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Third-order family of methods in Banach spaces

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ABSTRACT

Recently, Parida and Gupta [P.K. Parida, D.K. Gupta, Recurrence relations for a Newton-like method in Banach spaces, J. Comput. Appl. Math. 206 (2007) 873–877] used Rall's recurrence relation approach (from 1961) to approximate roots of nonlinear equations, by developing several methods, the latest of which is free of second derivative and it is of third order. In this paper, we use an idea of Kou and Li [J.-S. Kou, Y.-T. Li, Modified Chebyshev's method free from second derivative for non-linear equations, Appl. Math. Comput. 187 (2007) 1027–1032] and modify the approach of Parida and Gupta, obtaining yet another third-order method to approximate a solution of a nonlinear equation in a Banach space. We give several applications to our method.

1. Introduction

Newton's method and its variants are used to solve nonlinear operator equations $F(x) = 0$ or systems of nonlinear equations. The convergence of these methods was established using Kantorovich theorem (see e.g. [1–3]). The convergence of the sequences obtained by these methods in Banach spaces is derived from the convergence of majorizing sequences (see [4] and references therein). Rall [5] has suggested a different approach for the convergence of these methods, based on recurrence relations. Parida [6], and Parida and Gupta [7] used this idea for several third-order methods (see also the work of Candella and Marquina [8,9], Ezquerro and Hernández [10], and Gutiérrez and Hernández [11,12]).

Here we apply the idea to the third-order method free of second derivative proposed by Kou and Li [13]. They developed a family of methods for the solution of a nonlinear equation $f(x) = 0$ as follows

$$\begin{aligned} y_n &= x_n - \theta \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= x_n - \frac{f(y_n) + (\theta^2 + \theta - 1) f(x_n)}{\theta^2 f'(x_n)}. \end{aligned} \tag{1.1}$$

It turns out that this method is of third order when approximating a simple root.

2. Recurrence relations

In this section, we discuss a third-order method for solving nonlinear operator equations

$$F(x) = 0, \tag{2.1}$$

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where $F : \Omega \subseteq X \to Y$ is a nonlinear operator on an open convex subset Ω of a Banach space X with values in a Banach space Y. The third-order method [13] is defined as follows:

$$
\begin{aligned}
y_n &= x_n - \theta F'(x_n)^{-1} F(x_n), \\
x_{n+1} &= x_n - \frac{\theta^2 + \theta - 1}{\theta^2} F'(x_n)^{-1} F(x_n) - \frac{1}{\theta^2} F'(x_n)^{-1} F(y_n).
\end{aligned}
\tag{2.2}
$$

This family uses two evaluations of F and one evaluation of F'. In [7] they discuss a third-order method requiring one evaluation of F and two evaluations of F'. Several choices of θ were suggested in [14] and [15].

Let F be a twice Fréchet differentiable in Ω and $BL(Y, X)$ be the set of bounded linear operators from Y into X. Let us assume that $\Gamma_0 = F'(x_0)^{-1} \in BL(Y, X)$ exists at some $x_0 \in \Omega$ and the following conditions hold:

(1) $\|F'(x) - F'(y)\| \le k_1 \|x - y\|, \; x, y \in \Omega$,
(2) $\|F''(x)\| \le M, \; x \in \Omega$,
(3) $\|\Gamma_0\| \le \beta$,
(4) $\|\Gamma_0 F(x_0)\| \le \eta$.

Let us also denote

$$
\begin{aligned}
a &= k_1 \beta \eta, \\
\alpha &= \frac{|\theta^2 + \theta - 1| + |1 - \theta|}{\theta^2}, \\
\gamma &= \frac{M}{2} \beta \eta.
\end{aligned}
\tag{2.3}
$$

Now, we define the sequences

$$
\begin{aligned}
& a_0 = b_0 = 1, \qquad d_0 = \alpha + \gamma, \qquad b_{-1} = 0, \\
& a_{n+1} = \frac{a_n}{1 - a a_n d_n}, \\
& b_{n+1} = a_{n+1} \beta \eta C_n, \\
& d_{n+1} = \frac{|\theta^2 + \theta - 1|}{\theta^2} b_{n+1} + \frac{1}{\theta^2} a_{n+1} \beta \eta \left[|1 - \theta| C_n + \frac{M}{2} \theta^2 b_{n+1}^2 \right],
\end{aligned}
\tag{2.4}
$$

where

$$
C_n = \frac{M}{2} K_n^2 + k_1 |\theta| b_n K_n + \frac{M}{2} |\theta^2 - 1| b_n^2, \tag{2.5}
$$

with

$$
K_n = \frac{|\theta + 1|(\theta - 1)^2 + |1 - \theta|}{\theta^2} b_n + \frac{M}{2} a_n \beta b_n^2 \eta. \tag{2.6}
$$

Note that we can rewrite d_{n+1} also in the form

$$
d_{n+1} = \alpha b_{n+1} + \gamma a_{n+1} b_{n+1}^2, \tag{2.7}
$$

or, equivalently, as

$$
d_{n+1} = d_0 b_{n+1} + \gamma b_{n+1} \left(a_{n+1} b_{n+1} - 1 \right).
$$

The polynomials C_n and K_n can be rewritten as

$$
\begin{aligned}
C_n &= \left(P_0 + P_1 a_n b_n + P_2 a_n^2 b_n^2 \right) b_n^2, \\
K_n &= \left(Q_0 + Q_1 a_n b_n \right) b_n.
\end{aligned}
$$

Lemma 1. *Under the previous assumptions, we prove the following:*

(I_n) $\|\Gamma_n\| = \|F'(x_n)^{-1}\| \le a_n \beta$,
(II_n) $\|\Gamma_n F(x_n)\| \le b_n \eta$,
(III_n) $\|x_{n+1} - x_n\| \le d_n \eta$,
(IV_n) $\|x_{n+1} - y_n\| \le (d_n + 2K_{n-1} + \theta b_n) \eta$.

Proof. We use induction to prove the above claims. Notice that (I_0) and (II_0) follow immediately from the assumptions. To prove (III_0), we start with the first substep of (2.2),

$$F(y_0) = F(y_0) - \theta F(x_0) - F'(x_0)(y_0 - x_0). \tag{2.8}$$

This can be written as

$$\begin{aligned} F(y_0) &= (1-\theta)F(x_0) + F(y_0) - F(x_0) - F'(x_0)(y_0 - x_0) \\ &= (1-\theta)F(x_0) + \int_0^1 F''(x_0 + t(y_0 - x_0))(1-t)\mathrm{d}t(y_0 - x_0)^2. \end{aligned} \tag{2.9}$$

Now multiply by Γ_0 and use the assumptions, it follows that

$$\|\Gamma_0 F(y_0)\| \le |1-\theta|\eta + \frac{M}{2}\beta\theta^2\eta^2, \tag{2.10}$$

so that

$$\begin{aligned} \|x_1 - x_0\| &\le \frac{|\theta^2 + \theta - 1|}{\theta^2}\|\Gamma_0 F(x_0)\| + \frac{1}{\theta^2}\|\Gamma_0 F(y_0)\| \\ &\le \left(\frac{|\theta^2+\theta-1| + |1-\theta|}{\theta^2} + \frac{M}{2}\beta\eta\right)\eta = d_0\eta, \end{aligned} \tag{2.11}$$

and (III_0) holds.
We have

$$x_1 - y_0 = -\frac{(\theta+1)(\theta-1)^2}{\theta^2}\Gamma_0 F(x_0) - \frac{1}{\theta^2}\Gamma_0 F(y_0), \tag{2.12}$$

so that it follows from (2.10) that

$$\begin{aligned} \|x_1 - y_0\| &\le \frac{|\theta+1|(\theta-1)^2}{\theta^2}\|\Gamma_0 F(x_0)\| + \frac{1}{\theta^2}\|\Gamma_0 F(y_0)\| \\ &\le \left(\frac{|\theta+1|(\theta-1)^2 + |1-\theta|}{\theta^2} + \frac{M}{2}\beta\eta\right)\eta = d_0\eta \\ &\le (d_0 + 2K_{-1} + \theta b_0)\eta, \end{aligned} \tag{2.13}$$

and (IV_0) also holds.
Following an inductive procedure and assuming $x_n \in \Omega$ and $aa_nd_n < 1$, if $x_{n+1} \in \Omega$, we have

$$\|I - \Gamma_n F'(x_{n+1})\| \le \|\Gamma_n\|\,\|F'(x_n) - F'(x_{n+1})\| \le aa_nd_n < 1. \tag{2.14}$$

Now, we note that

$$\Gamma_{n+1}\left[I - \left(F'(x_n) - F'(x_{n+1})\right)\Gamma_n\right] = \Gamma_n. \tag{2.15}$$

Then Γ_{n+1} is defined and

$$\|\Gamma_{n+1}\| \le \frac{\|\Gamma_n\|}{1 - \|\Gamma_n\|\,\|F'(x_n) - F'(x_{n+1})\|} \le \frac{a_n\beta}{1 - aa_nd_n} = a_{n+1}\beta. \tag{2.16}$$

Hence, by induction, (2.16) holds for all n. This proves condition (I_n).
Using the first step of (2.2), we have

$$\begin{aligned} F(y_n) &= F(y_n) - \theta F(x_n) - F'(x_n)(y_n - x_n) \\ &= (1-\theta)F(x_n) + F(y_n) - F(x_n) - F'(x_n)(y_n - x_n) \\ &= (1-\theta)F(x_n) + \int_0^1 F''(x_n + t(y_n - x_n))(1-t)\mathrm{d}t(y_n - x_n)^2. \end{aligned} \tag{2.17}$$

Now subtract the first step of (2.2) from the second to get

$$F'(x_n)(x_{n+1} - y_n) = \frac{\theta^3 - \theta^2 - \theta + 1}{\theta^2}F(x_n) - \frac{1}{\theta^2}F(y_n). \tag{2.18}$$

Using (2.17) on the identity

$$F(x_{n+1}) = F'(x_n)(x_{n+1} - y_n) + F(y_n) + [F'(y_n) - F'(x_n)](x_{n+1} - y_n) + F(x_{n+1}) - F(y_n) - F'(y_n)(x_{n+1} - y_n) \tag{2.19}$$

and

$$F(x_{n+1}) - F(y_n) - F'(y_n)(x_{n+1} - y_n) = \int_0^1 F''(y_n + t(x_{n+1} - y_n))(1-t)\mathrm{d}t(x_{n+1} - y_n)^2, \tag{2.20}$$

we have

$$\begin{aligned} F(x_{n+1}) &= \frac{1-\theta}{\theta^2}F(x_n) - \frac{1}{\theta^2}F(y_n) + \int_0^1 F''(x_n + t(y_n - x_n))(1-t)\mathrm{d}t(y_n - x_n)^2 \\ &\quad + \int_0^1 F''(y_n + t(x_{n+1} - y_n))(1-t)\mathrm{d}t(x_{n+1} - y_n)^2 + [F'(y_n) - F'(x_n)](x_{n+1} - y_n), \end{aligned} \tag{2.21}$$

or

$$\begin{aligned} F(x_{n+1}) &= \frac{\theta^2 - 1}{\theta^2}\int_0^1 F''(x_n + t(y_n - x_n))(1-t)\mathrm{d}t(y_n - x_n)^2 \\ &\quad + \int_0^1 F''(y_n + t(x_{n+1} - y_n))(1-t)\mathrm{d}t(x_{n+1} - y_n)^2 + [F'(y_n) - F'(x_n)](x_{n+1} - y_n). \end{aligned} \tag{2.22}$$

Hence,

$$\|F(x_{n+1})\| \le \frac{M}{2}\frac{|\theta^2 - 1|}{\theta^2}\|y_n - x_n\|^2 + \frac{M}{2}\|x_{n+1} - y_n\|^2 + k_1\|y_n - x_n\|\,\|x_{n+1} - y_n\|. \tag{2.23}$$

From (2.17) we get

$$\|\Gamma_n F(y_n)\| \le |1-\theta|b_n\eta + \frac{M}{2}a_n\beta\|y_n - x_n\|^2, \tag{2.24}$$

so that since $\|y_n - x_n\| = \|\theta\Gamma_n F(x_n)\| \le |\theta|b_n\eta$, and combining (2.24) with (2.18), we have

$$\begin{aligned} \|x_{n+1} - y_n\| &\le \frac{|\theta^3 - \theta^2 - \theta + 1| + |1-\theta|}{\theta^2}b_n\eta + \frac{M}{2\theta^2}a_n\beta\|y_n - x_n\|^2 \\ &\le \frac{|\theta+1|(\theta-1)^2 + |1-\theta|}{\theta^2}b_n\eta + \frac{M}{2}a_n\beta b_n^2\eta^2 = K_n\eta. \end{aligned} \tag{2.25}$$

Hence,

$$\|F(x_{n+1})\| \le C_n\eta^2, \tag{2.26}$$

where C_n is given by (2.5).
Therefore

$$\begin{aligned} \|\Gamma_{n+1}F(x_{n+1})\| &\le \|\Gamma_{n+1}\|\,\|F(x_{n+1})\| \\ &\le a_{n+1}\beta C_n\eta^2 = b_{n+1}\eta, \end{aligned} \tag{2.27}$$

So, by induction, (2.27) holds for all n. This proves condition (II_n).
Using (2.17) with x_n and y_n replaced by x_{n+1} and y_{n+1} respectively,

$$\|F(y_{n+1})\| \le |1-\theta|C_n\eta^2 + \frac{M}{2}\theta^2 b_{n+1}^2\eta^2, \tag{2.28}$$

so that

$$\|\Gamma_{n+1}F(y_{n+1})\| \le \|\Gamma_{n+1}\|\,\|F(y_{n+1})\| \le a_{n+1}\beta\eta^2\left[|1-\theta|C_n + \frac{M}{2}\theta^2 b_{n+1}^2\right]. \tag{2.29}$$

Using (2.27) and (2.29), we therefore have

$$\begin{aligned} \|x_{n+2} - x_{n+1}\| &\le \frac{|\theta^2 + \theta - 1|}{\theta^2}\|\Gamma_{n+1}F(x_{n+1})\| + \frac{1}{\theta^2}\|\Gamma_{n+1}F(y_{n+1})\| \\ &\le \frac{|\theta^2 + \theta - 1|}{\theta^2}b_{n+1}\eta + \frac{1}{\theta^2}a_{n+1}\beta\eta^2\left[|1-\theta|C_n + \frac{M}{2}\theta^2 b_{n+1}^2\right] \\ &= \left(\frac{|\theta^2 + \theta - 1|}{\theta^2}b_{n+1} + \frac{1}{\theta^2}a_{n+1}\beta\eta\left[|1-\theta|C_n + \frac{M}{2}\theta^2 b_{n+1}^2\right]\right)\eta \\ &= d_{n+1}\eta. \end{aligned} \tag{2.30}$$

Hence, by induction, this inequality holds for all n. This proves condition (III_n).
Since

$$x_{n+1} - y_{n+1} = y_n - x_{n+1} + \frac{(\theta+1)(\theta-1)^2}{\theta^2}\Gamma_n F(x_n) \tag{2.31}$$

$$- \frac{1}{\theta^2}\Gamma_n F(y_n) + \theta \Gamma_{n+1} F(x_{n+1}), \tag{2.32}$$

we have from (2.24), (2.25), (2.27) that

$$\|x_{n+1} - y_{n+1}\| \le (2K_n + \theta b_{n+1})\eta. \tag{2.33}$$

Hence we have

$$\|x_{n+2} - y_{n+1}\| \le \|x_{n+2} - x_{n+1}\| + \|x_{n+1} - y_{n+1}\| \tag{2.34}$$

$$\le d_{n+1}\eta + (2K_n + \theta b_{n+1})\eta \tag{2.35}$$

$$= (d_{n+1} + 2K_n + \theta b_{n+1})\eta. \tag{2.36}$$

Hence, by induction, condition (IV_n) holds for all n. This proves condition (IV_n). □

3. Convergence analysis

In this section, we shall establish the convergence of our third-order method (2.2). To this end, we have to prove the convergence of the sequence x_n defined in a Banach space or, which is the same, to prove that d_n is a Cauchy sequence and that the following assumptions hold:

1. $x_n \in \Omega$,
2. $aa_n d_n < 1,\ n \in N$.

The next two lemmas will show the Cauchy property for the sequence d_n.

Lemma 2. *Assume that x_0 is chosen so as to satisfy $0 < d_0 < 1/a$, that is, $a(\alpha+\gamma) < 1$, where α and γ are given by (2.3). Then, the sequence $a_n \ge 1$ is increasing, as n increases.*

Proof. Now we show that all the involved sequences are positive. Under the imposed conditions, we see that a_0, b_0, d_0, C_0, K_0 are all positive, and also that $1 - aa_0d_0 > 0$. Assume, now, that all a_i, b_i, d_i, C_i, K_i, and $1 - aa_id_i$ are positive, for $i = 0, 1, \ldots, n$.

Since $C_n > 0$ and $b_{n+1} = a_{n+1}\beta\eta C_n$, it follows that a_{n+1}, b_{n+1} have the same sign, and so $a_{n+1}b_{n+1} > 0$. Further, from $d_{n+1} = b_{n+1}(\alpha + \gamma a_{n+1}b_{n+1})$, we get that d_{n+1} has the same sign as b_{n+1}, and so, all three terms $a_{n+1}, b_{n+1}, d_{n+1}$ share the same sign.

By absurd, we suppose that the implied sign is negative. Then $d_n + d_{n+1} < d_n$, and so,

$$1 - aa_n(d_n + d_{n+1}) > 1 - aa_nd_n,$$

which renders

$$1 - aa_{n+1}d_{n+1} = \frac{1 - aa_n(d_n + d_{n+1})}{1 - aa_nd_n} > 1,$$

which implies $aa_{n+1}d_{n+1} < 0$, but that is impossible since a_{n+1}, d_{n+1} have the same sign and $a > 0$.

Next, since $a_{n+1} = \frac{a_n}{1 - aa_nd_n}$, then

$$d_n = \frac{1}{a}\left(\frac{1}{a_n} - \frac{1}{a_{n+1}}\right),$$

and so, by telescoping, we get

$$\sum_{i=0}^{n-1} d_i = \frac{1}{a}\left(\frac{1}{a_0} - \frac{1}{a_n}\right), \quad \text{where } a_0 = 1.$$

This will render

$$a_n = \frac{1}{1 - a\sum_{i=0}^{n-1} d_i}.$$

Certainly, since $a > 0$, $d_i > 0$, for all $i \ge 0$, then $a\sum_{i=0}^{n-1} d_i$ increases as n increases, and so, $1 - a\sum_{i=0}^{n-1} d_i$ decreases as n increases, which implies that the reciprocal, namely, a_n is an increasing sequence, and consequently, $a_n \ge a_0 = 1$. □

We define the sequence $c_n = a_n b_n$. Then the sequences $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$ can be rewritten as

$$a_{n+1} = \frac{a_n}{1 - a a_n d_n} = \frac{a_n}{1 - a\left(\alpha c_n + \gamma c_n^2\right)},$$

$$b_{n+1} = a_{n+1}\beta\eta C_n = \frac{\beta\eta b_n c_n \left(P_0 + P_1 c_n + P_2 c_n^2\right)}{1 - a\left(\alpha c_n + \gamma c_n^2\right)},$$

$$c_{n+1} = a_{n+1} b_{n+1} = \frac{\beta\eta c_n^2 \left(P_0 + P_1 c_n + P_2 c_n^2\right)}{\left[1 - a\left(\alpha c_n + \gamma c_n^2\right)\right]^2},$$

$$\begin{aligned} d_{n+1} &= \alpha b_{n+1} + \gamma a_{n+1} b_{n+1}^2 \\ &= \frac{\beta\eta b_n c_n \left(P_0 + P_1 c_n + P_2 c_n^2\right)}{1 - a\left(\alpha c_n + \gamma c_n^2\right)} \left(\alpha + \gamma c_{n+1}\right). \end{aligned}$$

That the sequence $\{c_n\}$ is a decreasing sequence under the assumption that $a_1 b_1 < 1$ can be proved by using the mathematical induction. It is obvious that $c_1 = a_1 b_1 < 1 = c_0$. Assuming that $c_n < c_{n-1}$ for some $n > 0$, we have

$$\begin{aligned} c_{n+1} &= \frac{\beta\eta c_n^2 \left(P_0 + P_1 c_n + P_2 c_n^2\right)}{\left[1 - a\left(\alpha c_n + \gamma c_n^2\right)\right]^2} \\ &< \frac{\beta\eta c_{n-1}^2 \left(P_0 + P_1 c_{n-1} + P_2 c_{n-1}^2\right)}{\left[1 - a\left(\alpha c_{n-1} + \gamma c_{n-1}^2\right)\right]^2} = c_n. \end{aligned}$$

Therefore the sequence $\{c_n\}$ becomes a decreasing sequence with $c_n < 1$ for all n. If $0 < s < 1$ and $c_n \le s c_{n-1}$, then

$$\begin{aligned} c_{n+1} &= \frac{\beta\eta c_n^2 \left(P_0 + P_1 c_n + P_2 c_n^2\right)}{\left[1 - a\left(\alpha c_n + \gamma c_n^2\right)\right]^2} \\ &\le s^2 \frac{\beta\eta c_{n-1}^2 \left(P_0 + P_1 s c_{n-1} + P_2 s^2 c_{n-1}^2\right)}{\left[1 - a\left(\alpha s c_{n-1} + \gamma s^2 c_{n-1}^2\right)\right]^2} \\ &\le s^2 \frac{\beta\eta c_{n-1}^2 \left(P_0 + P_1 c_{n-1} + P_2 c_{n-1}^2\right)}{\left[1 - a\left(\alpha c_{n-1} + \gamma c_{n-1}^2\right)\right]^2} = s^2 c_n. \end{aligned}$$

Let $\zeta = \frac{c_1}{c_0} = c_1 = a_1 b_1$, then we have $0 < \zeta < 1$ and $c_1 \le \zeta c_0 = \zeta$, so that

$$\begin{aligned} & c_1 \le \zeta c_0, \\ & c_2 \le \zeta^2 c_1, \\ & c_3 \le \zeta^{2^2} c_2, \\ & c_4 \le \zeta^{2^3} c_3, \\ & \vdots \\ & c_{n+1} \le \zeta^{(2^n + 2^{n-1} + \cdots + 2^1 + 1)} c_0 = \zeta^{2^{n+1}} \cdot \frac{1}{\zeta}, \\ & \ddots \end{aligned}$$

On the other hand, with the sequence $\{d_n\}$ under the assumption that $a_1 b_1 < 1$ we have

$$\begin{aligned} d_n &= \frac{a a_n d_n}{a a_n} = \left(\alpha c_n + \gamma c_n^2\right) \frac{1}{a_n} \\ &\le \left(\alpha c_n + \gamma c_n^2\right) \frac{1}{a_0} = \alpha c_n + \gamma c_n^2 \\ &\le (\alpha + \gamma) c_n \\ &\le (\alpha + \gamma) \zeta^{2^n} \cdot \frac{1}{\zeta} \end{aligned}$$

since $\{a_n\}$ is an increasing sequence, and $a_0 \ge 1$.

We have thus proved the following estimates.

Lemma 3. *We assume that $a_1b_1 < 1$. Then the sequence $\{c_n\}$ is a decreasing sequence and for all $n \in \mathbf{N}$ we have the following estimates*

$$c_{n+1} \le \zeta^{2^{n+1}} \cdot \frac{1}{\zeta},$$

$$d_n \le (\alpha+\gamma)\zeta^{2^n} \cdot \frac{1}{\zeta}$$

where $0 < \zeta = a_1b_1 < 1$.

Lemma 4. *The sequence $d_n > 0$ is a convergent sequence and its limit is 0.*

Proof. Since $a_n \ge 1$ is increasing, then $1/a_n \le 1$ is a decreasing sequence and further $0 \le 1/a_n \le 1$. Therefore, a_n is convergent (it is monotonic in a compact set) to a limit L. Since $d_n = \frac{1}{a}\left(\frac{1}{a_n} - \frac{1}{a_{n+1}}\right)$, then d_n is convergent to the limit $\frac{1}{a}(L-L) = 0$. □

Remark 1. A similar approach would work for some of the lemmas in the paper [7], as well. Some of their results, like Lemmas 4–7 can be simplified using a similar approach: for instance, in Lemma 7 of [7], it is claimed that $\sum_{i=0}^{\infty} d_i < \infty$, but that is immediate, since $\sum_{i=0}^{\infty} d_i = \lim_{n\to\infty}\sum_{i=0}^{n-1} d_i = \lim_{n\to\infty}\frac{1}{a}\left(1-\frac{1}{a_n}\right) = \frac{1}{a}(1-L)$, where L would be the (finite) limit of $1/a_n$.

Now, we state the semilocal convergence of the method defined by (2.2).

Theorem 5. *Let X, Y be Banach spaces and F be a twice Fréchet differentiable in an open convex domain Ω of a Banach space X and $BL(Y, X)$ be the set of bounded linear operators from Y into X. Let us assume that $\Gamma_0 = F'(x_0)^{-1} \in BL(Y, X)$ exists at some $x_0 \in \Omega$ and the following conditions hold:*

(1) $\|F'(x) - F'(y)\| \le k_1\|x-y\|,\ x, y \in \Omega$
(2) $\|F''(x)\| \le M,\ x \in \Omega$,
(3) $\|\Gamma_0\| \le \beta$,
(4) $\|\Gamma_0 F(x_0)\| \le \eta$.

Let us denote $a = k_1\beta\eta$. Suppose that x_0 is chosen so as to satisfy $a(\alpha+\gamma) < 1$ and $a_1b_1 < 1$, where α and γ are given by (2.3). Then, if $\overline{B}(x_0, r\eta) \subset \Omega$, where $r = \sum_{n=0}^{\infty} d_n$, then the sequence $\{x_n\}$ defined by (2.2) and starting at x_0 converges to a solution x^* of the equation $F(x) = 0$. In this case, the solution x^* and the iterates x_n belong to $\overline{B}(x_0, r\eta)$, and x^* is the only solution of $F(x) = 0$ in $B(x_0, 2/(k_1\beta) - r\eta) \cap \Omega$.

Furthermore, the error bound on x^* depends on the sequence $\{d_n\}$ given by

$$\|x_{n+1} - x^*\| \le \sum_{k=n+1}^{\infty} d_k\eta \le \frac{(\alpha+\gamma)\eta}{\zeta}\sum_{k=n+1}^{\infty}\zeta^{2^k}, \quad \zeta = a_1b_1. \tag{3.1}$$

Proof. It is easy to see that the sequence $\{x_n\}$ is convergent. Hence, there exists a limit x^* such that $\lim_{n\to\infty} x_n = x^*$. The sequence $\{a_n\}$ is bounded above since

$$a_n = \frac{1}{1 - a\sum_{i=0}^{n-1} d_i} \le \frac{1}{1 - a\sum_{i=0}^{\infty} d_i}.$$

Since $\lim_{n\to\infty} d_n = 0$, by (2.7), we have $\lim_{n\to\infty} b_n = 0$. This indicates that $\lim_{n\to\infty} C_n = 0$. Thus, by (2.26) and by the continuity of F, we proved that

$$\|F(x^*)\| = 0.$$

Also,

$$\begin{aligned}\|x_{n+1} - x_0\| &\le \|x_{n+1} - x_n\| + \|x_n - x_{n-1}\| + \cdots + \|x_1 - x_0\| \\ &\le \sum_{k=0}^{n} d_k\eta \le r\eta,\end{aligned} \tag{3.2}$$

where $r = \sum_{n=0}^{\infty} d_n$. We conclude that x_n lies in $\overline{B}(x_0, r\eta)$ and taking limit as $n \to \infty$ we have $x^* \in \overline{B}(x_0, r\eta)$.

To show the uniqueness of the solution, let $y^* \in B(x_0, 2/(k_1\beta) - r\eta)$ be another solution of $F(x) = 0$. Then

$$0 = F(y^*) - F(x^*) = \int_0^1 F'(x^* + t(y^* - x^*))\mathrm{d}t(y^* - x^*). \tag{3.3}$$

To show that $y^* = x^*$, we have to show that the operator $\int_0^1 F'(x^* + t(y^* - x^*))\mathrm{d}t$ is invertible. Now, for

$$\begin{aligned}\|\Gamma_0\| \int_0^1 \|F'(x^* + t(y^* - x^*)) - F'(x_0)\|\mathrm{d}t &\le k_1\beta \int_0^1 \|x^* + t(y^* - x^*) - x_0\|\mathrm{d}t \\ &\le k_1\beta \int_0^1 ((1-t)\|x^* - x_0\| + t\|y^* - x_0\|)\mathrm{d}t \\ &\le \frac{k_1\beta}{2}\left(r\eta + \frac{2}{k_1\beta} - r\eta\right) = 1, \end{aligned} \tag{3.4}$$

it follows from Banach's Theorem [1] that the operator $\int_0^1 F'(x^* + t(y^* - x^*))\mathrm{d}t$ has an inverse. Therefore, $y^* = x^*$.
For every $m \ge n + 1$, we have

$$\begin{aligned}\|x_m - x_{n+1}\| &\le \|x_m - x_{m-1}\| + \|x_{m-1} - x_{m-2}\| + \cdots + \|x_{n+2} - x_{n+1}\| \\ &\le \sum_{k=n+1}^{m-1} d_k\eta \le r\eta. \end{aligned} \tag{3.5}$$

By taking $m \to \infty$, we get

$$\|x_{n+1} - x^*\| \le \sum_{k=n+1}^{\infty} d_k\eta < r\eta, \tag{3.6}$$

and from Lemma 3

$$\|x_{n+1} - x^*\| \le \sum_{k=n+1}^{\infty} d_k\eta \le \frac{(\alpha+\gamma)\eta}{\zeta} \sum_{k=n+1}^{\infty} \zeta^{2^k}, \quad 0 < \zeta < 1, \tag{3.7}$$

which shows that $\{x_n\}$ converges and proves (3.1). This completes the proof. □

4. Examples

In this section, we give some examples to illustrate the previous convergence result.

Example 4.1 ([7]). Let $X = C[0, 1]$ be the space of continuous functions on $[0, 1]$ and consider the integral equation $F(x) = 0$, where

$$F(x)(s) = -1 + x(s) + \lambda x(s) \int_0^1 \frac{s}{s+t} x(t)\mathrm{d}t, \tag{4.1}$$

where $s \in [0, 1]$, $x \in C[0, 1]$ and $0 < \lambda \le 2$. The norm is taken as sup-norm. We easily find

$$F'(x)u(s) = u(s) + \lambda u(s) \int_0^1 \frac{s}{s+t} x(t)\mathrm{d}t + \lambda x(s) \int_0^1 \frac{s}{s+t} u(t)\mathrm{d}t, \quad u \in \Omega, \tag{4.2}$$

and

$$F''(x)(uv)(s) = \lambda u(s) \int_0^1 \frac{s}{s+t} v(t)\mathrm{d}t + \lambda v(s) \int_0^1 \frac{s}{s+t} u(t)\mathrm{d}t, \quad u, v \in \Omega. \tag{4.3}$$

Since

$$\|[F'(x) - F'(y)](u)\| \le 2\lambda \ln 2\|u\|\,\|x - y\|, \tag{4.4}$$

we get $k_1 = 2\lambda \ln 2$.
Since

$$\|F''(x)\| \le 2\lambda \max_{0\le s\le 1} \left| \int_0^1 \frac{s}{s+t}\mathrm{d}t \right| = 2\lambda \ln 2, \tag{4.5}$$

we get $M = 2\lambda \ln 2$.

We also have from [7] that $\beta = \frac{1}{1-2\lambda \ln 2\|x_0\|}$ and $\eta = \frac{\|x_0 - 1\| + \lambda \ln 2\|x_0\|^2}{1-2\lambda \ln 2\|x_0\|}$.

In our case, we have $a = k_1\beta\eta = \frac{2\lambda \ln 2(\|x_0-1\|+\lambda \ln 2\|x_0\|^2)}{(1-2\lambda \ln 2\|x_0\|)^2}$.

If x_0 is chosen so that $a(\alpha+\gamma) < 1$, where $\alpha = \frac{|\theta^2+\theta-1|+|1-\theta|}{\theta^2}$, $\gamma = \frac{\lambda \ln 2(\|x_0-1\|+\lambda \ln 2\|x_0\|^2)}{(1-2\lambda \ln 2\|x_0\|)^2}$, the sequence $\{x_n\}$ defined by (2.2) and starting at such an x_0, converges to a solution x^* of the equation $F(x) = 0$.

Example 4.2. Consider the solution of the nonlinear equation $F(x) = x^3 + x - 10$ on [1,3]. We let $\theta = -2$. Now the initial condition is $x_0 = 1.7$, and it is easy to show that $F'(x_0) = 3x_0^2 + 1 = 9.67, \alpha = 1, \beta = \|F'(x_0)^{-1}\| = 0.1034126163$, $\eta = \|F'(x_0)^{-1}F(x_0)\| = |(0.1034126163)(-3.387)| = 0.3502585314$ and $\|F''(x)\| = \|6x\| \leq 18 = M$. Now $F'(x) - F'(y) = 3(x-y)(x+y)$ and therefore $k_1 = 18$. Therefore $a = k_1\beta\eta = 0.6519807199$ and $\gamma = \frac{M}{2}\beta\eta = 9(0.1034126163)(0.3502585314) = 0.3259903600$. Then the condition holds: $a(\alpha+\gamma) = 0.8645201495 < 1$. As a result, the solution of this nonlinear equation can be studied by Theorem 5.

Remark 2. One can take a larger interval, i.e. $[1, A]$ for $A > 3$ and still satisfy the condition. Suppose, we let $x_0 = A > 3$, then $F'(A) = 3A^2 + 1, \alpha = 1, \beta = \|F'(x_0)^{-1}\| = \frac{1}{3A^2+1}$, $\eta = \|F'(x_0)^{-1}F(x_0)\| = \frac{A^3+A-10}{3A^2+1}$ and $\|F''(x)\| = \|6x\| \leq 6A = M$. Now $F'(x) - F'(y) = 3(x-y)(x+y)$ and therefore $k_1 = 6A$. Therefore $a = k_1\beta\eta = 6A\frac{A^3+A-10}{(3A^2+1)^2}$ and $\gamma = \frac{M}{2}\beta\eta = 3A\frac{A^3+A-10}{(3A^2+1)^2}$. Then the condition holds: $a(\alpha+\gamma) = 6A\frac{A^3+A-10}{(3A^2+1)^2}\left[1 + 3A\frac{A^3+A-10}{(3A^2+1)^2}\right] \to \frac{8}{9} < 1$ as $A \to \infty$.

Clearly the number of iterations (say, n) required for convergence depends on how close A is to $x^* = 2$. Experimenting with various values of x_0 yields the following results:

$x_0 = 1.7 \quad n = 3$
$x_0 = 3.0 \quad n = 3$
$x_0 = 5.0 \quad n = 4$
$x_0 = 10.0 \quad n = 4.$

Example 4.3. Let us consider the system of three nonlinear equations $F(x, y, z) = 0$ where $F : \Omega \to \mathcal{R}^3$ where $\Omega = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$ is a domain of F containing a solution of this system, and

$$F(x,y,z) = (x^2+y^2+z^2-4,\ x^2+y^2+z^2-2x-2y-2,\ x^2+y^2+z^2-4y-2z-4). \tag{4.6}$$

Then we have

$$F'(x,y,z) = \begin{bmatrix} 2x & 2y & 2z \\ 2x-2 & 2y-2 & 2z \\ 2x & 2y-4 & 2z-2 \end{bmatrix}, \tag{4.7}$$

and

$$F'(x,y,z)^{-1} = \frac{1}{2(-y+2z+x)}\begin{bmatrix} -y+z+1 & y-2z & z \\ z-1+x & -x & -z \\ -x-y+2 & 2x & -x+y \end{bmatrix}. \tag{4.8}$$

We use the Frobenius norm in $\mathcal{R}^3$: $\|X\| = (x^2+y^2+z^2)^{1/2}$ for $X = (x,y,z) \in \mathcal{R}^3$. The corresponding norm on $A \in \mathcal{R}^3 \times \mathcal{R}^3$ is $\|A\| = \left(\sum_{i=1}^{3}\sum_{j=1}^{3}|a_{ij}|^2\right)^{1/2}$.

The second derivative is a bilinear operator on $\mathcal{R}^3$ given by

$$F''(x,y,z) = \begin{bmatrix} f_{1xx} & f_{1xy} & f_{1xz} \\ f_{1yx} & f_{1yy} & f_{1yz} \\ f_{1zx} & f_{1zy} & f_{1zz} \\ \hline f_{2xx} & f_{2xy} & f_{2xz} \\ f_{2yx} & f_{2yy} & f_{2yz} \\ f_{2zx} & f_{2zy} & f_{2zz} \\ \hline f_{3xx} & f_{3xy} & f_{3xz} \\ f_{3yx} & f_{3yy} & f_{3yz} \\ f_{3zx} & f_{3zy} & f_{3zz} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \\ ------ & ------ & ------ \\ 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \\ ------ & ------ & ------ \\ 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

For a bilinear operator $B : \mathcal{R}^3 \times \mathcal{R}^3 \to \mathcal{R}^3$, given two vectors $X, Y \in \mathcal{R}^3$, we have

$$\begin{aligned} B(X, Y) &:= (x_1, x_2, x_3) \begin{bmatrix} B_1 \\ - \\ B_2 \\ - \\ B_3 \end{bmatrix} \begin{bmatrix} y_1 \\ - \\ y_2 \\ - \\ y_3 \end{bmatrix} \\ &= \begin{bmatrix} b_1^{11}x_1 + b_1^{12}x_2 + b_1^{13}x_3 & b_1^{21}x_1 + b_1^{22}x_2 + b_1^{23}x_3 & b_1^{31}x_1 + b_1^{32}x_2 + b_1^{33}x_3 \\ b_2^{11}x_1 + b_2^{12}x_2 + b_2^{13}x_3 & b_2^{21}x_1 + b_2^{22}x_2 + b_2^{23}x_3 & b_2^{31}x_1 + b_2^{32}x_2 + b_2^{33}x_3 \\ b_3^{11}x_1 + b_3^{12}x_2 + b_3^{13}x_3 & b_3^{21}x_1 + b_3^{22}x_2 + b_3^{23}x_3 & b_3^{31}x_1 + b_3^{32}x_2 + b_3^{33}x_3 \end{bmatrix} \begin{bmatrix} y_1 \\ - \\ y_2 \\ - \\ y_3 \end{bmatrix}, \end{aligned} \tag{4.9}$$

where

$$B = \begin{bmatrix} B_1 \\ - \\ B_2 \\ - \\ B_3 \end{bmatrix} = \begin{bmatrix} b_1^{11} & b_1^{12} & b_1^{13} \\ b_1^{21} & b_1^{22} & b_1^{23} \\ b_1^{31} & b_1^{32} & b_1^{33} \\ ------ & ------ & ------ \\ b_2^{11} & b_2^{12} & b_2^{13} \\ b_2^{21} & b_2^{22} & b_2^{23} \\ b_2^{31} & b_2^{32} & b_2^{33} \\ ------ & ------ & ------ \\ b_3^{11} & b_3^{12} & b_3^{13} \\ b_3^{21} & b_3^{22} & b_3^{23} \\ b_3^{31} & b_3^{32} & b_3^{33} \end{bmatrix}. \tag{4.10}$$

We consider the norm of a bilinear operator B on $\mathcal{R}^3$ by

$$\|B\| = \sup_{\|u\|=1} \sqrt{\sum_{i=1}^{3} \sum_{j=1}^{3} \left(\sum_{k=1}^{3} b_i^{jk} u_k \right)^2}, \tag{4.11}$$

where $u = (u_1, u_2, u_3)$.

In our case, for any triple (x, y, z),

$$M = \|F''(x, y, z)\| = 2\sqrt{3} \sup_{\|u\|=1} \{|u_1|^2 + |u_2|^2 + |u_3|^2\}^{1/2} = 2\sqrt{3}. \tag{4.12}$$

Now, since

$$F'(x, y, z) - F'(u, v, w) = 2 \begin{bmatrix} x-u & y-v & z-w \\ x-u & y-v & z-w \\ x-u & y-v & z-w \end{bmatrix}, \tag{4.13}$$

we have

$$\begin{aligned} \|F'(x, y, z) - F'(u, v, w)\| &= 2\sqrt{3} \left[(x-u)^2 + (y-v)^2 + (z-w)^2 \right]^{1/2} \\ &= 2\sqrt{3} \|(x, y, z) - (u, v, w)\|, \end{aligned}$$

and therefore $k_1 = 2\sqrt{3}$.

We let $\theta = -2$, and so, $\alpha = 1$. If we choose $x_0 = -0.2, y_0 = 1.2, z_0 = -2.1$, then $\beta = \|F'(x_0, y_0, z_0)^{-1}\| = 0.675981901098132166$, $\eta = \|F'(x_0, y_0, z_0)^{-1}F(x_0, y_0, z_0)\| = 0.28125$. Therefore $a = k_1\beta\eta = 0.6585946863$ and $\gamma = \frac{M}{2}\beta\eta = 0.3292973430$. Then the left side of the condition holds $a(\alpha + \gamma) = 0.8754681666 < 1$. As a result, the convergence of this system of equations can be studied by Theorem 5.

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