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Some modification of Newton's method by the method of undetermined coefficients

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ABSTRACT

In this paper, we construct some modifications of Newton's method for solving nonlinear equations, which is based on the method of undetermined coefficients. It is shown by way of illustration that the method of undetermined coefficients is a promising tool for developing new methods, and reveals its wide applicability by obtaining some existing methods as special cases. Two new sixth-order methods are developed. Numerical examples are given to support that the methods thus obtained can compete with other iterative methods.

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1. Introduction

Solving nonlinear equations is one of the most important problems in numerical analysis. To solve nonlinear equations, iterative methods such as Newton's method are usually used. Throughout this paper we consider iterative methods to find a simple root ξ, i.e., $f(\xi)=0$ and $f'(\xi)\neq 0$, of a nonlinear equation $f(x)=0$ that uses no higher than the second derivative of f.

Newton's method for the calculation of ξ is probably the most widely used iterative method defined by

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}. \tag{1}$$

It is well known (see e.g. Traub [1]) that this method is quadratically convergent.

Several third-order methods based on quadratures are given in the literature. A third-order variant of Newton's method appeared in Weerakoon and Fernando [2] where rectangular and trapezoidal approximations to the integral in Newton's theorem

$$f(x)=f(x_n)+\int_{x_n}^{x} f'(t)\mathrm{d}t \tag{2}$$

were considered to rederive Newton's method and to obtain the cubically convergent method

$$x_{n+1}=x_n-\frac{2f(x_n)}{f'(x_n)+f'(y_n)}, \tag{3}$$

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respectively, where from here on

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{4}$$

Frontini and Sormani [3] considered the midpoint rule for the integral of (2) to obtain the third-order method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'\left(\frac{x_n+y_n}{2}\right)}. \tag{5}$$

It should be mentioned that the method (5) has been derived by Homeier [4] independently

$$x_{n+1} = x_n - \frac{f(x_n)}{f'\left(x_n - \frac{f(x_n)}{2f'(x_n)}\right)}. \tag{6}$$

In [5], Homeier derived the following cubically convergent iteration scheme

$$x_{n+1} = x_n - \frac{f(x_n)}{2}\left(\frac{1}{f'(x_n)} + \frac{1}{f'(y_n)}\right) \tag{7}$$

by applying Newton's theorem to the inverse function $x = f(y)$ instead of $y = f(x)$. It should be pointed out that this method has also been derived in [6] independently and it is now known as the harmonic mean Newton method.

In [7], Kou et al. observed that the midpoint method (5) can be obtained by using the midpoint value $f'(\frac{1}{2}(x_n + y_n))$ instead of the arithmetic mean of $f'(x_n)$ and $f'(y_n)$ in the method of Weerakoon and Fernando (3). That is, they applied the approximation

$$f'\left(\frac{1}{2}(x_n + y_n)\right) \approx \frac{f'(x_n) + f'(y_n)}{2}, \tag{8}$$

or

$$f'(y_n) \approx 2f'\left(\frac{1}{2}(x_n + y_n)\right) - f'(x_n), \tag{9}$$

to Homeier's method (7) to obtain a modification of Newton's method.

Note, if one takes Simpson's rule to approximate the integral in (2), the resulting method is only quadratic. A modified method based on Simpson's rule will be

$$x_{n+1} = x_n - \frac{bf(x_n)}{f'(x_n) + (b-2)f'((x_n + y_n)/2) + f'(y_n)} \tag{10}$$

where b is a free parameter. This method requires more function-evaluations for the same order and thus it is not efficient. Similarly, methods based on Gaussian quadratures are not efficient.

Recently, Neta [8] used the method of undetermined coefficients to obtain a new efficient modification of Popovski's methods [9] by considering an idea of removing the second derivative. In this paper, we further investigate the use of the undetermined coefficients in developing methods. We rederive the existing methods from this point of view and propose new methods. For example, the approximation (8) can be easily obtained by using the method of undetermined coefficients. To see that, we let

$$f'\left(\frac{1}{2}(x_n + y_n)\right) = Af'(x_n) + Bf'(y_n) \tag{11}$$

and expand the second term $f'(y_n)$ about the point x_n. By comparing the coefficients of the derivatives of f at x_n up to second derivatives, we can easily obtain

$$A = B = \frac{1}{2}, \tag{12}$$

this yielding (8).

In [10], Kou and Li considered an iteration scheme consisting of Jarratt's iterate z_n defined by

$$z_n = x_n - J_f(x_n)\frac{f(x_n)}{f'(x_n)} \tag{13}$$

and followed by a Newton iterate, and use of the linear approximation

$$f'(z_n) \approx \frac{z_n - x_n}{v_n - x_n}f'(v_n) + \frac{z_n - v_n}{x_n - v_n}f'(x_n), \tag{14}$$

where $v_n = x_n - \frac{2}{3}f(x_n)/f'(x_n)$, and $J_f(x_n) = \frac{3f'(v_n)+f'(x_n)}{6f'(v_n)-2f'(x_n)}$ to obtain an improvement of Jarratt's method [11]. The method is given by

$$\begin{aligned} v_n &= x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)} \\ z_n &= x_n - J_f(x_n)\frac{f(x_n)}{f'(x_n)} \\ x_{n+1} &= z_n - \frac{f(z_n)}{\frac{3}{2}J_f(x_n)f'(v_n) + (1-\frac{3}{2}J_f(x_n))f'(x_n)}. \end{aligned} \tag{15}$$

The approximation (14) can also be easily obtained by applying the method of undetermined coefficients with

$$f'(z_n) = Af'(v_n) + Bf'(x_n), \tag{16}$$

as in the above. The method (15) is of order six. Another sixth-order improved Jarratt's method is given by the first author in [12].

Many other iterative methods in the literature can also be derived from each other through the method of undetermined coefficients. For example, Nedzhibov's third-order method (see [14] or [13]) defined by[1]

$$x_{n+1} = x_n - \frac{f(x_n)}{\frac{1}{4}\left(f'(y_n) + 2f'(\frac{x_n+y_n}{2}) + f'(x_n)\right)} \tag{17}$$

Hasanov's third-order method (see [15] or [13])[2]

$$x_{n+1} = x_n - \frac{f(x_n)}{\frac{1}{6}\left(f'(y_n) + 4f'(\frac{x_n+y_n}{2}) + f'(x_n)\right)} \tag{18}$$

and the Newton-secant method (see [16] or [13])

$$x_{n+1} = x_n - \frac{f(x_n)}{\frac{f'(x_n)(f(x_n)-f(y_n))}{f(x_n)}} \tag{19}$$

can all be derived from (5). To show this in the case of (17), we apply the method of undetermined coefficients a little differently, that is, we search for the expression d_n satisfying

$$f'(d_n) = \frac{1}{4}\left[f'(y_n) + 2f'\left(\frac{x_n+y_n}{2}\right) + f'(x_n)\right]. \tag{20}$$

Expanding the terms $f'(d_n), f'(y_n)$ and $f'(\frac{x_n+y_n}{2})$ of (20) about the point x_n up to second derivatives, using (4), and then comparing the coefficients of the derivatives of f at x_n, we easily obtain the equation after simplifications

$$d_n - x_n = -\frac{1}{2}\frac{f(x_n)}{f'(x_n)}, \tag{21}$$

or

$$d_n = x_n - \frac{1}{2}\frac{f(x_n)}{f'(x_n)} = \frac{1}{2}(x_n + y_n). \tag{22}$$

We, therefore, derived (5) from Nedzhibov's method, and the other way around. Similarly we can show the equivalence of (5) and Hasanov's method (18).

This can also be done in the case of the Newton-secant method, we seek to find d_n in the equation

$$f'(d_n) = \frac{f'(x_n)(f(x_n) - f(y_n))}{f(x_n)}, \tag{23}$$

or, equivalently

$$f'(d_n)f(x_n) = f'(x_n)[f(x_n) - f(y_n)]. \tag{24}$$

[1] This is a special case of (10) with $b = 4$.
[2] This is a special case of (10) with $b = 6$.

If we expand the terms $f'(d_n)$ and $f(y_n)$ of (24) about the point x_n up to second derivatives, and then compare the coefficients of the derivatives of f at x_n, we can obtain the same equation as in (22)

$$d_n = \frac{1}{2}(x_n + y_n). \tag{25}$$

We thus showed that (5) is equivalent to the Newton-secant method.

We can continue to derive new or existing methods from methods available. If we consider (7) in our application with the form

$$f'(d_n) = \frac{2f'(x_n)f'(y_n)}{f'(x_n) + f'(y_n)}, \tag{26}$$

we can obtain the approximating expression

$$d_n = x_n - \frac{f(x_n)f'(x_n)}{2f'(x_n)^2 - f(x_n)f''(x_n)}. \tag{27}$$

This suggests a new third-order method defined by

$$\begin{aligned} y_n &= x_n - \frac{1}{2}\frac{f(x_n)}{f'(x_n)}\frac{1}{1 - \frac{f(x_n)f''(x_n)}{2f'(x_n)^2}} \\ x_{n+1} &= x_n - \frac{f(x_n)}{f'(y_n)}. \end{aligned} \tag{28}$$

The order was found using Maple software. This method is inefficient since it requires one function- and three derivative-evaluation. The efficiency of this method is the same as the schemes by Hasanov (18) and by Nedzhibov (17), which are special cases of (10).

Traub–Ostrowski's fourth-order method (see [1]) is given by

$$x_{n+1} = x_n - \frac{f(y_n) - f(x_n)}{2f(y_n) - f(x_n)}\frac{f(x_n)}{f'(x_n)}. \tag{29}$$

If we look for d_n through the equation

$$f'(d_n) = \frac{f'(x_n)[2f(y_n) - f(x_n)]}{f(y_n) - f(x_n)} \tag{30}$$

or, equivalently

$$f'(d_n)[f(y_n) - f(x_n)] = f'(x_n)[2f(y_n) - f(x_n)] \tag{31}$$

then after expanding the terms $f'(d_n)$ and $f(y_n)$ of (31) about the point x_n up to second derivatives, and then comparing the coefficients of the derivatives of f at x_n, we can obtain exactly the same expression as in (27), thereby again obtaining the same method as (28). This shows that Homeier's method (7) and Traub–Ostrowski's method are closely connected through the method of undetermined coefficients.

Chebyshev–Halley methods [17] are a family of third-order methods defined by

$$x_{n+1} = x_n - \left(1 + \frac{1}{2}\frac{L_f(x_n)}{1 - \alpha L_f(x_n)}\right)\frac{f(x_n)}{f'(x_n)}, \tag{32}$$

where $L_f(x_n) = \frac{f(x_n)f''(x_n)}{f'(x_n)^2}$. This family includes Chebyshev's method ($\alpha = 0$), Halley's method ($\alpha = 1/2$) and the super-Halley method ($\alpha = 1$). With this family, let us consider seeking the approximating expression d_n such that

$$f'(d_n) = f'(x_n)\frac{2[1 - \alpha L_f(x_n)]}{2 + (1 - 2\alpha)L_f(x_n)} \tag{33}$$

or

$$f'(d_n)[2f'(x_n)^2 + (1 - 2\alpha)f(x_n)f''(x_n)] = 2f'(x_n)[f'(x_n)^2 - \alpha f(x_n)f''(x_n)]. \tag{34}$$

If we expand the term $f'(d_n)$ of (34) about the point x_n up to second derivative, and then compare the coefficients of the derivatives of f at x_n, we can easily obtain

$$d_n = x_n - \frac{f(x_n)f'(x_n)}{2f'(x_n)^2 + (1 - 2\alpha)f(x_n)f''(x_n)}. \tag{35}$$

When $\alpha = 1/2$, (35) reduces to (22), thus showing that (5) is also equivalent to the Halley method. When $\alpha = 1$, (35) reduces to (27), thus implying that Homeier's method (7) is equivalent to the super-Halley method. When $\alpha = 0$, (35) reduces to

$$d_n = x_n - \frac{f(x_n)f'(x_n)}{2f'(x_n)^2 + f(x_n)f''(x_n)}. \tag{36}$$

It should be remarked that different values of α would result in different new third-order methods.

Before proceeding on, it should be emphasized that many approximations that were used in deriving existing iterative methods can be considered as results of appropriately applying the method of undetermined coefficients as illustrated above. In this contribution, it is noteworthy that almost all of known third-order methods in the literature are equivalent to each other in the context of the method of undetermined coefficients. This also reveals the potential of the method of undetermined coefficients as a powerful means of developing iterative methods for solving nonlinear equations.

2. Development of methods and convergence analysis

For the sake of simplicity and illustration, let us consider the iteration scheme of the form

$$x_{n+1} = u_{n+1} - \frac{f(u_{n+1})}{f'(u_{n+1})}, \tag{37}$$

where $u_{n+1} = g_3(x_n)$ stands for any third-order modification of Newton's method. We would like to mention that in [18], Kou et al. considered a variant of (37)

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ x_{n+1} &= u_{n+1} - \frac{f(u_{n+1})}{f'(y_n)}, \end{aligned} \tag{38}$$

with u_{n+1} given by one of (3), (5) or (7) to obtain three different fifth-order methods. Using the approximation (9) one can obtain other less efficient fifth-order methods.

To derive the new method, we consider the expression

$$f'(u_{n+1}) = Af'(x_n) + Bf'(y_n) + Cf(u_{n+1}) + Df(x_n), \tag{39}$$

for application of the method of undetermined coefficients.

Expand the terms $f'(u_{n+1}), f'(y_n)$ and $f(u_{n+1})$ about the point x_n up to third derivatives and collect terms. Upon comparing the coefficients of the derivatives of f at x_n, we have the following system of equations for the unknowns $A, \ldots, D$

$$C + D = 0 \tag{40}$$

$$A + B + \alpha C = 1 \tag{41}$$

$$\beta B + \frac{1}{2}\alpha^2 C = \alpha \tag{42}$$

$$\frac{1}{2}\beta^2 B + \frac{1}{6}\alpha^3 C = \frac{1}{2}\alpha^2 \tag{43}$$

where $\alpha = u_{n+1} - x_n$, and $\beta = y_n - x_n$. Solving the equations (42) and (43), we get

$$B = \frac{\alpha^2}{\beta(3\beta - 2\alpha)} \tag{44}$$

$$C = \frac{6(\beta - \alpha)}{\alpha(3\beta - 2\alpha)}. \tag{45}$$

Substituting in Eqs. (40) and (41), we get

$$A = \frac{-\alpha^2 + 4\alpha\beta - 3\beta^2}{\beta(3\beta - 2\alpha)} \tag{46}$$

$$D = -\frac{6(\beta - \alpha)}{\alpha(3\beta - 2\alpha)}. \tag{47}$$

The method is now

$$x_{n+1} = u_{n+1} - \frac{\alpha\beta(3\beta - 2\alpha)f(u_{n+1})}{\gamma f'(x_n) + \alpha^3 f'(y_n) + 6\beta(\beta - \alpha)\left(f(u_{n+1}) - f(x_n)\right)}, \tag{48}$$

where $\gamma = \alpha(-\alpha^2 + 4\alpha\beta - 3\beta^2)$ and u_{n+1} is computed by a third-order method such as (3), or (7) and y_n is given by (4). If we decide to use (5) then it is more efficient to expand $f'(u_{n+1})$ using

$$f'(u_{n+1}) = Af'(x_n) + Bf'\left(\frac{x_n + y_n}{2}\right) + Cf(u_{n+1}) + Df(x_n). \tag{49}$$

Now (40)–(43) are replaced by

$$C + D = 0 \tag{50}$$

$$A + B + \alpha C = 1 \tag{51}$$

$$\frac{\beta}{2}B + \frac{1}{2}\alpha^2 C = \alpha \tag{52}$$

$$\frac{1}{8}\beta^2 B + \frac{1}{6}\alpha^3 C = \frac{1}{2}\alpha^2 \tag{53}$$

for the same α and β above. The solution is

$$A = \frac{-4\alpha^2 + 8\alpha\beta - 3\beta^2}{\beta(3\beta - 4\alpha)} \tag{54}$$

$$B = \frac{4\alpha^2}{\beta(3\beta - 4\alpha)} \tag{55}$$

$$C = \frac{6(\beta - 2\alpha)}{\alpha(3\beta - 4\alpha)} \tag{56}$$

$$D = -\frac{6(\beta - 2\alpha)}{\alpha(3\beta - 4\alpha)}. \tag{57}$$

The method is now

$$x_{n+1} = u_{n+1} - \frac{\alpha\beta(3\beta - 4\alpha)f(u_{n+1})}{\delta f'(x_n) + 4\alpha^3 f'\left(\frac{x_n+y_n}{2}\right) + 6\beta(\beta - 2\alpha)\left(f(u_{n+1}) - f(x_n)\right)}, \tag{58}$$

where $\delta = \alpha(-4\alpha^2 + 8\alpha\beta - 3\beta^2)$ and u_{n+1} is computed by the third-order method (5) and y_n is given by (4). Note that Neta [19] has developed a sixth-order method requiring three function- and one derivative-evaluation per step. The new methods we developed here require two function- and two derivative-evaluation per step. The efficiency of the three methods is the same, unless the cost of function-evaluation is different from the cost of derivative-evaluation. It should also be pointed out that the method (38) has been improved in the order from five to six by the approach of the method of undetermined coefficients.

For the method defined by (48), we have the following analysis of convergence. A similar analysis can be done for (58).

Theorem 2.1. *Let $\alpha \in I$ be a simple zero of a sufficiently differentiable function $f : I \to \mathbf{R}$ for an open interval I. Let $u_{n+1} = g_3(x_n)$ be any third-order method and assume that it satisfies*

$$u_{n+1} - \xi = Ae_n^3 + O(e_n^4), \tag{59}$$

for some $A \neq 0$, and $e_n = x_n - \xi$. Then the new method defined by (48) is of sixth order.

Proof. Let $c_k = (1/k!)f^{(k)}(\alpha)/f'(\alpha)$, $k = 2, 3, \ldots$. We assume that

$$u_{n+1} - \xi = Ae_n^3 + Be_n^4 + O(e_n^5). \tag{60}$$

Using the Taylor expansion and taking into account $f(\xi) = 0$ and by simple calculations, we easily obtain

$$f(u_{n+1}) = f'(\xi)[(u_{n+1} - \xi) + c_2(u_{n+1} - \xi)^2 + O(e_n^9)], \tag{61}$$

$$f(x_n) = f'(\xi)[e_n + c_2e_n^2 + c_3e_n^3 + O(e_n^4)], \tag{62}$$

$$f'(x_n) = f'(\xi)[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + O(e_n^4)], \tag{63}$$

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} = \xi + c_2e_n^2 - 2(c_2^2 - c_3)e_n^3 + O(e_n^4), \tag{64}$$

$$f'(y_n) = f'(\xi)[1 + 2c_2^2e_n^2 - 4c_2(c_2^2 - c_3)e_n^3 + O(e_n^4)], \tag{65}$$

$$\alpha = u_{n+1} - x_n = -e_n + Ae_n^3 + Be_n^4 + O(e_n^5) \tag{66}$$

$$\beta = y_n - x_n = -e_n + c_2e_n^2 + 2(c_3 - c_2^2)e_n^3 + (3c_4 + 4c_2^3 - 7c_2c_3)e_n^4 + O(e_n^5). \tag{67}$$

Hence we have

$$3\beta - 2\alpha = -e_n + 3c_2e_n^2 + 2(3c_3 - 3c_2^2 - A)e_n^3 + (9c_4 + 12c_2^3 - 21c_2c_3 - 2B)e_n^4 + O(e_n^5) \quad (68)$$

$$\alpha\beta = e_n^2 - c_2e_n^3 + (-2c_3 + 2c_2^2 - A)e_n^4 + (-3c_4 - 4c_2^3 + 7c_2c_3 + Ac_2 - B)e_n^5 + O(e_n^6) \quad (69)$$

$$\alpha\beta(3\beta - 2\alpha) = -e_n^3 + 4c_2e_n^4 + (8c_3 - 11c_2^2 - A)e_n^5 + (12c_4 + 28c_2^3 - 40c_2c_3 - 2Ac_2 - B)e_n^6 + O(e_n^7). \quad (70)$$

We then obtain

$$\alpha\beta(3\beta - 2\alpha)f(u_{n+1}) = f'(\xi)[-(u_{n+1} - \xi)e_n^3 + 4c_2(u_{n+1} - \xi)e_n^4 + (8c_3 - 11c_2^2 - A)(u_{n+1} - \xi)e_n^5 - c_2(u_{n+1} - \xi)^2e_n^3 + (12c_4 + 28c_2^3 - 40c_2c_3 - 2Ac_2 - B)(u_{n+1} - \xi)e_n^6 + O(e_n^{10})]. \quad (71)$$

We also get

$$\beta - \alpha = c_2e_n^2 + (2c_3 - 2c_2^2 - A)e_n^3 + (3c_4 + 4c_2^3 - 7c_2c_3 - B)e_n^4 + O(e_n^5) \quad (72)$$

$$6\beta(\beta - \alpha) = -6c_2e_n^3 + (6A - 12c_3 + 18c_2^2)e_n^4 + (-18c_4 - 48c_2^3 + 66c_2c_3 + 6B - 6Ac_2)e_n^5 + O(e_n^6), \quad (73)$$

so that it follows from (61) and (62) that

$$6\beta(\beta - \alpha)f(u_{n+1}) = f'(\xi)[-6c_2(u_{n+1} - \xi)e_n^3 + (6A - 12c_3 + 18c_2^2)(u_{n+1} - \xi)e_n^4 + O(e_n^8)] \quad (74)$$

$$6\beta(\beta - \alpha)f(x_n) = f'(\xi)[-6c_2e_n^4 + (6A - 12c_3 + 12c_2^2)e_n^5 + (48c_2c_3 - 30c_2^3 - 18c_4 + 6B)e_n^6 + O(e_n^7)]. \quad (75)$$

On the other hand, we can obtain

$$\alpha^2 = e_n^2 - 2Ae_n^4 - 2Be_n^5 + O(e_n^6) \quad (76)$$

$$\alpha^3 = -e_n^3 + 3Ae_n^5 + 3Be_n^6 + O(e_n^7) \quad (77)$$

$$\beta^2 = e_n^2 - 2c_2e_n^3 + (5c_2^2 - 4c_3)e_n^4 + (-6c_4 - 12c_2^3 + 18c_2c_3)e_n^5 + O(e_n^6) \quad (78)$$

$$-\alpha^2 + 4\alpha\beta - 3\beta^2 = 2c_2e_n^3 + (4c_3 - 7c_2^2 - 2A)e_n^4 + (-2B + 6c_4 + 20c_2^3 - 26c_2c_3 + 4Ac_2)e_n^5 + O(e_n^6) \quad (79)$$

$$\alpha(-\alpha^2 + 4\alpha\beta - 3\beta^2) = -2c_2e_n^4 + (-4c_3 + 7c_2^2 + 2A)e_n^5 + (2B - 6c_4 - 20c_2^3 + 26c_2c_3 - 2Ac_2)e_n^6 + O(e_n^7), \quad (80)$$

so that we get from (63) and (65) that

$$\alpha(-\alpha^2 + 4\alpha\beta - 3\beta^2)f'(x_n) = f'(\xi)[-2c_2e_n^4 + (3c_2^2 - 4c_3 + 2A)e_n^5 + (2B - 6c_4 - 6c_2^3 + 12c_2c_3 + 2Ac_2)e_n^6 + O(e_n^7)] \quad (81)$$

$$\alpha^3f'(y_n) = f'(\xi)[-e_n^3 + (3A - 2c_2^2)e_n^5 + (4c_2^3 - 4c_2c_3 + 3B)e_n^6 + O(e_n^7)]. \quad (82)$$

Thus, from (74), (75), (81) and (82), we have

$$\alpha(-\alpha^2 + 4\alpha\beta - 3\beta^2)f'(x_n) + \alpha^3f'(y_n) + 6\beta(\beta - \alpha)(f(u_{n+1}) - f(x_n)) = f'(\xi)[-e_n^3 + 4c_2e_n^4 + (-11c_2^2 + 8c_3 - A)e_n^5 + (-B + 12c_4 + 28c_2^3 - 40c_2c_3 - 4Ac_2)e_n^6 + O(e_n^7)]. \quad (83)$$

Dividing (71) by (83), we get

$$\frac{\alpha\beta(3\beta - 2\alpha)f(u_{n+1})}{\alpha(-\alpha^2 + 4\alpha\beta - 3\beta^2)f'(x_n) + \alpha^3f'(y_n) + 6\beta(\beta - \alpha)(f(u_{n+1}) - f(x_n))} = (u_{n+1} - \xi) - 2Ac_2(u_{n+1} - \xi)e_n^3 + c_2(u_{n+1} - \xi)^2 + O(e_n^7). \quad (84)$$

Thus,

$$\begin{aligned} e_{n+1} &= u_{n+1} - \xi - [(u_{n+1} - \xi) - 2Ac_2(u_{n+1} - \xi)e_n^3 + c_2(u_{n+1} - \xi)^2 + O(e_n^7)] \\ &= 2Ac_2(u_{n+1} - \xi)e_n^3 - c_2(u_{n+1} - \xi)^2 + O(e_n^7) \\ &= A^2c_2e_n^6 + O(e_n^7). \end{aligned} \quad (85)$$

This means that the method defined by (48) is of sixth order. This completes the proof. □

3. Numerical examples

We present some numerical test results for various sixth-order convergent iterative schemes in Table 1. The following methods were compared: the Newton method (NM), the method of Neta [19] (BM) defined by

$$z_n = y_n - \frac{f(y_n)}{f'(x_n)} \frac{f(x_n) - \frac{1}{2}f(y_n)}{f(x_n) - \frac{5}{2}f(y_n)}, \tag{86}$$

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(x_n)} \frac{f(x_n) - f(y_n)}{f(x_n) - 3f(y_n)}, \tag{87}$$

the method of Kou [20] (KM) defined by

$$z_n = x_n - \frac{2f(x_n)}{f'(x_n) + f'(y_n)}, \tag{88}$$

$$x_{n+1} = z_n - \frac{f'(y_n) + f'(x_n)}{3f'(y_n) - f'(x_n)} \frac{f(z_n)}{f'(x_n)}, \tag{89}$$

the method of Grau et al. [21] (GM) defined by

$$z_n = y_n - \frac{f(x_n)}{f(x_n) - 2f(y_n)} \frac{f(y_n)}{f'(x_n)}, \tag{90}$$

$$x_{n+1} = z_n - \frac{f(x_n)}{f(x_n) - 2f(y_n)} \frac{f(z_n)}{f'(x_n)}, \tag{91}$$

and our new methods (48) with (3) (OM1) and (58) (OM2) as well as the sixth-order methods given in [12].

All computations were done using MAPLE using 128 digit floating point arithmetics (Digits := 128). We accept an approximate solution rather than the exact root, depending on the precision (ϵ) of the computer. We use the following stopping criteria for computer programs: (i) $|x_{n+1} - x_n| < \epsilon$, (ii) $|f(x_{n+1})| < \epsilon$, and so, when the stopping criterion is satisfied, x_{n+1} is taken as the exact root α computed. For numerical illustrations in this section we used the fixed stopping criterion $\epsilon = 10^{-25}$. We used the test functions as Weerakoon and Fernando [2] and the test functions in Neta [22]

Test	Function	x_0	x_*
1	$x^3 + 4x^2 - 10$	1.6	1.3652300134140968457608068290
2	$\sin^2(x) - x^2 + 1$	1.0	1.4044916482153412260350868178
3	$x^2 - e^x - 3x + 2$	2.0	0.25753028543986076045536730494
4	$\cos(x) - x$	1.5	0.73908513321516064165531208767
5	$(x-1)^3 - 1$	3.5	2.0
6	$x^3 - 10$	4.0	2.1544346900318837217592935665
7	$xe^{x^2} - \sin^2(x) + 3\cos(x) + 5$	−1.0	−1.2076478271309189270094167584
8	$e^{x^2+7x-30} - 1$	4.0	3.0
9	$\sin(x) - \frac{x}{2}$	2.0	1.8954942670339809471440357381
10	$x^5 + x - 10000$	4.0	6.3087771299726890947675717718
11	$\sqrt{(x)} - \frac{1}{x} - 3$	1.0	9.6335955628326951924063127092
12	$e^x + x - 20$	0.0	2.8424389537844470678165859402
13	$\ln(x) + \sqrt{(x)} - 5$	1.0	8.3094326942315717953469556827
14	$x^3 - x^2 - 1$	0.5	1.4655712318767680266567312252

As convergence criterion, it was required that the distance of two consecutive approximations δ for the zero was less than 10^{-25}. Also displayed are the number of iterations to approximate the zero (IT), the number of functional evaluations (NFE) counted as the sum of the number of evaluations of the function itself plus the number of evaluations of the derivative, the approximate zero x_*, and the value $f(x_*)$. Note that the approximate zeroes were displayed only up to the 28th decimal places, so making all look the same though they may in fact differ.

The test results in Table 1 show that for most of the functions we tested, the methods introduced in the present presentation for numerical tests have equal or better performance compared to the other methods of the same order. Notice that in some test cases we had divergence, but our methods always converged.

Table 1
Comparison of various sixth-order convergent iterative schemes and the Newton method

	IT	NFE	$f(x_*)$	δ
1				
NM	6	12	1.29e−61	1.26e−31
BM	3	12	−6.0e−127	3.79e−47
KM	3	12	−6.0e−127	4.71e−38
GM	3	12	−6.0e−127	1.14e−34
OM1	3	12	−6.0e−127	7.43e−35
OM2	3	12	1.0e−126	6.85e−36
[12]-1	3	12	−6.0e−127	5.65e−41
[12]-2	3	12	−6.0e−127	8.22e−39
2				
NM	7	14	−1.04e−50	7.33e−26
BM	4	16	−1.0e−127	4.42e−104
KM	4	16	−1.0e−127	5.35e−95
GM	4	16	−1.0e−127	2.98e−82
OM1	4	16	−1.0e−127	5.54e−79
OM2	4	16	−1.0e−127	3.94e−86
[12]-1	4	16	−1.0e−127	5.30e−126
[12]-2	4	16	−1.0e−127	0.00e+00
3				
NM	6	12	2.93e−55	9.1e−28
BM	5	20	−1.0e−127	4.16e−116
KM	4	16	0	2.89e−64
GM	4	16	1.0e−127	1.15e−63
OM1	4	16	−1.0e−127	9.74e−91
OM2	4	16	1.0e−127	3.0e−128
[12]-1	4	16	1.0e−127	3.519e−32
[12]-2	4	16	1.0e−127	3.85e−64
4				
NM	6	12	−3.76e−64	3.19e−32
BM	3	12	0	3.13e−27
KM	3	12	0	3.88e−28
GM	3	12	0	3.76e−26
OM1	3	12	0	1.10e−31
OM2	3	12	0	2.49e−31
[12]-1	3	12	0	5.01e−26
[12]-2	4	16	0	0.00e+00
5				
NM	9	18	1.41e−84	6.86e−43
BM	4	16	0	1.63e−68
KM	4	16	0	4.65e−48
GM	4	16	0	3.16e−34
OM1	4	16	0	4.15e−34
OM2	4	16	0	1.88e−37
[12]-1	4	16	0	1.04e−42
[12]-2	Div			
6				
NM	8	16	5.44e−72	9.17e−37
BM	4	16	0	2.07e−115
KM	4	16	0	6.95e−78
GM	4	16	0	4.67e−59
OM1	4	16	0	1.11e−58
OM2	4	16	0	2.18e−63
[12]-1	4	16	0	2.30e−75
[12]-2	4	16	0	7.07e−55
7				
NM	7	14	−2.27e−63	8.63e−33
BM	4	16	−1.1e−126	6.65e−120
KM	4	16	−1.1e−126	1.22e−96
GM	3	12	−1.1e−126	1.05e−26
OM1	4	16	1.2e−126	3.90e−95
OM2	4	16	1.2e−126	7.07e−112
[12]-1	3	12	−1.1e−126	6.76e−29
[12]-2	3	12	1.2e−126	7.97e−29
8				
NM	21	42	9.09e−78	3.26e−40
BM	6	24	0	1.08e−71

Table 1 (*continued*)

	IT	NFE	$f(x_*)$	δ
KM	7	28	0	0
GM	9	36	0	7.76e−121
OM1	11	44	0	4.68e−72
OM2	9	36	0	7.06e−42
[12]-1	10	40	0	5.56e−72
[12]-2	10	40	0	5.97e−72
9				
NM	6	12	−1.5e−80	1.80e−40
BM	3	12	−2.0e−128	3.70e−52
KM	3	12	−2.0e−128	1.55e−44
GM	3	12	−2.0e−128	1.98e−42
OM1	3	12	−2.0e−128	2.67e−46
OM2	3	12	−2.0e−128	3.39e−45
[12]-1	3	12	−2.0e−128	2.68e−44
[12]-2	3	12	−2.0e−128	3.26e−43
10				
NM	10	20	1.74e−62	2.63e−33
BM	7	28	0	1.22e−59
KM	4	16	0	1.01e−45
GM	Div			
OM1	5	20	0	2.35e−39
OM2	5	20	0	1.56e−78
[12]-1	4	16	0	3.81e−30
[12]-2	4	16	0	3.91e−30
11				
NM	8	16	−5.0e−67	9.75e−33
BM	Div			
KM	Div			
GM	Div			
OM1	5	20	0	5.59e−78
OM2	4	16	0	1.07e−35
[12]-1	6	24	0	6.20e−104
[12]-2	6	24	0	1.81e−103
12				
NM	14	28	6.1e−54	8.42e−28
BM	Div			
KM	4	16	0	1.34e−30
GM	5	20	0	8.54e−50
OM1	8	32	0	1.76e−74
OM2	7	28	0	2.92e−86
[12]-1	5	20	0	5.40e−29
[12]-2	5	20	0	1.97e−30
13				
NM	8	16	−2.5e−79	4.46e−39
BM	Div			
KM	5	20	−1.0e−127	4.44e−47
GM	4	16	0	1.25e−35
OM1	5	20	1.0e−127	0
OM2	4	16	1.0e−127	2.54e−48
[12]-1	5	20	−1.0e−127	0
[12]-2	5	20	1.0e−127	0
14				
NM	13	26	1.7e−51	2.23e−26
BM	15	60	−1.0e−127	0
KM	9	36	2.0e−127	7.27e−35
GM	10	40	−1.0e−127	1.15e−115
OM1	13	52	−1.0e−127	3.26e−44
OM2	9	36	−1.0e−127	5.63e−29
[12]-1	10	40	−1.0e−127	1.92e−96
[12]-2	16	64	−1.0e−127	5.35e−47

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