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STUDIES IN A SHALLOW WATER FLUID MODEL WITH TOPOGRAPHY

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1. INTRODUCTION

In this paper we develop a numerical model water for a single layer of fluid with the shallow approximation which flows over variable bottom topography. The motion is confined in a channel with cyclic boundary conditions. The Galerkin finite element method is used for the spatial variation and, the time discretization is accomplished with semi-implicit finite differencing. In our experiments we use bilinear basis functions on rectangles.

We also analyze the linearized version of the model. In this analysis we compare four spatial schemes, bilinear basis functions on rectangles, linear basis functions on isosceles triangles and second and fourth order finite differences. The time will not be discretized in this analysis.

2. NUMERICAL MODEL

The system of equations referred to as the shallow water equations consists of three equations with three forcast variables ϕ, u and v. The equations are

$$\frac{\partial \phi}{\partial t} + \frac{\partial}{\partial x}[u(\phi - \phi_B)] + \frac{\partial}{\partial y}[v(\phi - \phi_B)] = 0, \quad (1.1)$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv + \frac{\partial \phi}{\partial x} = 0, \quad (1.2)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu + \frac{\partial \phi}{\partial y} = 0, \quad (1.3)$$

where ϕ = gh is the geopotential height, (h = height of free surface) ϕ_B is the bottom topography (assumed to be independent of time), u is the east/west component of the wind, and f is the Coriolis parameter. (See e.g., Staniforth and Mitchell [5]). By expanding ϕ into a mean value Φ and a deviation ϕ', the equations (1.1)-(1.3) become

$$\frac{\partial \phi'}{\partial t} + \phi D + \frac{\partial}{\partial x}[u(\phi' - \phi_B)] + \frac{\partial}{\partial y}[v(\phi' - \phi_B)] = 0, \quad (1.4)$$

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$$\frac{\partial u}{\partial t} + \frac{\partial \phi'}{\partial x} + \frac{\partial K}{\partial x} - vQ = 0, \quad (1.5)$$

$$\frac{\partial v}{\partial t} + \frac{\partial \phi'}{\partial y} + \frac{\partial K}{\partial y} + uQ = 0, \quad (1.6)$$

where

$D = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$ is the divergence, $K = \frac{1}{2}(u^2 + v^2)$ is the kinetic energy per unit mass, and Q is the absolute vorticity. The primes will be dropped for the rest of the paper.

Cullen and Hall [1] showed that the accuracy of the Galerkin finite element solution was better for the vorticity-divergence formulation of the shallow-water equations than for an increase in resolution with the primitive formulation (1.1)-(1.3). Williams and Schoenstadt [6] noted that staggered variable formulation of the primitive equations and the unstaggered vorticity-divergence formulation gave the best treatment of geostrophic adjustment for small-scale features.

The vorticity-divergence form of the equations is

$$\frac{\partial \phi}{\partial t} + \Phi D + \frac{\partial}{\partial x}[u(\phi - \phi_B)] + \frac{\partial}{\partial y}[v(\phi - \phi_B)] = 0, \quad (1.7)$$

$$\frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x}(uQ) + \frac{\partial}{\partial y}(vQ) = 0, \quad (1.8)$$

$$\frac{\partial D}{\partial t} + \nabla^2\phi + \nabla^2 K - \frac{\partial}{\partial x}(vQ) + \frac{\partial}{\partial y}(uQ) = 0, \quad (1.9)$$

where $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ is the relative vorticity.

The velocity can be written as the sum of the rotational and irrotational components as

$$\underset{\sim}{V} = \underset{\sim}{V}_{\psi} + \underset{\sim}{V}_{\chi} . \quad (1.10)$$

The equations (1.7) - (1.9) can be rewritten using $D = \nabla^2\chi$ and $\zeta = \nabla^2\psi$ as follows

$$\frac{\partial \phi}{\partial t} + \phi\nabla^2\chi = -\frac{\partial}{\partial x}[u(\phi - \phi_B)] - \frac{\partial}{\partial y}[v(\phi - \phi_B)], \quad (1.11)$$

$$\frac{\partial}{\partial t}\nabla^2\psi = -\frac{\partial}{\partial x}(uQ) - \frac{\partial}{\partial y}(vQ), \quad (1.12)$$

$$\nabla^2 \left(\frac{\partial \chi}{\partial t} + \phi \right) = \frac{\partial}{\partial x} \left(vQ - \frac{\partial K}{\partial x} \right) - \frac{\partial}{\partial y} \left(uQ + \frac{\partial K}{\partial y} \right). \tag{1.13}$$

The domain of integration is a channel with east-west cyclic conditions. The boundary condition at the wall is

$$\underset{\sim}{V} \cdot \underset{\sim}{n} = 0, \tag{1.14}$$

where $\underset{\sim}{n}$ is an outward normal vector. Along the northern and southern walls, the v component is equal to zero, so that the v equation of motion (1.3) reduces to

$$\frac{\partial \phi}{\partial y} = - fu. \tag{1.15}$$

The zonal and meridional components of the wind can be written as

$$u = - \frac{\partial \psi}{\partial y} + \frac{\partial \chi}{\partial x} , \tag{1.16}$$

$$v = \frac{\partial \psi}{\partial x} + \frac{\partial \chi}{\partial y} . \tag{1.17}$$

Then, along the north/south walls where v equals zero, the boundary condition is

$$\frac{\partial \psi}{\partial x} + \frac{\partial \chi}{\partial y} = 0. \tag{1.18}$$

The above condition is imposed by setting

$$\psi = \text{Constant} \tag{1.19}$$

when solving the vorticity equation, and

$$\frac{\partial \chi}{\partial y} = 0 \tag{1.20}$$

when solving the divergence equation. This is an overspecification but (1.18) would be difficult to apply.

The semi-implicit scheme is implemented by evaluating all the terms on the left hand side of the equation as an average at time level $t + \Delta t$ and and $t - \Delta t$ or with a centered time diffrence as appropriate. All the terms on the right hand side are evaluated at time t. The equations become

$$\nabla^2 \bar{\phi} - \frac{\bar{\phi}}{\Phi (\Delta t)^2} = \frac{\partial}{\partial x} \left(vQ - \frac{\partial K}{\partial x} \right) - \frac{\partial}{\partial y} \left(uQ + \frac{\partial K}{\partial y} \right)$$

$$- \frac{\phi(t - \Delta t)}{\Phi (\Delta t)^2} + \frac{\nabla^2 \chi(t - \Delta t)}{\Delta t}$$

$$+ \frac{1}{\Phi \Delta t} \left\{ \frac{\partial}{\partial x} [u(\phi - \phi_B)] + \frac{\partial}{\partial y} [v(\phi - \phi_B)] \right\}, \tag{1.21}$$

$$\nabla^2 \left(\bar{\phi} + \frac{\partial \chi}{\partial t} \right) = \frac{\partial}{\partial x} \left(vQ - \frac{\partial K}{\partial x} \right) - \frac{\partial}{\partial y} \left(uQ + \frac{\partial K}{\partial y} \right), \tag{1.22}$$

$$\nabla^2 \left(\frac{\partial \chi}{\partial t} \right) = - \frac{\partial}{\partial x} (uQ) - \frac{\partial}{\partial y} (vQ), \tag{1.23}$$

where $\bar{\phi} = [\phi(t + \Delta t) - \phi(t - \Delta t)]/2.$

The solution procedure involves solving (1.21) for a new $\bar{\phi}$. The divergence equation (1.22) is then solved for $\bar{\phi} + \frac{\partial \chi}{\partial t}$. Finally, the vorticity equation (1.23) is solved for $\frac{\partial \psi}{\partial t}$. We choose to use ϕ, u and v as history-carrying variables. They are updated after each time step by (see [5]),

$$\phi(t + \Delta t) = 2\bar{\phi} - \phi(t - \Delta t),$$

$$u(t + \Delta t) = 2 \Delta t \left(\frac{\partial}{\partial x} \frac{\partial \chi}{\partial t} - \frac{\partial}{\partial y} \frac{\partial \psi}{\partial t} \right) + u(t - \Delta t), \tag{1.24}$$

$$v(t + \Delta t) = 2 \Delta t \left(\frac{\partial}{\partial y} \frac{\partial \chi}{\partial t} + \frac{\partial}{\partial x} \frac{\partial \psi}{\partial t} \right) + v(t - \Delta t).$$

Implementation of the Galerkin finite element is accomplished as in Hinsman [2].

In the next section we describe some of the numerical experiments performed. We measure the phase speed and compare it to the analytical speed obtained by analytical results as described in Section 4.

3. NUMERICAL SIMULATIONS

In our numerical simulations we used the model described above. The basis functions are bilinear on rectangular elements. We have measured the phase speed and compared it to the value obtained from the formula in the Appendix, i.e.

$$c = \frac{\sigma}{\mu} \tag{3.1}$$

where

$$\sigma = \mu U + \frac{f \mu \frac{\partial \gamma}{\partial y}}{\gamma (\mu^2 + \nu^2 - \rho^2) + f^2 - \rho \frac{\partial \gamma}{\partial y}} \tag{3.2}$$

and

$$\rho = \frac{\frac{\partial \gamma}{\partial y}}{-2\,\gamma}$$

In our first set of experiments we examined small amplitude wave motion in a channel with a constant bottom slope in the y direction. Thus $\frac{\partial \gamma}{\partial x} = 0$. The results are summarized in Table 1.

Mean Flow (m/s)	Slope of Bottom	Exact Phase	approximate phase for ϕ		V		ζ	
0	2000	-39.4	-39.3	.25%	-39.7	.66%	-39.6	.40%
0	1000	-18.4	-18.3	.60%	-18.6	.86%	-18.4	.22%
10	1000	45.3	43.3	4.5%	43.3	4.4%	43.4	4.2%
10	0	66.5	64.7	2.7%	65.2	2.0%	65.1	2.1%

TABLE 1

The percentages indicate the relative error after 48 hours for a 12 x 12 rectangular grid on a channel 5653.51 km long and 4896.08 km wide. The phase speed is given in degrees per 48 hours.

In the next Table we compare the value of phase speed computed by (3.2)-(3.3) and the formulae in Section 4 obtained under further simplifications.

Mean Flow (m/s)	Slope of Bottom	(3.2)-(3.3) (Deg/Day)	(4.16) $V = \frac{\partial \gamma}{\partial x} = 0$	Relative Error
0	2000	-5.1875	-5.1807	.13%
0	1000	-2.4833	-2.4826	.03%
10	1000	9.12646	9.12649	.0003%
10	0	4.0033	4.0124	.23%

TABLE 2

The results here show that the simplifications taken in Section 4 are justified.

4. PHASE SPEED

In this section we derive expressions for the phase speed using various spatial discretizations. Our starting point is the linearized vorticity-divergence formulation

$$\frac{\partial \phi}{\partial t} + U\frac{\partial \phi}{\partial x} + V\frac{\partial \phi}{\partial y} + \gamma D = u\frac{\partial \gamma}{\partial x} + v\frac{\partial \gamma}{\partial y} \tag{4.1}$$

$$\frac{\partial \zeta}{\partial t} + U\frac{\partial \zeta}{\partial x} + V\frac{\partial \zeta}{\partial y} + fD = 0 \tag{4.2}$$

$$\nabla^2 \phi = f\,\zeta \tag{4.3}$$

with the geostrophic relations

$$fu = -\frac{\partial \phi}{\partial y} \tag{4.4}$$

$$fv = \frac{\partial \phi}{\partial x} \tag{4.5}$$

where

$$\gamma = \Phi - \phi_B. \tag{4.6}$$

Note that we simplified the dynamics further by dropping the divergence terms in (4.3) which eliminates the gravity waves. Combining (4.1)-(4.2) and (4.4)-(4.5) we obtain

$$\frac{\partial \zeta}{\partial t} + U\frac{\partial \zeta}{\partial x} + V\frac{\partial \zeta}{\partial y} - \frac{f}{\gamma}\left(\frac{\partial \phi}{\partial t} + U\frac{\partial \phi}{\partial x} + V\frac{\partial \phi}{\partial y}\right) + \frac{1}{\gamma}\left(\frac{\partial \gamma}{\partial x}\frac{\partial \phi}{\partial y} - \frac{\partial \gamma}{\partial y}\frac{\partial \phi}{\partial x}\right) = 0. \tag{4.7}$$

After discretization of (4.7) and (4.3) one has

$$P\dot{\underline{z}} + UP_x\underline{z} + VP_y\underline{z} - \frac{f}{\gamma}(P\dot{\underline{F}} + UP_x\,\underline{F} + V\,P_y\,\underline{F}) + \frac{1}{\gamma}\left(\frac{\partial \gamma}{\partial x}P_y\,\underline{F} - \frac{\partial \gamma}{\partial y}P_x\,\underline{F}\right) = 0. \tag{4.8}$$

$$P_{xx}\,\underline{F} = fP\underline{z}, \tag{4.9}$$

where the matrices P, P_x, P_y, P_{xx} depend on the discretization used.

The analytical phase speed of (4.7), (4.3) may be obtained by substituting

$$\zeta = Z\,e^{i(\mu x + \nu y)} \tag{4.10}$$

$$\phi = F\,e^{i(\mu x + \nu y)} \tag{4.11}$$

into these equations to obtain

$$\dot{Z} + i(\mu U + \nu V)Z - \frac{f}{\gamma}[\dot{F} + i(\mu U + \nu V)F] + \frac{i}{\gamma}\left(\nu\frac{\partial \gamma}{\partial x} - \mu\frac{\partial \gamma}{\partial y}\right) = 0, \tag{4.12}$$

$$-(\mu^2 + \nu^2)\,F = fZ. \tag{4.13}$$

Substituting (4.13) and its derivative into (4.12) one has

$$\dot{Z} + i(\mu U + \nu V)Z - i\,\frac{f(\nu \frac{\partial\gamma}{\partial x} - \mu \frac{\partial\gamma}{\partial y})}{f^2 + \gamma\ (\mu^2 + \nu^2)}\,Z=0 \quad (4.14)$$

Let

$$Z = z_o\, e^{-i\sigma t}, \quad (4.15)$$

then the phase speed c is given by

$$c = \frac{\sigma}{\mu} = (\mu U + \nu V + \frac{\frac{f}{\gamma}(\mu\frac{\partial\gamma}{\partial y} - \nu\frac{\partial\gamma}{\partial x})}{\mu^2 + \nu^2 + \frac{f^2}{\gamma}})/\mu . \quad (4.1)$$

4.1. Isosceles Triangles

The matrices P, P_x and P_y were computed by Neta and Williams [3]. The entries of the matrix P_{xx} can be computed in a similar fashion. It is easily shown that for ζ, ϕ given by (4.10)-(4.11) the following holds

$$\begin{aligned} P\underline{z} &= \alpha\,\frac{\Delta x\,\Delta y}{6}\,z_o,\\ P_x\underline{z} &= i\,\theta\,z_o,\\ P_y\underline{z} &= i\,\beta\,z_o,\\ P_{xx}\underline{F} &= -\,(2\,\frac{\Delta y}{\Delta x}\,\delta + \frac{1}{2}\frac{\Delta x}{\Delta y}\,\varepsilon)\,F_o, \end{aligned} \quad (4.17)$$

where

$$\begin{aligned} \alpha &= 3 + \cos \mu\Delta x + 2 \cos \mu\,\frac{\Delta x}{2}\,\cos \nu\Delta y\\ \beta &= (\cos \mu\,\frac{\Delta x}{2}\,\sin \nu\Delta y)\,\Delta x \end{aligned} \quad (4.18)$$

$$\theta = \frac{2}{3}\Delta y\ (\sin \mu\Delta x + \sin \mu\,\frac{\Delta x}{2}\,\cos \nu\Delta y)$$

$$\delta = 1-\cos \mu\Delta x,$$

$$\varepsilon = 3 + \cos \mu\Delta x - 4 \cos \mu\,\frac{\Delta x}{2}\,\cos \nu\Delta y.$$

Substituting (4.17) into (4.8)-(4.9) and eliminating F_o from both, yields

$$\dot{z}_o + i(\frac{4\theta}{\alpha\Delta x}\,U + \frac{6\,\beta}{\alpha\Delta y}V)\,z_o$$

$$+i\,\frac{\frac{f}{\gamma}(\frac{2}{3}\,\theta\Delta y\,\frac{\partial\gamma}{\partial y} - \beta\Delta x\,\frac{\partial\gamma}{\partial x}\,)}{2\,\frac{\Delta y}{\Delta x}\,\delta + \frac{1}{2}\frac{\Delta x}{\Delta y}\,\varepsilon + \frac{f^2}{\gamma}\,\frac{\alpha\Delta x\,\Delta y}{6}}\,z_o=0. \quad (4.19)$$

The approximate phase speed is then

$$c_T = (\frac{4\,\theta}{\alpha\Delta x}\,U + \frac{6\,\beta}{\alpha\Delta y}\,V + \frac{\frac{f}{\gamma}(\frac{2}{3}\,\theta\Delta y\frac{\partial\gamma}{\partial y} - \beta\Delta x\,\frac{\partial\gamma}{\partial x})}{2\,\frac{\Delta y}{\Delta x}\,\delta + \frac{1}{2}\frac{\Delta x}{\Delta y}\,\varepsilon + \frac{f^2}{\gamma}\,\frac{\alpha\Delta x\,\Delta y}{6}})/\mu . \quad (4.20)$$

4.2. Bilinear Rectangular Elements

Relations (4.17) are now

$$\begin{aligned} &P\underline{z} = \alpha\,\frac{\Delta x\,\Delta y}{9}\,z_o, \quad P_x\underline{z} = i\theta\,z_o,\\ &P_y\underline{z} = i\beta\,z_o, \quad P_{xx}\underline{F} = -\frac{2}{3}(\frac{\Delta y}{\Delta x}\ \delta + \frac{\Delta x}{\Delta y}\,\varepsilon)F_o, \end{aligned} \quad (4.21)$$

where

$$\begin{aligned} \alpha &= (2 + \cos \mu\Delta x)(2 + \cos \nu\Delta y),\\ \beta &= \frac{1}{3}\Delta x\ (\sin\nu\Delta y\ (2 + \cos \mu\Delta x)),\\ \theta &= \frac{1}{3}\Delta y\ (\sin \mu\Delta x\ (2 + \cos \nu\Delta y),\\ \delta &= (2 + \cos\nu\Delta y)(1 - \cos \mu\Delta x),\\ \varepsilon &= (2 + \cos\mu\Delta x)(1 - \cos\nu\Delta y), \end{aligned} \quad (4.22)$$

and the phase speed is

$$c_R = (\frac{3\,\theta}{\alpha\Delta x}\,U + \frac{3\,\beta}{\alpha\Delta y}\,V + \frac{\frac{f}{\gamma}\,(\theta\Delta y\,\frac{\partial\gamma}{\partial y} - \beta\Delta x\,\frac{\partial\gamma}{\partial x})}{2\,(\frac{\Delta y}{\Delta x}\,\delta + \frac{\Delta x}{\Delta y}\,\varepsilon) + \frac{f^2}{\gamma}\,\frac{\alpha\Delta x\,\Delta y}{3}})/\mu . \quad (4.23)$$

4.3. Second Order Finite Differences of the Vorticity-Divergence Formulation

In this case we approximate the fires derivatives by centered differences. Relations (4.17)-(4.18) become

$$P\underline{z} = \alpha\ z_o,\ P_x\underline{z} = i\theta\ z_o,\ P_y\underline{z} = i\beta\ z_o, \quad (4.24)$$

$$P_{xx}\underline{F} = -\ 2(\delta + \varepsilon)\ F_o,$$

$$\alpha = 1,\ \ \beta = \frac{\sin\nu\Delta y}{\Delta y},\ \ \theta = \frac{\sin \mu\Delta x}{\Delta x},$$

$$\delta = \frac{-\cos \mu\Delta x+1}{(\Delta x)^2},\quad \varepsilon = \frac{-\cos\nu\Delta+1}{(\Delta y)^2}, \quad (4.25)$$

$$c_2 = (\beta V + \theta U + \frac{\frac{f}{\gamma}(\theta \frac{\partial \gamma}{\partial y} - \beta \frac{\partial \gamma}{\partial x})}{-2(\delta + \epsilon) + \frac{f^2}{\gamma}})/\mu. \quad (4.26)$$

4.4 Fourth Order Finite Differences

A fourth order approximation for the Laplacian $\Delta^2\phi$ is given by

$$\Delta^2\phi = -\frac{30\phi_{mn} - 16(\phi_{m+1\,n} + \phi_{m-1n}) + \phi_{m+2n} + \phi_{m-2n}}{12(\Delta x)^2}, \quad (4.27)$$

$$-\frac{30\phi_{mn} - 16(\phi_{mn+1} + \phi_{mn-1}) + \phi_{mn+2} + \phi_{mn-2}}{12(\Delta y)^2}.$$

It is then easy to show that (4.17) is exactly (4.24), where

$$\alpha = 1, \quad \beta = \frac{4}{3}\frac{\sin\nu\Delta y}{\Delta y} - \frac{1}{6}\frac{\sin 2\nu\Delta y}{\Delta y},$$

$$\theta = \frac{4}{3}\frac{\sin\mu\Delta x}{\Delta x} - \frac{1}{6}\frac{\sin 2\mu\Delta x}{\Delta x}, \quad (4.28)$$

$$\delta = \frac{\cos 2\mu\Delta x - 16\cos\mu\Delta x + 15}{12(\Delta x)^2}$$

$$\epsilon = \frac{\cos 2\nu\Delta y - 16\cos\nu\Delta y + 15}{12(\Delta y)^2},$$

and the phase speed is given again by (4.26).

In the following figures we plotted the relative phase speed for the channels used in Section 3. The first set of 4 curves corresponds to the first example in Table 1. The second set corresponds to the last example in that Table. Note that in the first example for which the mean flow is zero the phase speed is over estimated by both second and fourth order finite differences. The accuracy is the highest for low x and y wave numbers. The finite elements approximate the phase speed better than the finite differences. The rectangles show slightly better results than the triangles. In the second example, in which there is no topography, one can see the finite elements again perform better than the finite diferences with one exception. There is a small region where the fourth order finite differences perform better than the isosceles triangles, that is large y wave number and large x wave number. The rectangular elements perform better than the isosceles triangles except for some small region for small y wave number and large x wave number.

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APPENDIX

In this appendix we derive (3.2)-(3.3). We start by Linearizing (1.1)-(1.3). Let

$$u = U + u', \quad v = V = v', \quad \phi = \Phi + \phi', \quad (A.1)$$

where U,V are constant mean flow and Φ is independent of time. We assume that U,V related to Φ via the geostrophic relations.

$$U = -\frac{1}{f}\frac{\partial\Phi}{\partial y}, \quad V = \frac{1}{f}\frac{\partial\Phi}{\partial x}. \quad (A.2)$$

Substituting (A.1) into (1.1)-(1.3) and using (A.2) one obtains, after supressing the primes,

$$\frac{\partial\phi}{\partial t} + U\frac{\partial\phi}{\partial x} + V\frac{\partial\phi}{\partial y} + \frac{\partial}{\partial x}[(\Phi - \phi_B)u] + \frac{\partial}{\partial y}[(\Phi - \phi_B)v] = 0, \quad (A.3)$$

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + V\frac{\partial u}{\partial y} - fv + \frac{\partial\phi}{\partial x} = 0. \quad (A.4)$$

$$\frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} + V\frac{\partial v}{\partial y} + fu + \frac{\partial\phi}{\partial y} = 0, \quad (A.5)$$

where we assumed that the flow is along the topography, i.e.,

$$U \frac{\partial \phi_B}{\partial x} + V \frac{\partial \phi_B}{\partial y} = 0. \qquad (A.6)$$

Since in our numerical experiments, we assumed ϕ_B to depend only on y, i.e.,

$$\frac{\partial \phi_B}{\partial x} = 0. \qquad (A.7)$$

Thus from (A.6) and (A.2) we have

$$V = \frac{\partial \Phi}{\partial x} = 0. \qquad (A.8)$$

Following Pedloski [6], we assume that the linearized system (A.3)-(A.5) with the assumptions (A.6)-(A.8), admits a solution of the form

$$u = u_o e^{i(\mu x + \nu y - \sigma t)} e^{\rho y},$$

$$v = v_o e^{i(\mu x + \nu y - \sigma t)} e^{\rho y}, \qquad (A.9)$$

$$\phi = \phi_o e^{i(\mu x + \nu y - \sigma t)} e^{\rho y}.$$

In order for (A.9) to be a solution, one must have

$$\lambda = -\sigma + \mu U \qquad (A.10)$$

satisfying

$$i\lambda\left[(i\lambda)^2 - \gamma(\rho + i\nu)^2 - (\rho + i\nu)\frac{\partial \gamma}{\partial y}\right] - f\left[-fi\lambda - i\mu(\rho + i\nu)\gamma - i\mu\frac{\partial \gamma}{\partial y}\right] + i\mu\gamma\left[-f(\rho + i\nu) - i\mu(i\lambda)\right] = 0. \qquad (A.11)$$

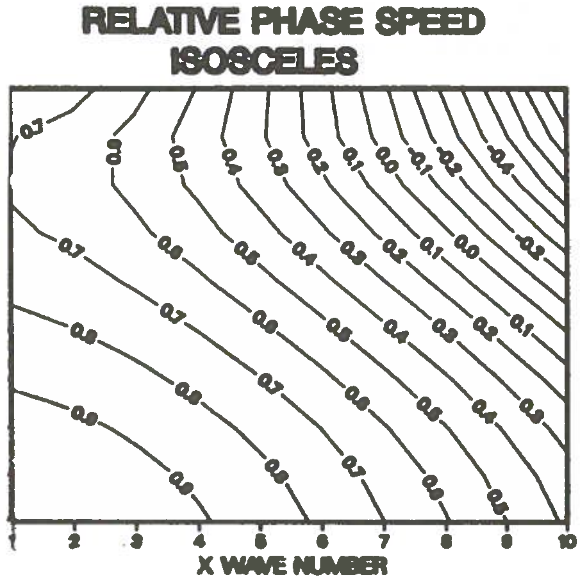
The real part of (A.11) yields (3.3), whereas the imaginary part is a cubic equation for λ.

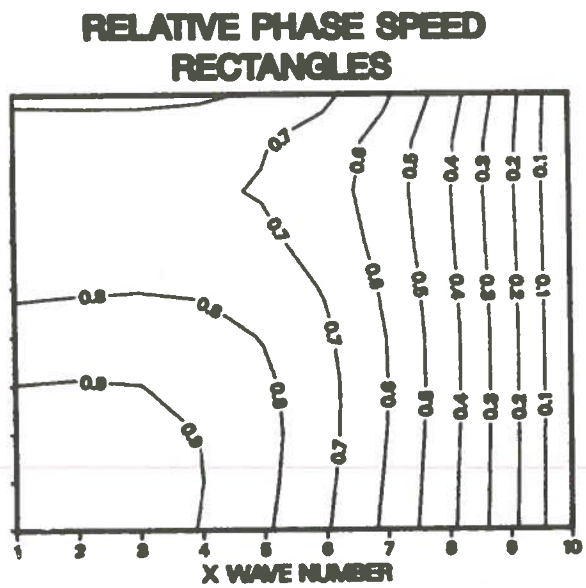
$$\lambda^2 - \lambda\{\gamma(\mu^2 + \nu^2 - \rho^2) + f^2 - \rho\frac{\partial \gamma}{\partial y}\} - \mu f \frac{\partial \gamma}{\partial y} = 0. \qquad (A.12)$$

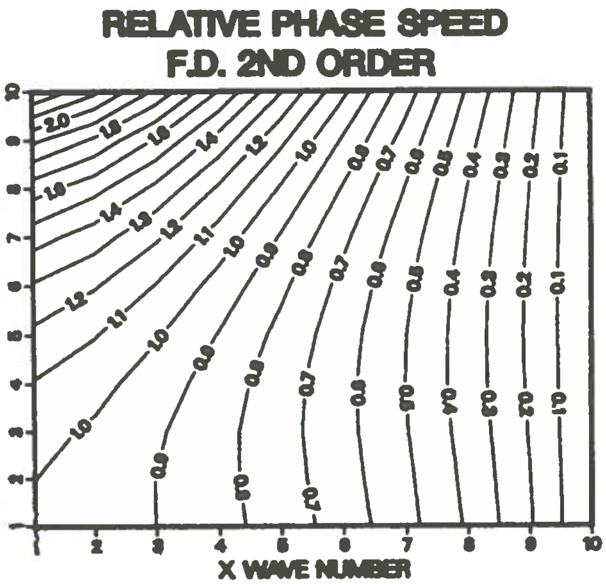
To obtain the phase speed of Rossby waves, we drop λ^2 term and solve

$$-\lambda = \frac{\mu f \frac{\partial \gamma}{\partial y}}{\gamma(\mu^2 + \nu^2 - \rho^2) + f^2 - \rho\frac{\partial \gamma}{\partial y}}. \qquad (A.13)$$

Combining (A.13) and (A.10) gives (3.2)



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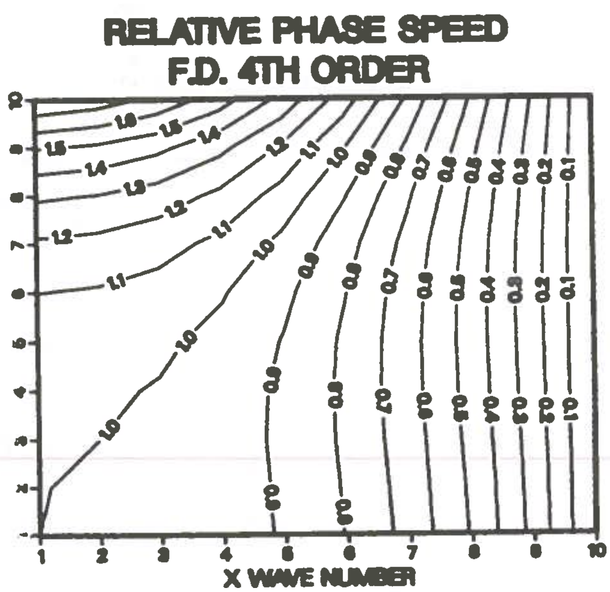
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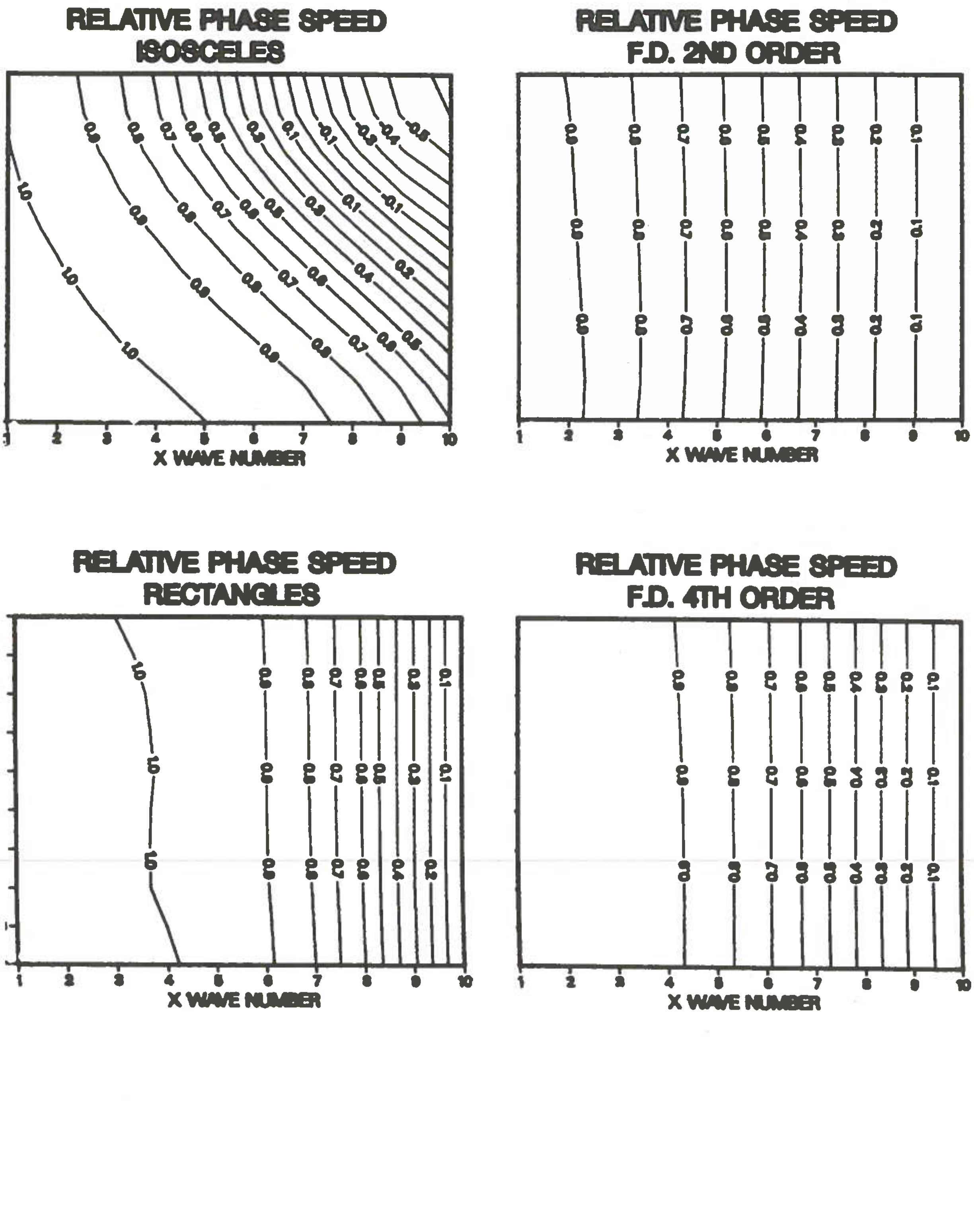
RELATIVE PHASE SPEED
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X WAVE NUMBER
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RELATIVE PHASE SPEED
RECTANGLES
X WAVE NUMBER
RELATIVE PHASE SPEED
F.D. 4TH ORDER
X WAVE NUMBER