometric polynomials exploiting information on the periodicity of the solution.

In the next section we formulate the problem. Methods for first order ordinary differential equations and second order not containing the first order derivative will be discussed in separate sections. We also included a discussion of symmetric methods for second order initial value problems. The last section will give numerical experiments with several techniques for periodic and quasi periodic problems.

PROBLEM FORMULATION

Let $\vec{a}(t,\vec{r},\vec{v})$ be the total acceleration in the equations of motion described by

$$\frac{d^2\vec{r}}{dt^2} = \vec{a}(t,\vec{r},\vec{v}) \qquad (1)$$

subject to the given initial conditions

$$\vec{r}(t_0) = \vec{r}_0, \qquad \vec{v}(t_0) = \vec{v}_0. \qquad (2)$$

Without loss of generality, we assume that the position, velocity and acceleration vectors are given in Earth centered inertial coordinate system. The total acceleration includes central gravity, oblateness, drag, thrust, solar radiation pressure, and n-body gravity.

The second order differential equations (1)-(2) can be rewritten as a first order system as follows

$$\begin{pmatrix} \frac{d\vec{r}}{dt} \\ \\ \frac{d\vec{v}}{dt} \end{pmatrix} = \begin{pmatrix} \vec{v} \\ \\ \vec{a}(t, \vec{r}, \vec{v}) \end{pmatrix} \qquad (3)$$

subject to the given initial conditions

$$\begin{pmatrix} \vec{r} \\ \\ \vec{v} \end{pmatrix} (t_0) = \begin{pmatrix} \vec{r}_0 \\ \\ \vec{v}_0 \end{pmatrix}. \qquad (4)$$

The direct integration of (1)-(2) or (3)-(4) is called Cowell method. The comparative study by Der[20] includes Runge-Kutta-Fehlberg and Adams-Bashforth Adams-Moulton predictor corrector integrators for the first order initial value problem (3)-(4) and Nyström and Gauss-Jackson-Fox integrators for the second order (1)-(2).

METHODS FOR FIRST ORDER EQUATIONS

Gautschi[25] was the first to develop numerical integrators based on trigonometric polynomials for problems with oscillatory solutions whose frequency is known. The resulting methods depend on a parameter $\nu = h\omega$, where h is the step size and ω is the known frequency. These methods are of (explicit and implicit) Adams type and reduce to the classical ones if $\nu \to 0$.

To be specific, we consider the linear multistep methods

$$\rho(E)y_n - h\sigma(E)\dot{y}_n = 0 \qquad (5)$$

for the integration of

$$\dot{y}(t) = f(t, y(t)), \qquad (6)$$

in cases where the exact solution is known to be approximately of the form

$$y(t) \sim \alpha_0 + \sum_{j=1}^{m} \alpha_j e^{i\omega_j t} \qquad (7)$$

where the frequencies ω_j are such that the solution is periodic or quasi periodic, that is $y(t) \sim y(t + 2\pi/\omega_0)$ for some a priori given frequency ω_0. The operator E is the forward operator, i.e.

$$Ey_n = y_{n+1}. \qquad (8)$$

The first characteristic polynomial $\rho(\zeta)$ depends on the class of methods used. For Adams type it is

$$\rho(\zeta) = \zeta^k - \zeta^{k-1}. \qquad (9)$$

The second characteristic polynomial $\sigma(\zeta)$ is of the same degree k for implicit methods and of degree $k-1$ for explicit ones.

Let

$$\phi(z) = \rho(e^z) - z\sigma(e^z), \qquad (10)$$

then the local truncation error at t_{n+k} is given by (cf. e.g. Lambert[26]) by

$$T_{n+k} = \phi(h\frac{d}{dt})y(t)|_{t=t_n}. \qquad (11)$$

Because of the consistency condition $\phi(0) = 0$ and from (7)

$$|T_{n+k}| \leq \sum_{j=1}^{m} |\alpha_j|\,|\phi(i\nu_j)|, \qquad \nu_j = h\omega_j. \qquad (12)$$

In the case $y(t)$ is a periodic or a quasi periodic function with frequency ω_0, we may replace $y(t)$ by the Fourier series

$$y(t) = \sum_{\ell=0}^{\infty} \hat{\alpha}_\ell e^{i\ell\omega_0 t} \qquad (13)$$

and obtain the inequality

$$|T_{n+k}| \leq \sum_{\ell=0}^{m} |\hat{\alpha}_\ell|\ |\phi(i\ell\nu_0)|. \qquad (14)$$

The inequalities (12) and (14) suggest essentially three approaches (van der Houwen and Sommeijer[27]) for adapting linear multistep method to the additional information available on the exact solution. The first approach is that of Gautschi[25] and Neta and Ford[28]. The resulting method is said to be of trigonometric order q and algebraic order $2q$ and it is obtained by exponentially fitting at the points $i\ell\omega_0, \quad \ell = 1, \ldots, q$, i.e. solving

$$\phi(i\ell\omega_0) = 0, \ell = 1, 2, \ldots, q. \tag{15}$$

Gautschi concluded based on some numerical experiments that one can overestimate the period or underestimate it somewhat and still get better results. This is not encouraging, since one doesn't have exact value for the frequency.

Neta and Ford[28] considered Nyström and generalized Milne-Simpson type methods for first order ordinary differential equations. Here the first characteristic polynomial is

$$\rho(\zeta) = \zeta^k - \zeta^{k-2}. \tag{16}$$

Their methods are restricted to problems whose Jacobian matrix $\frac{\partial f}{\partial y}$ have purely imaginary eigenvalues. In those cases, it was shown numerically that the methods are not sensitive to changes in the frequency.

The second approach assumes there are several dominant frequencies ω_j. One has to start with a linear multistep method containing sufficiently many free parameters in order to achieve

$$\phi(i\omega_j h) = 0, \qquad \text{for all } \omega_j. \tag{17}$$

This approach was taken by Lyche[29], Bettis[30], Stieffel and Bettis[31] and others. One of the disadvantages of such an approach is that a rather detailed knowledge of the dominant solution components is required. In nonlinear problems the frequencies may vary over one integration step which will decrease the accuracy. Therefore Van der Houwen and Sommeijer[27] took a slightly different approach. They have developed a family of linear multistep methods that minimize those terms in the local truncation error which correspond to the oscillatory solution components. They have shown that if one takes the q zeros, $\nu^{(\ell)}$, in the interval $\nu_m \le \nu \le \nu_M$ as

$$\nu^{(\ell)} = \nu_1 + \nu_2 \cos\frac{2\ell - 1}{2q}\pi, \qquad \ell = 1, 2, \ldots, q. \tag{18}$$

where $\nu_1 = \frac{1}{2}(\nu_m + \nu_M)$, and $\nu_2 = \frac{1}{2}(\nu_M - \nu_m)$. Then the free coefficients in the function ϕ (equivalently, the coefficients of the second characteristic polynomial) can be determined by solving the linear system

$$\phi(i\nu^{(\ell)}) = 0, \qquad \ell = 1, 2, \ldots, q. \tag{19}$$

They have developed methods of trigonometric order 3 and algebraic order 6 of Adams-Moulton, Milne-Simpson and backward differentiation types. The latter ones are useful when the problem is stiff, i.e. some components of the solution decay very fast.

METHODS FOR SECOND ORDER EQUATIONS

In this section, we discuss the second order initial value problem

$$\ddot{y}(t) = f(t, y(t)), \qquad y(t_0) = y_0, \quad \dot{y}(t_0) = \dot{y}_0. \tag{20}$$

The linear multistep method is characterized by the same polynomials ρ and σ. The local truncation error is given by (11), but now ϕ is defined as

$$\phi(z) = \rho(e^z) - z^2\sigma(e^z). \tag{21}$$

In the first approach, taken by Gautschi[25], explicit and implicit Störmer Cowell type methods of trigonometric order q were developed in such a way that

$$\phi(ihj\omega_0) = 0, \qquad j = 0, 1, \ldots, q. \qquad (22)$$

The first characteristic polynomial of such methods is

$$\rho(\zeta) = \zeta^k - 2\zeta^{k-1} + \zeta^{k-2}. \qquad (23)$$

Note that these methods are not symmetric for $k \geq 3$ and consequently don't have the optimal algebraic order. This is true even for the slightly more general case

$$\rho(\zeta) = \zeta^k + \alpha_1\zeta^{k-1} + \alpha_2\zeta^{k-2}. \qquad (24)$$

For example, see the method $S_5(\nu)$ listed in the numerical examples section. Sommeijer, van der Houwen and Neta[32] decided to develop methods based on this approach but have an optimal order as well as take the third approach mentioned in the previous section, that is to replace the fitting condition (22) by

$$\phi(0) = 0, \quad \phi(ih\omega^{(j)}) = 0, \quad 1 \leq j \leq q, \qquad (25)$$

where the $\omega^{(j)}$ are appropriately chosen points in the interval $\omega_m \leq \omega \leq \omega_M$. An advantage of this so called minimax approach over the fitting approach is the increased accuracy in cases where no accurate estimate of ω_0 is available or when the frequency is varying in time. In order to facilitate the use of these methods they also implemented a simple mechanism to estimate the frequency during the numerical integration. At every step n, we evaluate $\omega(n)$ using

$$\omega(n) = \frac{f_{n-1} - f_n}{y_{n-1} - y_n} \qquad (26)$$

and take

$$\omega_0 = \frac{1}{3}(\omega(n-2) + \omega(n-1) + \omega(n)). \qquad (27)$$

For the minimax type methods, we use the interval $[.95\omega_0, 1.05\omega_0]$.

SYMMETRIC METHODS FOR SECOND ORDER EQUATIONS

A symmetric method for the solution of second order ordinary differential equations (missing the first derivative of the unknown) are necessarily implicit, have an even step, have the modulus of all the roots of the first characteristic polynomial equal one and have a symmetric second characteristic polynomial (see e.g. Sommeijer et al[32]). Therefore the Störmer type method suggested by Gautschi[25] for $k = 3$ is not of optimal order.

The above implies that the first characteristic polynomial must be of the form

$$\rho(\zeta) = (\zeta - 1)^2 \Pi_{j=1}^{(k-2)/2} (\zeta - e^{i\theta_j})(\zeta - e^{-i\theta_j}), \qquad (28)$$

where $0 < \theta_j < 2\pi$. The θ_j are free parameters, restriced only by zero stability. The second characteristic polynomial has $\frac{1}{2}k+1$ free coefficients, β_j. To achieve order $p = k + 2$, we have to satisfy $k + 4$ conditions two of which are zero stability, i.e. $\rho(1) = \rho'(1) = 0$. Thus the trigonometric order is $\frac{1}{2}k + 1$, exactly as the number of free parameters.

Let's rewrite the function $\phi(z)$ in (21) as

$$\phi(z) = \frac{1}{2}\sum_{j=0}^{k}(a_j - b_j z^2)(e^{(k-j)z} + e^{jz}), \qquad (29)$$

or

$$\phi(z) = e^{\frac{1}{2}kz}\sum_{j=0}^{k}(a_j - b_j z^2)\cosh((\frac{k}{2} - j)z). \qquad (30)$$

The fitting condition (22) assumes the form (since $z = irh\omega_0$)

$$\sum_{j=0}^{k}(a_j + b_j(rh\omega_0)^2)\cos((\frac{k}{2} - j)rh\omega_0) = 0, \quad 1 \leq r \leq q, \qquad (31)$$

Sommeijer, van der Houwen and Neta[32] have shown that for $k = 4$, the symmetric Störmer type method is

$$\rho(\zeta) = (\zeta - 1)^2(\zeta^2 - \alpha\zeta + 1), \quad -2 \le \alpha < 2 \tag{32}$$

and the coefficients (β_j) for the second characteristic polynomial are given in terms of $x = \cos\nu_0$, $\nu_0 = h\omega_0$ by

$$\begin{aligned} \beta_0 &= -\frac{x-1}{36\nu_0^2}\Big\{2x(16x^3 + 38x^2 + 24x + 3) \\ &+ \alpha(5x+4)\}/\{x(x+1)(2x+1) \\ &\quad (4x^2 + 2x - 1)\Big\}, \end{aligned} \tag{33}$$

$$\begin{aligned} \beta_1 &= -\frac{x-1}{9\nu_0^2}\Big\{2x(20x^4 + 60x^3 + 40x^2 - 3) \\ &- \alpha(18x^3 + 14x^2 - 3x - 2)\Big\}/ \\ &\quad \{x(2x+1)(4x^2 + 2x - 1)\}, \end{aligned} \tag{34}$$

$$\begin{aligned} \beta_2 &= \frac{x-1}{18\nu_0^2}\Big\{2x(40x^5 + 12x^4 - 56x^3 \\ &\quad -20x^2 + 6x - 3)\Big\}\Big\{\alpha(108x^4 + \\ &\quad + 170x^3 + 42x^2 - 25x - 4)\Big\}/ \\ &\quad \Big\{x(x+1)(4x^2 + 2x - 1)\Big\}. \end{aligned} \tag{35}$$

For the minimax, the frequencies should be taken as roots of Chebyshev polynomials

$$\nu^{(j)} = \sqrt{\nu_1 + \nu_2 \cos(\frac{2j-1}{2q})\pi}, \quad 1 \le j \le q. \tag{36}$$

where $\nu_1 = \frac{1}{2}(\nu_m^2 + \nu_M^2)$ and $\nu_2 = \frac{1}{2}(\nu_M^2 - \nu_m^2)$.

NUMERICAL EXAMPLES

In this section, we apply several linear multistep methods for the solution of three problems, the first is the orbit equation, the second describes the orbit of an object slowly spiralling outwards, and the third is an almost periodic problem involving Bessel functions. We will show the benefit of using the knowledge of the frequency, even in almost periodic cases.

The methods to be used for first order systems are:

1. Adams Moulton of order 6, AM_6

$$\rho(\zeta) = \zeta^5 - \zeta^4 \tag{37}$$

$$\begin{aligned} \sigma(\zeta) &= \frac{1}{1440}\Big(475\zeta^5 + 1427\zeta^4 \\ &\quad -798\zeta^3 + 482\zeta^2 - 173\zeta + 27\Big) \end{aligned} \tag{38}$$

2. Milne Simpson of order 6, MS_6

$$\rho(\zeta) = \zeta^5 - \zeta^3 \tag{39}$$

$$\begin{aligned} \sigma(\zeta) &= \frac{1}{90}\Big(28\zeta^5 + 129\zeta^4 \\ &\quad +14\zeta^3 + 14\zeta^2 - 6\zeta + 1\Big) \end{aligned} \tag{40}$$

3. Adams Moulton of order 6, $AM_6(\nu)$

   The same $\rho(\zeta)$ as in AM_6 and $\sigma(\zeta)$ is determined by (15) with $q = 3$.

4. Milne Simpson of order 6, $MS_6(\nu)$

   The same $\rho(\zeta)$ as in MS_6 and $\sigma(\zeta)$ is determined by (15) with $q = 3$.

5. Minimax Adams Moulton of order 6, $AM_6(\nu_m, \nu_M)$

$$\rho(\zeta) = \zeta^5 - \zeta^4 \tag{41}$$

   and $\sigma(\zeta)$ is determined by (18),(19) with $q = 3$

6. Minimax Milne Simpson of order 6, $MS_6(\nu_m, \nu_M)$

$$\rho(\zeta) = \zeta^5 - \zeta^3 \tag{42}$$

   and $\sigma(\zeta)$ is determined by (25) with $q = 3$

The methods to be used for second order systems are:

1. Lambert and Watson of order 6, LW_6

$$\begin{aligned}\rho(\zeta) &= \zeta^4 - (2+a)\zeta^3 + (2+2a)\zeta^2 \\ &\quad -(2+a)\zeta + 1 \qquad (43)\end{aligned}$$

$$\begin{aligned}\sigma(\zeta) &= \frac{1}{240}\Big((18+a)\zeta^4 \\ &+ 8(26-3a)\zeta^3 + (28-194a)\zeta^2 \\ &+ 8(26-3a)\zeta + (18+a)\Big) \qquad (44)\end{aligned}$$

The parameter a is taken as zero.

2. Störmer of order 5, $S_5(\nu)$

3. Symmetric (optimized) Störmer of order 6, $SO_6(\nu)$

$\rho(\zeta)$ is given by (32) and

$$\sigma(\zeta) = \sum_{j=0}^{5} \beta_j y''_{n+1-j}$$

with $\beta_0 = \beta_5$ given by (33), $\beta_1 = \beta_4$ given by (34), and $\beta_2 = \beta_3$ given by (35).

4. Symmetric minimax Störmer of order 6, $SO_6(\nu_m, \nu_M)$

The same ρ, but need to solve (25) with $h\omega^{(j)} = \nu^{(j)}$ given by (36).

The first example is a system of first order initial value problems

$$\begin{pmatrix} \frac{dy_1}{dt} \\ \frac{dy_2}{dt} \\ \frac{dy_3}{dt} \\ \frac{dy_4}{dt} \end{pmatrix} = \begin{pmatrix} y_3 \\ y_4 \\ -\frac{y_1}{(y_1^2+y_2^2)^{(3/2)}} \\ -\frac{y_2}{(y_1^2+y_2^2)^{(3/2)}} \end{pmatrix} \qquad (45)$$

subject to the initial condition

$$\begin{pmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \\ y_4(0) \end{pmatrix} = \begin{pmatrix} 1-\epsilon \\ 0 \\ 0 \\ \sqrt{\frac{1+\epsilon}{1-\epsilon}} \end{pmatrix} \qquad (46)$$

where ϵ is the eccentricity of the orbit. We will take $\epsilon = .01$ in our experiments. Clearly $\omega = 1$. The exact solution $\vec{y}_e$ can be written in terms of ϵ and τ as follows

$$\begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ y_4(t) \end{pmatrix} = \begin{pmatrix} \cos\tau - \epsilon \\ -\frac{\sin\tau}{1-\epsilon\cos\tau} \\ \sqrt{1-\epsilon^2}\sin\tau \\ \frac{\sqrt{1-\epsilon^2}\cos\tau}{1-\epsilon\cos\tau} \end{pmatrix} \qquad (47)$$

where τ satisfies

$$\tau - \epsilon\sin\tau = t. \qquad (48)$$

This example is one of those used by Hull et al[19] for their comparative studies and also used by van der Houwen and Sommeijer[27].

The following linear multistep method will be compared: Adams Moulton of order 6 (see e.g. Gear[6], p. 113), (denoted AM_6), Milne Simpson of order 6 (MS_6) and the corresponding methods using the knowledge of the frequency as discussed in the first approach ($AM_6(\nu)$, $MS_6(\nu)$) and the third approach ($AM_6(\nu_m, \nu_M)$, $MS_6(\nu_m, \nu_M)$). We will use the number of significant figures as defined by van der Houwen and Sommeijer[27],

$$sd = -\log_{10}(L_2 \text{ norm of the error at end}) \qquad (49)$$

The results of integration from $t = 0$ to $t = 12\pi$ using a fixed step size $h = \pi/25$ are as follows

AM_6	$AM_6(h)$	$AM_6(.9h, 1.1h)$
4.34	7.68	5.01
AM_6	$AM_6(.9h)$	$AM_6(.8h, 1.h)$
4.34	3.73	4.94

MS_6	$MS_6(h)$	$MS_6(.9h, 1.1h)$
3.09	5.69	3.69
MS_6	$MS_6(.9h)$	$MS_6(.8h, 1.h)$
3.09	3.06	3.62

It is clear that the first approach shows a dramatic gain in the case $\omega = 1$, but no gain when the frequency is underestimated ($\omega = .9$), exactly as in Gautschi[25]. In the third approach, we took an interval around ω and find that there is no difference in the two cases (a moderate gain over the original methods).

The second problem is given by a system of two second order ordinary differential equations and can be written also as a system of four first order. This example is taken from Stiefel and Bettis[31] and also solved in Sommeijer et al[32]. The problem is

$$\frac{d^2z}{dt^2} + z = .001e^{it}, \qquad 0 \le t \le 40\pi \quad (50)$$

subject to the initial condition

$$\begin{pmatrix} z(0) \\ \frac{dz}{dt}(0) \end{pmatrix} = \begin{pmatrix} 1 \\ .9995i \end{pmatrix} \quad (51)$$

where $i = \sqrt{-1}$, and the exact solution is

$$z(t) = \cos t + .0005t \sin t + i(\sin t - .0005t \cos t) \quad (52)$$

One can also solve this problem by using methods for second order problems. We compare Lambert-Watson method (LW_6) with original (S_5) and optimized (SO_6) Gautschi and optimized minimax (all but the original Gautschi's method, which is of Störmer type, are of order six) as found in Sommeijer et al[32]. The results for $h = \pi/12$ at final time of 40π are given in the next table (where in the first row the frequency or interval are given and in the second row the program adjusts the data)

LW_6	$S_5(h)$	$SO_6(h)$	$SO_6(.9h, 1.1h)$
4.5	3.4	6.1	8.0
4.5	3.4	7.3	9.2

One can see that the fifth order Störmer method due to Gautschi couldn't compete with Lambert and Watson (difference in order), but the optimized Gautschi's method obtains more than one digit of accuracy relative to LW_6. On the other hand the minimax methods due to Sommeijer, van der Houwen and Neta[32] yield about twice the number of digits of accuracy.

Neta and Ford[28] have compared AM_6 and MS_6 for the solution of the first order system resulting in this example. The results for $h = \frac{\pi}{60}$ are 5.8 for AM_6 and 8.0 for MS_6. Even though the step size is finer, the results are not better than the minimax method $SO_6(.9h, 1.1h)$.

The third example is a second order almost periodic equation with $\omega \sim 10$

$$\ddot{y} + \left(100 + \frac{1}{4t^2}\right) y = 0, \qquad 1 \le t \le 9 \quad (53)$$

The initial conditions are chosen so that the exact solution is given in terms of Bessel function J_0, (i.e. the coefficient of Y_0 term is zero)

$$y_e = \sqrt{t} J_0(10t) \quad (54)$$

This example was used by Gautschi[25], Neta and Ford[28] and Sommeijer et al[32].

The results when solving the second order system using $h = 1/50$ are given (at the final time $t = 10$) in the next table:

LW_6	$S_5(h)$	$SO_6(h)$	$SO_6(.9h, 1.1h)$
6.0	4.9	8.2	11.0
6.0	4.9	7.9	11.0

Of course, the second order equation can be written as a system of two first order equations. The results of solution of this system using methods for first order equations are listed below:

AM_6	$AM_6(h)$	$AM_6(.9h, 1.1h)$
4.57	6.89	8.60

MS_6	$MS_6(h)$	$MS_6(.9h, 1.1h)$
5.14	6.80	8.73

Notice the quality of minimax methods relative to all others. Using the information on periodicity can yield almost twice the number of digits of accuracy relative to traditional schemes.

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